Sensitivity Analysis of the Eisenberg-Noe Model of Contagion

Ming Liu and Jeremy Staum[1]

Department of Industrial Engineering and Management Sciences, Robert R. McCormick School of Engineering and Applied Science, Northwestern University, Evanston, IL 60208-3119, U.S.A.

Abstract

We use linear programming to provide a sensitivity analysis of Eisenberg and Noe's one-period model of contagion via direct bilateral links. We provide a formula for the sensitivities of clearing payments and the terminal wealth of each node to initial wealth of each node.

Keywords: systemic risk, contagion, clearing payment, sensitivity analysis

1. Introduction

Eisenberg and Noe [1] present a one-period model of contagion via direct bilateral links. They provide an algorithm for computing the payments that each node makes. Elsinger [2] extends their model to include cross-holdings of equity among nodes and multiple levels of seniority for debt. We use linear programming to provide a sensitivity analysis of the Eisenberg-Noe model, showing how the payments made and the terminal wealth of nodes are influenced by small changes in the initial wealth of each node. We use the same

[1]Corresponding author. Email address: j-staum@northwestern.edu

techniques to address multiple levels of seniority for debt and sensitivities to the nodes' liabilities in Liu and Staum [4].

In systemic risk management, it is vital to address scenarios that involve large changes to wealth, but the analysis of sensitivity to small changes also has an important role to play. Sensitivity analysis is a crucial ingredient in optimization and risk allocation methods that are based on gradients. We envision applications in which sensitivity analysis is applied in each of many scenarios sampled from a distribution describing shocks to the financial system. An average sensitivity across scenarios can be used to describe the impact of a small change in initial wealth on the expected performance of the financial system. For example, in Liu and Staum [4], we use this scenario-by-scenario sensitivity analysis of the Eisenberg-Noe model in a risk allocation method for deposit insurance.

2. The Eisenberg-Noe Model and Algorithm

In the Eisenberg-Noe model, there are N nodes which have promised to make certain payments to each other. Node i has initial wealth e_i and has total liabilities of $\bar{p}_i$. The fraction of its total liabilities owed to node j is Π_{ij}. A node has no liabilities to itself, so $\Pi_{ii} = 0$ for all $i = 1, \ldots, N$. Also, if $\bar{p}_i = 0$, let $\Pi_{ij} = 0$. If node i pays p_i in total, then it pays $p_i \Pi_{ij}$ to node j because of the equal priority of all liabilities. Then the terminal wealth of node j is

$$v_j = e_j + \sum_{i \neq j} p_i \Pi_{ij} - p_j. \tag{1}$$

Let $\boldsymbol{e}$, $\boldsymbol{p}$, $\bar{\boldsymbol{p}}$, and $\boldsymbol{v}$ be the vectors whose ith components are respectively e_i, p_i, $\bar{p}_i$, and v_i; let $\boldsymbol{\Pi}$ be the matrix whose (i, j)th element is Π_{ij}. In this notation, Equation (1) can be rewritten as

$$\boldsymbol{v} = \boldsymbol{e} + (\boldsymbol{\Pi}^\top - \boldsymbol{I})\boldsymbol{p}. \tag{2}$$

Eisenberg and Noe [1] are concerned with the existence, uniqueness, and computation of a clearing payment vector satisfying the following conditions for all $j = 1, \ldots, N$:

- the total payment node j makes is nonnegative and does not exceed its total liabilities: $0 \leq p_i \leq \bar{p}_i$,
- limited liability of equity: $v_j \geq 0$, and
- priority of debt over equity: $v_j > 0$ only if $p_j = \bar{p}_j$.

(These conditions on $\boldsymbol{p}$ and $\boldsymbol{v}$ can be interpreted as conditions on the payment vector $\boldsymbol{p}$ alone by making use of Equation (2).) Under mild assumptions on $\boldsymbol{\Pi}$, $\boldsymbol{e}$, and $\bar{\boldsymbol{p}}$, the clearing payment vector is unique. One sufficient condition is to have $e_i > 0$ for all $i = 1, \ldots, N$ [1, §2.4]. We assume this condition holds and let $\boldsymbol{p}^*$ denote the unique clearing payment vector, and $\boldsymbol{v}^* = \boldsymbol{e} + (\boldsymbol{\Pi}^\top - \boldsymbol{I})\boldsymbol{p}^*$ denote the resulting vector of terminal wealth.

Eisenberg and Noe [1, §3.2] prove that, for any row vector $\boldsymbol{c}$ whose coefficients are all strictly positive, an optimal solution of the linear program

$$P(\boldsymbol{c}, \boldsymbol{e}): \quad \max_{\boldsymbol{p}} \boldsymbol{c}\boldsymbol{p} \text{ such that } (\boldsymbol{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} \leq \boldsymbol{e},\ \boldsymbol{0} \leq \boldsymbol{p} \leq \bar{\boldsymbol{p}}$$

is a clearing payment vector. Indeed, $\boldsymbol{p}^*$ is the unique optimal solution of this linear program. That is, $\boldsymbol{p}^*$ can be found by maximizing a weighted sum of all nodes' payments, subject to the constraints that a node's payment can exceed what it receives from other nodes by no more than its initial wealth, and its payment is nonnegative and can not exceed its promised payment. Our sensitivity analysis is based on a reformulation of the linear program $P(\boldsymbol{c}, \boldsymbol{e})$.

For analysis of computational complexity, it is preferable to consider another algorithm for computing the clearing payment vector $\boldsymbol{p}^*$, the fictitious default algorithm of Eisenberg and Noe [1, §3.1]. Where N is the number of nodes, the fictitious default algorithm requires $\mathcal{O}(N)$ iterations, each of which involves $\mathcal{O}(N^2)$ operations, for a computational complexity of $\mathcal{O}(N^3)$. A brute-force method of computing sensitivities to the initial wealth of each of N nodes involves applying the fictitious default algorithm $\mathcal{O}(N)$ times, each time with a sufficiently small perturbation to the initial wealth of one node. The computational complexity of this brute-force method is $\mathcal{O}(N^4)$. Our method of sensitivity analysis takes the clearing payment vector as an input, and its computational complexity is $\mathcal{O}(N^3)$, dominated by the inversion of an $N \times N$ matrix in Equation (3). There are thousands of banks in the United States alone, so our method could be thousands of times faster than the brute-force method in large-scale applications. For applications in which sensitivity analysis is performed in each of many scenarios, the increased speed would be practically significant.

3. Alternative Linear Programs

For purposes of sensitivity analysis, $P(\boldsymbol{c}, \boldsymbol{e})$ can be reformulated by introducing the vector $\boldsymbol{v}$ of slack variables for the inequality constraint $(\boldsymbol{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} \leq \boldsymbol{e}$. The slack variables allow the formulation of an equality constraint $(\boldsymbol{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} + \boldsymbol{v} = \boldsymbol{e}$ which, with the bound $\boldsymbol{v} \geq 0$, implies the inequality constraint $(\boldsymbol{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} \leq \boldsymbol{e}$. The equality constraint $(\boldsymbol{I} - \boldsymbol{\Pi}^\top)\boldsymbol{p} + \boldsymbol{v} = \boldsymbol{e}$ is equivalent to Equation (2), i.e. the slack variable v_j is the terminal wealth of node j if the payment vector is $\boldsymbol{p}$. This justifies the use of the notation $\boldsymbol{v}$ for the vector of slack variables. Let

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{bmatrix}, \quad \tilde{\boldsymbol{c}} = \begin{bmatrix} \boldsymbol{c} & \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{\Pi}^\top & \boldsymbol{I} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{u} = \begin{bmatrix} \bar{\boldsymbol{p}} \\ \infty \end{bmatrix},$$

where each sub-matrix or sub-vector is of size N. The linear program $P(\boldsymbol{c}, \boldsymbol{e})$ is equivalent to the linear program

$$\tilde{P}(\boldsymbol{c}, \boldsymbol{e}): \quad \max_{\boldsymbol{x}} \tilde{\boldsymbol{c}}\boldsymbol{x} \ \text{ such that } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{e}, \ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u},$$

whose optimal solution $\boldsymbol{x}^*$ is the transpose of $[(\boldsymbol{p}^*)^\top;\ (\boldsymbol{v}^*)^\top]$.

Because we are not primarily interested in the sensitivity of the objective function value $\tilde{\boldsymbol{c}}\boldsymbol{x}^* = \boldsymbol{c}\boldsymbol{p}^*$ for a vector $\boldsymbol{c}$ whose components are all positive, we also consider linear programs in which the objective is to maximize the payment made by a single node or the terminal wealth of a single node. Let $\boldsymbol{\xi}^j$ be a row vector of length N whose jth element is 1 and whose other elements are zero. For $j = 1, \ldots, N$, $\tilde{P}(\boldsymbol{\xi}^j, \boldsymbol{e})$ has objective function $[\boldsymbol{\xi}^j\ \mathbf{0}]\boldsymbol{x} = \boldsymbol{\xi}^j\boldsymbol{p} = p_j$. We consider these linear programs to learn about the sensitivity of p_j^* and v_j^* to $\boldsymbol{e}$. The following proposition implies that $\boldsymbol{x}^*$ is also an optimal solution to all such linear programs.

Proposition 1. *If $\boldsymbol{c}$ is a non-zero, non-negative row vector of length N, then $\boldsymbol{x}^*$ is an optimal solution of $\tilde{P}(\boldsymbol{c}, \boldsymbol{e})$.*

Proof. Suppose that $\boldsymbol{x} = [\boldsymbol{p};\ \boldsymbol{v}]$ is not an optimal solution of $\tilde{P}(\boldsymbol{c}, \boldsymbol{e})$. Because $\tilde{P}(\boldsymbol{c}, \boldsymbol{e})$ is equivalent to $P(\boldsymbol{c}, \boldsymbol{e})$, this implies that $\boldsymbol{p}$ is not an optimal solution of $P(\boldsymbol{c}, \boldsymbol{e})$. If $\boldsymbol{p}$ is infeasible, then it is not the clearing payment vector $\boldsymbol{p}^*$, which is feasible. If $\boldsymbol{p}$ is feasible but not optimal, then there exists a feasible $\boldsymbol{p}'$ such that $p'_j > p_j$ for some j.

Suppose that $\boldsymbol{p}$ is feasible and $p'_j > p_j$. Let $\boldsymbol{c}'$ be the row vector of length N defined by $c'_j = 1$ and $c'_i = (p'_j - p_j)/(1 + \sum_{k=1}^{N} \bar{p}_k)$ for all $i \neq j$. Its components are all strictly positive. Because $\boldsymbol{p}' \geq 0$ and $\boldsymbol{p} \leq \bar{\boldsymbol{p}}$,

$$\begin{aligned} \boldsymbol{c}'\boldsymbol{p}' - \boldsymbol{c}'\boldsymbol{p} &= (p_j' - p_j) + \frac{p_j' - p_j}{1 + \sum_{k=1}^N \bar{p}_k} \sum_{i \neq j} (p_i' - p_i) \\ &\geq (p_j' - p_j)\left(1 - \frac{\sum_{i \neq j} \bar{p}_i}{1 + \sum_{k=1}^N \bar{p}_k}\right) > 0. \end{aligned}$$

Because $\boldsymbol{p}'$ is feasible and $\boldsymbol{c}'\boldsymbol{p}' > \boldsymbol{c}'\boldsymbol{p}$, $\boldsymbol{p}$ is not an optimal solution of $P(\boldsymbol{c}', \boldsymbol{e})$. At the end of Section 1, we observed that the clearing payment vector $\boldsymbol{p}^*$ is an optimal solution of $P(\boldsymbol{c}', \boldsymbol{e})$ for all $\boldsymbol{c}'$ with strictly positive components. Therefore $\boldsymbol{p} \neq \boldsymbol{p}^*$.

Thus, if $\boldsymbol{x} = [\boldsymbol{p};\ \boldsymbol{v}]$ is not an optimal solution of $\tilde{P}(\boldsymbol{c}, \boldsymbol{e})$, then whether or not $\boldsymbol{p}$ is feasible, $\boldsymbol{p} \neq \boldsymbol{p}^*$, so $\boldsymbol{x} \neq \boldsymbol{x}^*$. $\square$

Our approach is to solve $\tilde{P}(\tilde{\boldsymbol{c}}, \boldsymbol{e})$ once, for a single value of $\boldsymbol{c}$ whose components are all strictly positive (for example, all equal to one), to get $\boldsymbol{x}^*$, and then to compute sensitivities with respect to $e_1, \ldots, e_N$ by performing sensitivity analysis on $\tilde{P}(\boldsymbol{\xi}^1, \boldsymbol{e}), \ldots, \tilde{P}(\boldsymbol{\xi}^N, \boldsymbol{e})$ at $\boldsymbol{x}^*$. The reason to do this is that $\tilde{P}(\boldsymbol{c}, \boldsymbol{e})$ has a unique optimal solution which is also optimal for each problem $\tilde{P}(\boldsymbol{\xi}^j, \boldsymbol{e})$, whereas it is possible for $\tilde{P}(\boldsymbol{\xi}^j, \boldsymbol{e})$ to have multiple optimal solutions, some of which are not optimal for $\tilde{P}(\boldsymbol{\xi}^i, \boldsymbol{e})$ where $i \neq j$.

4. Sensitivity Analysis

To perform sensitivity analysis on $\tilde{P}(\boldsymbol{\xi}^j, \boldsymbol{e})$, we consider bases for the optimal solution $\boldsymbol{x}^*$, which we characterize by introducing a classification of nodes. Because of the priority of debt over equity (discussed in Section 1), each node j falls into one of three mutually exclusive sets:

1. $\mathcal{V}_+$, positive terminal wealth: $v_j^* > 0$ and $p_j^* = \bar{p}_j$,
2. $\mathcal{V}_-$, default: $v_j^* = 0$ and $p_j^* < \bar{p}_j$, or
3. $\mathcal{V}_0$, borderline: $v_j^* = 0$ and $p_j^* = \bar{p}_j$.

The variable v_j is non-binding if $j \in \mathcal{V}_+$, while p_j is non-binding if $j \in \mathcal{V}_-$. These variables must be basic. If there are borderline nodes, forming a basis of size N requires including some binding variables in the basis, in which case $\boldsymbol{x}^*$ is a degenerate solution and has multiple bases. We consider two bases $\mathcal{B}^+$ and $\mathcal{B}^-$ of $\boldsymbol{x}^*$. If node j has positive terminal wealth, v_j is a basic variable in $\mathcal{B}^+$ and $\mathcal{B}^-$; if node j defaults, p_j is a basic variable in $\mathcal{B}^+$ and $\mathcal{B}^-$; and if node j is borderline, v_j is a basic variable in $\mathcal{B}^+$ and p_j is a basic variable in $\mathcal{B}^-$. If there are no borderline nodes, $\boldsymbol{x}^*$ is a non-degenerate solution and has a single basis $\mathcal{B} = \mathcal{B}^+ = \mathcal{B}^-$. To express this formally, let the variables be ordered as $[p_1, \ldots, p_N, v_1, \ldots, v_N]$, so that j is the index of the variable p_j and $N + j$ is the index of the variable v_j. Then

$$\mathcal{B}^+ = \mathcal{V}_- \cup \{N + j : j \in \mathcal{V}_0 \cup \mathcal{V}_+\} \quad \text{and} \quad \mathcal{B}^- = \mathcal{V}_- \cup \mathcal{V}_0 \cup \{N + j : j \in \mathcal{V}_+\}.$$

The following proposition uses the bases $\mathcal{B}^+$ and $\mathcal{B}^-$ to provide the desired sensitivity analysis. First, we define some notation. When a basis such as $\mathcal{B}$ is used as a subscript of a vector or matrix, the result is a vector or matrix formed by selecting the rows or columns whose indices are in $\mathcal{B}$. For example, the basic matrix $\boldsymbol{B} = \boldsymbol{A}_{\cdot\mathcal{B}}$ is the square matrix formed of the columns of $\boldsymbol{A}$ corresponding to variables in the basis $\mathcal{B}$, while $[\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}}$ is the square matrix whose (h,i)th element is one if $x_h = p_h$ is the ith of N variables in the basis $\mathcal{B}$, and is zero otherwise. Let $\partial^-/\partial e_i$ and $\partial^+/\partial e_i$ be the left and right derivatives with respect to e_i. A matrix of the form $\partial \boldsymbol{p}^*/\partial \boldsymbol{e}$ has (h,i)th element equal to $\partial p_h^*/\partial e_i$.

Proposition 2. *The partial derivatives of the clearing payments with respect to initial wealth are given by*

$$\frac{\partial^+ \boldsymbol{p}^*}{\partial \boldsymbol{e}} = [\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^+} \left(\boldsymbol{B}^+\right)^{-1} \quad \text{and} \quad \frac{\partial^- \boldsymbol{p}^*}{\partial \boldsymbol{e}} = [\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^-} \left(\boldsymbol{B}^-\right)^{-1}. \tag{3}$$

If there is no borderline node, then $\partial \boldsymbol{p}^*/\partial \boldsymbol{e} = [\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}} \boldsymbol{B}^{-1}$.

Proof. For any $h = 1, 2, \ldots, N$, by Proposition 1, $\boldsymbol{x}^*$ is an optimal solution to $\tilde{P}(\boldsymbol{\xi}^h, \boldsymbol{e})$, whose objective function is $[\boldsymbol{\xi}^h\ \boldsymbol{0}]\boldsymbol{x} = \boldsymbol{\xi}^h \boldsymbol{p} = p_h$. For any $i = 1, 2, \ldots, N$, let $\varepsilon_i = \min\{\bar{p}_k - p_k^* : k \in \mathcal{V}_-\}/2$, which is strictly positive. Consider the LP $\tilde{P}(\boldsymbol{\xi}^h; \boldsymbol{e}')$ where $\boldsymbol{e}' = \boldsymbol{e} + \varepsilon_i \boldsymbol{\xi}^i$, representing an increase in the initial wealth e_i to $e_i' = e_i + \varepsilon_i$. Let $\mathcal{V}_-'$, $\mathcal{V}_+'$, and $\mathcal{V}_0'$ represent the sets of defaulting nodes, nodes with positive terminal wealth, and borderline nodes, respectively, in the scenario where initial wealth is $\boldsymbol{e}'$. The same nodes default in both scenarios, $\mathcal{V}_-' = \mathcal{V}_-$, and the original nodes with positive terminal wealth still have positive terminal wealth, $\mathcal{V}_+ \subseteq \mathcal{V}_+'$. The original borderline nodes may remain borderline or may have positive terminal wealth in the new scenario: $\mathcal{V}_0 \subseteq \mathcal{V}_0' \cup \mathcal{V}_+'$. From $\mathcal{V}_-' = \mathcal{V}_-$ and $\mathcal{V}_0 \cup \mathcal{V}_+ = \mathcal{V}_0' \cup \mathcal{V}_+'$ it follows that $\mathcal{B}^+$ is still a basis for $\boldsymbol{x}^*$ in $\tilde{P}(\boldsymbol{\xi}^j;\boldsymbol{e}')$; indeed, $\mathcal{B}^+ = (\mathcal{B}^+)'$. Because e_i can be increased without changing the basis $\mathcal{B}^+$, $\partial^+ p_h^*/\partial e_i = \partial^+ x_h^*/\partial e_i = [\boldsymbol{\xi}^h \ \boldsymbol{0}]_{\mathcal{B}^+}(\boldsymbol{B}^+)^{-1}(\boldsymbol{\xi}^i)^\top$ [3, Proposition 7]. The proof for the left derivatives is similar. The last assertion follows from $\mathcal{B} = \mathcal{B}^+ = \mathcal{B}^-$ in the absence of borderline nodes. □

With Equations (2) and (3), we can compute the partial derivatives of terminal wealth with respect to initial wealth:

$$\frac{\partial^+ \boldsymbol{v}^*}{\partial \boldsymbol{e}} = \boldsymbol{I} + (\boldsymbol{\Pi}^\top - \boldsymbol{I})\frac{\partial^+ \boldsymbol{p}^*}{\partial \boldsymbol{e}} \quad \text{and} \quad \frac{\partial^- \boldsymbol{v}^*}{\partial \boldsymbol{e}} = \boldsymbol{I} + (\boldsymbol{\Pi}^\top - \boldsymbol{I})\frac{\partial^- \boldsymbol{p}^*}{\partial \boldsymbol{e}}. \qquad (4)$$

If there is no borderline node, then $\partial \boldsymbol{v}^*/\partial \boldsymbol{e} = \boldsymbol{I} + (\boldsymbol{\Pi}^\top - \boldsymbol{I})\partial \boldsymbol{p}^*/\partial \boldsymbol{e}$.

5. Example

We illustrate our method with a simple example in which there are $N = 3$ nodes with

$$\boldsymbol{\Pi} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0.75 \\ 0.5 & 0.5 & 0 \end{bmatrix}, \quad \bar{\boldsymbol{p}} = \begin{bmatrix} 80 \\ 80 \\ 10 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{e} = \begin{bmatrix} 41 \\ 42 \\ 50 \end{bmatrix}.$$

The clearing vector is $\boldsymbol{p}^* = [66, 80, 10]^\top$. Choosing $\boldsymbol{c} = [1, 1, 1]$, the optimal solution of $\tilde{P}(\boldsymbol{c})$ is given by $\boldsymbol{p}^* = [66, 80, 10]$ and $\boldsymbol{v}^* = [0, 0, 133]^\top$. The first node defaults, the second is borderline, and the third has positive terminal wealth. Our bases are $\mathcal{B}^+ = [1, 5, 6]$, containing p_1, v_2, and v_3, and $\mathcal{B}^- =$

$[1, 2, 6]$, containing p_1, p_2, and v_3. By selecting columns from the matrix

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} - \boldsymbol{\Pi}^\top & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} 1 & -0.25 & -0.5 & 1 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 1 & 0 \\ -0.5 & -0.75 & 1 & 0 & 0 & 1 \end{bmatrix}$$

we derive the corresponding basis matrices

$$\boldsymbol{B}^+ = \boldsymbol{A}_{\cdot\mathcal{B}^+} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}^- = \boldsymbol{A}_{\cdot\mathcal{B}^-} = \begin{bmatrix} 1 & -0.25 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & -0.75 & 1 \end{bmatrix}.$$

The matrices $[\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^+}$ and $[\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^-}$ are

$$[\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^+} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{I}\ \boldsymbol{0}]_{\cdot\mathcal{B}^-} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From Equation (3),

$$\frac{\partial^+ \boldsymbol{p}^*}{\partial \boldsymbol{e}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial^- \boldsymbol{p}^*}{\partial \boldsymbol{e}} \approx \begin{bmatrix} 1.14 & 0.29 & 0 \\ 0.57 & 1.14 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From Equation (4),

$$\frac{\partial^+ \boldsymbol{v}^*}{\partial \boldsymbol{e}} = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \frac{\partial^- \boldsymbol{v}^*}{\partial \boldsymbol{e}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

These sensitivities indicate some facts which may interest an analyst of this three-node system. The sensitivity $\partial^- p_1^*/\partial e_1 \approx 1.14$ shows that a decrease of \$1 in the initial wealth of node 1 causes the clearing payment made by that node to drop by more than \$1. This happens because such a decrease also reduces the wealth flowing into node 2, making node 2 default, and thus reduces the clearing payment from node 2 to node 1, and hence the wealth flowing into node 1. From the sensitivities of $\boldsymbol{v}^*$ to $\boldsymbol{e}$, we see that whereas a decrease in the initial wealth of any node results in a loss of terminal wealth entirely borne by node 3, an increase in the initial wealth of node 1 results in a gain in terminal wealth split evenly between nodes 2 and 3, while an increase in the initial wealth of node 2 or 3 only increases the terminal wealth of that node itself.

Acknowledgements

The authors thank an anonymous referee for comments that helped to improve the presentation of this article. The second author gratefully acknowledges the support of an IBM faculty award.

References

[1] Eisenberg, L., Noe, T. H., 2001. Systemic risk in financial systems. Management Science 47 (2), 236–249.

[2] Elsinger, H., February 2007. Financial networks, cross holdings, and limited liability, available at: `http://ssrn.com/abstract=916763`.

[3] Ho, J. K., 2000. Computing true shadow prices in linear programming. Informatica 11 (4), 421–434.

[4] Liu, M., Staum, J., 2010. Systemic risk components and deposit insurance premia, working paper, Northwestern University.