Stochastic Programming
Models in Design

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OUTLINE

• Models
  • General - Farming
  • Structural design
  • Design portfolio
  • General
• Approximations
• Solutions
• Revisions
European Farming

Decision:
• How to plant 500 acres with wheat, corn and sugar beets?

<table>
<thead>
<tr>
<th>Wheat</th>
<th>Beets</th>
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<tr>
<td>Corn</td>
<td></td>
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• Issues: Livestock needs, quotas, costs, yields, prices

Farm Parameters

- **Livestock requirements**
  - 200 Tons of wheat
  - 240 Tons of corn

- **Prices**
  - Wheat: $170/ton to sell/ $238/ton to buy
  - Corn: $150/ton to sell/ $210/ton to buy
  - Beets: $36/ton up to 6000 ton (quota); $10/ton if over

- **Planting costs**
  - Wheat: $150/acre; Corn: $230/acre
  - Beets: $260/acre

- **Yields (means)**
  - Wheat: 2.5 tons/acre; Corn: 3 tons/acre
  - Beets: 20 tons/acre
Deterministic Farmer’s Problem

Formulation

\[
\begin{align*}
\text{Min} & \quad 150 x_1 + 230 x_2 + 260 x_3 + 238 a_1 - 170 v_1 + 210 a_2 - 150 v_2 -36 v_3 - 10 v_4 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 500 \quad \text{(acres)} \\
& \quad 2.5 x_1 + a_1 - v_1 \geq 200 \quad \text{(wheat)} \\
& \quad 3 x_2 + a_2 - v_2 \geq 240 \quad \text{(corn)} \\
& \quad 20 x_3 - v_3 - v_4 \geq 0 \quad \text{(beets)} \\
& \quad v_3 \leq 6000 \quad \text{(quota)}
\end{align*}
\]

SOLUTION: WHEAT CORN BEETS

ACRES (xi)= 120 80 300
YIELD = 300 240 6000
PROFIT = $118,600 per season

Scenario Solutions

Random Factor: Weather
  • Yield variations: +/- 20% of the mean

Scenario Approach
  • A - Optimistic - Assume +20%
  • B - Pessimistic - Assume -20%

SOLUTION: WHEAT CORN BEETS

ACRES (xi)= 183 67 250
YIELD = 550 240 6000
PROFIT = $167,667 per season

ACRES (xi)= 100 25 375
YIELD = 200 60 6000
PROFIT = $59,950 per season

Tight constraints
**STOCHASTIC PROGRAM**

- **ASSUME**: Plant without knowing future
  - Suppose each scenario equally likely (prob. = 1/3 each)
  - Place in single mathematical program
- **GOAL**: maximize expected profits
  - (risk neutral)
- **FORMULATION:**

\[
\begin{align*}
\text{Min } & 150x_1 + 230x_2 + 380x_3 + \frac{1}{3} \sum_{i=1,2} (238a_{1i} - 170v_{1i} + 210a_{2i} - 150v_{2i} - 36v_{3i} - 10v_{4i}) \\
\text{s.t. } & x_1 + x_2 + x_3 \leq 500 \text{ (acres)} \\
& (1+0.2(2-i))2.5x_1 + a_{1i} - v_{1i} \geq 200 \text{ (wheat)} \\
& (1+0.2(2-i))3x_2 + a_{2i} - v_{2i} \geq 240 \text{ (corn)} \\
& (1+0.2(2-i))20x_3 - v_{3i} - v_{4i} \geq 0 \text{ (beets)} \\
& v_{3i} \leq 6000 \text{ (quota)} \\
& x_1, x_2, x_3, a_{1i}, a_{2i}, v_{1i}, v_{2i}, v_{3i}, v_{4i} \geq 0
\end{align*}
\]

**Axle Design Example**

**Figure 1.** An axle of length \(l\), diameter \(d\), with a central load \(P\).

- Random: \(d(\hat{d})\)
- Density

\[
f_{d}(d) = \begin{cases} 
\frac{100}{0.9d}(d - 0.9\hat{d}) & \text{if } 0.9\hat{d} \leq d < \hat{d}; \\
\frac{100}{1.1d - d}(1.1d - d) & \text{if } \hat{d} \leq d \leq 1.1\hat{d}; \\
0 & \text{otherwise.} 
\end{cases}
\]  

(2.4.1)

- Decision: \(d \leq d_{\text{max}}\) and \(l \leq l_{\text{max}}\)
- Selling price:

\[
s(1 - e^{-0.1l}),
\]

(2.4.2)

- Manufacturing cost:

\[
\frac{c(l\pi\hat{d}^2)}{l}. 
\]

(2.4.3)
• Stress constraint: \[
\frac{l}{h^2} \leq 39.27. \quad (2.4.4)
\]

• Deflection constraint: obtain: \[
\frac{l^3}{h^4} \leq 63169. \quad (2.4.5)
\]

• Nonlinear recourse function: \[
Q(t, d, d) := \min_y \left\{ \frac{l}{h^2} - y \leq 39.27, \frac{l^3}{h^4} - 300y \leq 63169 \right\}, \quad (2.4.6)
\]

• Expected recourse function: \[
Q(t, d) = \int_d Q(t, d, d)f_d(d)dd. \quad (2.4.7)
\]

**Full Problem**

\[
\max \text{ (total revenue per item } - \text{ manufacturing cost per item} - \text{ expected future cost per item).} \quad (2.4.9)
\]

\[
\max z(l, d) = s(1 - e^{-0.1l}) - c\left(\frac{ln\frac{(l/2)^3}{4}}{l^3}\right) - Q(t, d),
\]

\[
s. t. \quad 0 \leq l \leq l_{max}, 0 \leq d \leq d_{max}. \quad (2.4.10)
\]

**Stochastic Solution**

\[
l_{max} = 35, d_{max} = 1.25, s = 10, c = .025, w = 1
\]

\[
l^* = 33.6, d^* = 1.063, z^* = z(l^*, d^*) = 8.94
\]

**Deterministic Solution**

\[
l^{Det} = 35.0719, d^{Det} = 0.963, z^{Det}(l^{Det}, d^{Det}) = 9.07, z(l^{Det}, d^{Det}) = 5.88
\]

\[
\text{Value of the Stochastic Solution}
\]

\[
z^* - z(l^{Det}, d^{Det}) = 3.06
\]
Example for Yacht Design

- Yacht velocity prediction program - A. Philpott
  - Determines velocity based on input parameters
  - Can be optimized for various conditions
  - Includes design parameters
- Stochastic variables
  - Wind velocity
  - Angle to wind
  - Hydrodynamic resistance

Deterministic Problem - Example

- Decision variables:
  - Length: Long, medium, short
- Conditions:
  - Wind: Strong or light
- Outcomes:

<table>
<thead>
<tr>
<th>Wind</th>
<th>Length</th>
<th>Prob. of Win</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strong</td>
<td>L</td>
<td>0.8 -Optimal /Strong</td>
</tr>
<tr>
<td>Strong</td>
<td>M</td>
<td>0.6</td>
</tr>
<tr>
<td>Strong</td>
<td>S</td>
<td>0.2</td>
</tr>
<tr>
<td>Light</td>
<td>L</td>
<td>0.2</td>
</tr>
<tr>
<td>Light</td>
<td>M</td>
<td>0.6</td>
</tr>
<tr>
<td>Light</td>
<td>S</td>
<td>0.8 -Optimal /Light</td>
</tr>
</tbody>
</table>
Deterministic Value

- Suppose equal likelihood on conditions
- Prob. of win:
  - If Long, \((1/2)(0.8) + (1/2)(0.2) = 0.5\)
  - If Short, \((1/2)(0.8) + (1/2)(0.2) = 0.5\)
  - If Medium, \((1/2)(0.6) + (1/2)(0.6) = 0.6\)
- Note: Deterministic is not optimal, also no deterministic opt. value is opt. overall
- Value of Stochastic Solution = 0.6 - 0.5 = 0.1

Portfolio Problem

- Suppose two boats possible
- Suppose choice given previous conditions,
  - If strong now, then \(P(\text{Strong at race}) = 0.8\)
  - If light now, then \(P(\text{Light at race}) = 0.8\).
- Prob. of win =
  - If Strong now, choose Long, \(P(\text{Win/Lo and St}) = (0.8)(0.8) + (0.2)(0.2) = 0.68\)
  - If Light now, choose Short, \(P(\text{Win/Sh and Li}) = 0.68\)
  - If Light or Strong now, choose Medium, \(P(\text{Win/Medium}) = 0.6\)
Portfolio Observations

- Portfolio allows:
  - Increased prob. of win
  - Use of learning about uncertainty
  - Partial hedging
- Note: Change in solution structure
- Difficulties in probability evaluation and integration with yacht design problem

General Multistage Model

- FORMULATION:
  \[
  \begin{align*}
  \text{MIN} & \quad E \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}) \right] \\
  \text{s.t.} & \quad x_t \in X_t \\
  & \quad x_t \text{ nonanticipative} \\
  & \quad P( h_t(x_t, x_{t+1}) \leq 0 ) \geq a \text{ (chance constraint)}
  \end{align*}
  \]

EXAMPLES:

- FARM: Linear functions, continuous variables
- AXLE: Nonlinear plus continuous variables
- YACHT: Nonlinear objective, integer variables
DYNAMIC PROGRAMMING VIEW

- **STAGES**: t=1,...,T
- **STATES**: $x_t \rightarrow B x_t$ (or other transformation)
- **VALUE FUNCTION**:
  - $\Psi_t(x_t) = E[\psi_t(x_t, \xi_t)]$ where
  - $\xi_t$ is the random element and
  - $\psi_t(x_t, \xi_t) = \min f_t(x_t, x_{t+1}, x_t) + \Psi_{t+1}(x_{t+1})$
    - s.t. $x_{t+1} \in X_{t+1}(\xi_t)$ $x_t$ given
- **SOLVE**: iterate from T to 1
- **PROBLEM**: How to find $E[\psi_t(x_t, \xi_t)]$?
  - $\xi_t$ may have high dimension

ALTERNATIVES FOR FINDING $\Psi_t$

- **DIRECT NUMERICAL INTEGRATION**
  - Possible only if very small or special structure
  - Not applicable to general, large problems
- **SIMULATION**
  - Limited convergence rate ($1/\sqrt{n}$ error for n samples)
  - Difficult estimates of confidence intervals on solutions
- **BOUNDING APPROXIMATIONS**
  - Find $\Psi_t^{l,k}$ and $\Psi_t^{u,k}$ such that:
    - $\Psi_t^{l,k} \leq \Psi_t \leq \Psi_t^{u,k}$
    - $\lim_k \Psi_t^{l,k} = \Psi_t = \lim_k \Psi_t^{u,k}$
    - where limit is “epigraphical”
**BOUNDING APPROXIMATIONS**

- **GOALS**
  - Maintain solvable system
  - Ensure solution value within bounds
  - Convergence of bounds

- **BASIC IDEA**
  - Use convexity/duality
  - Construct feasible:
    - Dual solutions
      - Lower bounds
    - Primal solutions
      - Upper rounds

- **CONVERGENCE**
  - No duality gap
  - Improving refinements

**DISCRETIZATIONS**

- **SIMPLIFY THE DISTRIBUTION**
  - Replace P by P^k which has finite support:

  ![Diagram](image)

  MAIN PROCEDURES:
  - Lower: Jensen (mean)
  - Upper: Edmundson-Madansky (extreme points)
BOUND IMPROVEMENTS

- PARTITIONING
  - SPLIT $\xi$ (SUPPORT OF RANDOM VECTOR) INTO SUBREGIONS
  - MAKE FUNCTION $\psi$ AS LINEAR AS POSSIBLE ON EACH SUBREGION

ENFORCE SEPARABILITY:
- FIND SEPARABLE RESPONSES TO ALL RANDOM PARAMETER CHANGES

SOLVING AS LARGE-SCALE MATHEMATICAL PROGRAMS

- ORIGIN:
  - DISCRETIZATION LEADS TO MATHEMATICAL PROGRAM BUT LARGE-SCALE
  - USE STANDARD METHODS BUT EXPLOIT STRUCTURE

- DIRECT METHODS
  - TAKE ADVANTAGE OF SPARSITY STRUCTURE
    • SOME EFFICIENCIES
  - USE SIMILAR SUBPROBLEM STRUCTURE
    • GREATER EFFICIENCY - DECOMPOSITION

- SIZE
  - UNLIMITED (INFINITE NUMBERS OF VARIABLES)
  - STILL SOLVABLE (CAUTION ON CLAIMS)
DECOMPOSITION METHODS

BENDERS IDEA
- FORM AN OUTER LINEARIZATION OF $\Psi_t$

ADD CUTS ON FUNCTION:

LINEARIZATION AT ITERATION $k$

USE AT EACH STAGE TO APPROXIMATE VALUE FUNCTION
- ITERATE BETWEEN STAGES UNTIL ALL MIN = $\Psi_t$

DECOMPOSITION IMPLEMENTATION

NESTED DECOMPOSITION
- LINEARIZATION OF VALUE FUNCTION AT EACH STAGE
- DECISIONS ON WHICH STAGE TO SOLVE, WHICH PROBLEMS AT EACH STAGE

LINEAR PROGRAMMING SOLUTIONS
- USE OSL FOR LINEAR SUBPROBLEMS
- USE MINOS FOR NONLINEAR PROBLEMS

PARALLEL IMPLEMENTATION
- USE NETWORK OF RS6000S
- PVM PROTOCOL
RESULTS

- SCAGR7 PROBLEM SET

![Graph showing log (CPUs) vs log (no. of variables)]

- PARALLEL: 60-80% EFFICIENCY IN SPEEDUP
- OTHER PROBLEMS: SIMILAR RESULTS
  - ONLY < ORDER OF MAGNITUDE SPEEDUP WITH STORM
  - TWO-STAGES - LITTLE COMMONALITY IN SUBPROBLEMS
  - STILL ABLE TO SOLVE ORDER OF MAGNITUDE LARGER PROBLEMS

SOME OPEN ISSUES

- MODELS
  - IMPACT ON METHODS
  - RELATION TO OTHER AREAS
- APPROXIMATIONS
  - USE WITH SAMPLING METHODS
  - COMPUTATION CONSTRAINED BOUNDS
  - SOLUTION BOUNDS
- SOLUTION METHODS
  - EXPLOIT SPECIFIC STRUCTURE
  - MASSIVELY PARALLEL ARCHITECTURES
  - LINKS TO APPROXIMATIONS
CRITICISMS

- **UNKNOWN COSTS OR DISTRIBUTIONS**
  - Find all available information
  - Can construct bounds over all distributions
    - Fitting the information
  - Still have known errors but alternative solutions
- **COMPUTATIONAL DIFFICULTY**
  - Fit model to solution ability
  - Size of problems increasing rapidly (over 20 million variables)

CONCLUSIONS

- **STOCHASTIC PROGRAMS CAN BE:**
  - Linear, nonlinear, integer programs
  - Continuous or discrete r.v.’s
  - Of significant value (VSS) over deterministic models
  - Integration into design problems
  - Portfolios to Hedge
- **RANDOMNESS =>**
  - Value of modeling
  - Difficulty in evaluating objectives
  - Motivation for approximation
- **SOLUTIONS**
  - Decomposition for linear problems
  - Speedups of orders of magnitude