PRICING AND HEDGING IN INCOMPLETE MARKETS: FUNDAMENTAL THEOREMS AND ROBUST UTILITY MAXIMIZATION

JEREMY STAUM

Abstract. We prove fundamental theorems of asset pricing for good deal bounds in incomplete markets, relating arbitrage-freedom and uniqueness of prices to existence and uniqueness of a pricing kernel with appropriate properties. The technology employed is duality of convex optimization in locally convex linear topological spaces. The concepts investigated are closely related to convex and coherent risk measures, exact functionals, and coherent lower previsions in the theory of imprecise probabilities. We apply the results to analyze a specific method for constructing good deal bounds, based on robust expected utility involving a unanimity rather than a maxmin criterion.

1. Introduction

The problem of pricing and hedging in incomplete markets demands a synthesis of the approaches of mathematical finance and economics: how does one hedge risks and establish preferences over residual, unhedgeable risks, and what implications does this have for pricing risks? At the same time, one must take account of the cost of hedging, as determined by current market prices, and of beliefs about future market prices and of fundamental preferences, which do not derive solely from current or historical market prices. In mathematical terms, the problem of pricing in incomplete markets is the problem of extending a function that gives the prices of marketed cashflows to a larger space of cashflows. The cashflows in the larger space but not the smaller marketed space are potential over-the-counter securities. The extension should have economic justification and be suitable for implementation by financial decision makers. The problem is important because the incorporation of features such as price jumps, transaction costs, and illiquidity into a model often yields incompleteness.

One approach to this problem arises from the consideration of equivalent martingale measures (EMMs) in no-arbitrage pricing theory. Under some conditions, market prices equal expected discounted terminal values, with the expectation taken under an EMM (DS99). In incomplete markets, there can be many EMMs, and one may propose criteria for selecting one. Expectation under this most-favored EMM is then the chosen extension of the market price function. Two criteria that have attracted extensive attention are minimization

Date: March 1, 2003; first version August 1, 2002.

This material is based upon work supported by the National Science Foundation under Grant No. DMS-0202958. The author is grateful for discussions with Philippe Artzner, Mark Broadie, Peter Carr, Freddy Delbaen, Terry Fine, Hans Föllmer, Paul Glasserman, Jean Jacod, Keith Lewis, Philip Protter, and Leandro Rêgo, and especially for the many helpful comments of Stefan Jaschke. He retains full responsibility for all errors.

of hedging residual variance (Sch96) and minimization of the relative entropy between the EMM and a subjective probability measure (Fri00b).

Another approach arises from consideration of the lower and upper no-arbitrage bounds for prices of nonredundant contingent claims. It analyzes tighter *good deal bounds*, which arise from the exclusion not only of arbitrages, but a larger *acceptance set* of *good deals* (CSR00). Recently, research in this area has taken inspiration from the work on coherent risk measures of Artzner et al. (ADEH99). It explicitly aims at creating a theory that occupies an intermediate position between no-arbitrage and expected utility theory, being more useful than the former and more robust than the latter. Recent papers include Carr et al. (CGM01), Černý and Hodges (ČH01), Jaschke and Küchler (JK00), and Roorda (Roo02). An investigation that similarly seeks to interpolate between no-arbitrage and expected utility theories, although not explicitly treating price bounds, is Frittelli (Fri00a). See (CGM01) and (ČH01) for further discussion of the relative merits and disadvantages of no-arbitrage and expected utility theories, as well as references to previous work along the same lines by economists not drawing on the coherent risk measure concept. The present paper continues the approach inspired by coherent risk measures, but is not restricted to the coherent case.

The first part of this paper (§§2–7) is an extension of the results of Jaschke and Küchler (JK00). We drop some assumptions of coherence and resolve some difficulties surrounding the converse in the fundamental theorem of asset pricing. The main tool is the duality theory of optimization in locally convex linear topological spaces. We use as a recurring example the case of bounded random variables, which several authors have treated in different contexts: Delbaen (Del02) and Föllmer and Schied (FS02a; FS02b; FS02c) on coherent and convex risk measures, Maaß (Maa02) on exact functionals, and Walley (Wal91) on imprecise probabilities. The second part of the paper (§§8–10) applies these results to a concrete proposal for pricing and hedging derivative securities in incomplete markets using imprecise beliefs and utilities. This grounds a generalization of the results of Carr et al. (CGM01) in expected utility concepts. It is also similar to Föllmer and Schied's (FS02b) use of Gilboa and Schmeidler's (GS89) maxmin expected utility to create an example of a convex risk measure. The important economic differences from that approach are that the method proposed here relies on a unanimity criterion which is more conservative when used for derivative security pricing, and that pricing depends on the trader's current portfolio.

The rest of the paper is organized as follows. In Section 2 we formulate no-good-deal price bounds and conditions for them to avoid arbitrage. Section 3 covers financial interpretations of the mathematical hypotheses needed for subsequent results. Sections 4 and 5 develop respectively the dual and primal results required for proving versions of the fundamental theorems of asset pricing, which occupy Section 6. In Section 7 we focus on the important special case of bounded random variables. A discussion of the proposed criterion of unanimity in expected utility occupies Section 8, while Section 9 analyzes this proposal, drawing on the results of Part I. We briefly discuss issues surrounding calibration and computation in Section 10. We conclude and discuss directions for future research in Section 11.

PART I: FUNDAMENTAL THEOREMS

2. Acceptance and Pricing

Let L be a linear space of cashflows for which we desire to establish bid and ask prices. We will derive results for pricing where L is endowed with a locally convex topology and paired with a dual space: see Section 4. The reader may consult the appendix of Jaschke and Küchler (JK00) for an introduction to locally convex linear topological spaces and their duality theory.

Let $L_+ \subset L$ be the subset of nonnegative cashflows. We assume that it is a cone, meaning a convex, absolutely homogeneous set, where absolutely homogeneous means positively homogeneous and containing 0. We do not assume that it has any "nice" topological properties, such as closedness or nonempty interior; see Example 3.1 for an example where it has empty interior.

Standing Assumption 2.1. L is a linear space and $L_+ \subseteq L$ is a cone.

We model the market with a function $\pi : L \mapsto (-\infty, \infty]$ giving market prices. It has the interpretation of an ask price, that is, if you purchase x, you must pay $\pi(x)$, whereas if you sell x, you receive $\pi(-x)$. Naturally, the 0 cashflow costs 0. To avoid some trivial cases, we assume there exist cashflows of both positive and negative price.

Standing Assumption 2.2. There is a market ask pricing function $\pi : L \to (-\infty, \infty]$ taking both positive and negative values, and with $\pi(0) = 0$.

The *effective domain* of π is the subset $R \subseteq L$ of cashflows that are marketed, or replicable; elsewhere π takes the value ∞. Our goal is to establish bid and ask prices for over-the-counter securities providing cashflows in $L \setminus R$.

Important special cases are those in which π is linear on a linear effective domain R, sublinear (convex and absolutely homogeneous), or convex. Linearity corresponds to frictionless markets. Sublinearity allows for proportional transaction costs, i.e. a fixed bid-ask spread for any transaction size. Convexity is consistent with more general transaction costs, trading constraints, and liquidity effects.

The set of cashflows you can have for free is $M{:=}\{x|\pi(x) \leq 0\} - L_+$, while the set of valuable and riskless cashflows is $L_+ \setminus \{0\}$, and the set of cashflows you can sell for cash now is $C{:=}\{x|\pi(-x) < 0\}$. An element of $M \cap (L_+ \setminus \{0\})$ is an *arbitrage*. An element of $M \cap C$ will be called a *cashout*. It is sometimes also called an arbitrage, but the distinction between these concepts is important enough here to warrant different names. Let a *near-arbitrage* be an element of $\mathrm{cl}(M) \cap (L_+ \setminus \{0\})$. The financial significance of being "near" depends on the topology in which the closure is taken.

Remark 2.1. If the topology is the strong topology of the $\|\cdot\|_\infty$-norm, a near-arbitrage is known as a *free lunch with vanishing risk*: see Delbaen and Schachermayer (DS99).

Let $A \subseteq L$ denote an *acceptance set*, that is, the set of cashflows that one is willing to accept without compensation. Say a set A is *monotone* when $A + L_+ \subseteq A$. We will assume the acceptance set is monotone; this represents a modicum of financial rationality.

Standing Assumption 2.3. A is nonempty and monotone.

For purposes of derivative security pricing, one interprets $x \in L$ as a change in wealth, so 0 is the status quo. Then it makes sense to have $0 \in A$. (From the perspective of portfolio optimization, $x \in L$ is a wealth, and 0 might very well not be acceptable.) Monotonicity and $0 \in A$ imply $L_+ \subseteq A$. Pure losses should be unacceptable: $A \cap (L_- \setminus \{0\}) = \emptyset$. Given our interpretation of contingent claims as changes in wealth, we will call an acceptance set A *rational* when it satisfies these three properties: monotonicity, acceptability of the status quo, and unacceptability of pure losses. These are equivalent to a subset of the axioms for coherent risk measures (ADEH99). Another interesting property is convexity of A, which corresponds to risk-aversion. We do not assume that any of these properties other than monotonicity holds, but they feature as hypotheses of some results.

Remark 2.2. Both π and A may depend on one's current portfolio. For instance, considerations of credit risk suggest that the price received for issuing liabilities in a state of the world depends on one's wealth in that state. A contingent claim's acceptability may depend on whether it hedges or exacerbates risks already present in the portfolio.

When we are willing to accept any claim $x \in A$, from our counterparty's point of view, the set of cashflows to be had for free is $M - A$. We can also describe the set $A - M$ as our hedging-aware acceptance set. We would like this set to satisfy the conditions:

- NC(π, A): $(M - A) \cap C = \emptyset$
- NA(π, A): $(M - A) \cap L_+ \setminus \{0\} = \emptyset$
- NNA(π, A): $\mathrm{cl}(M - A) \cap L_+ \setminus \{0\} = \emptyset$

These stand for "No Cashout," "No Arbitrage," and " No Near-Arbitrage," respectively.

Remark 2.3. Although the concepts are not quite equivalent, the condition NA(π, A) relates to (the absence of) Jaschke and Küchler's (JK00) good deals of the first kind, while NC(π, A) relates to their good deals of the second kind, or good deals simply.

Define our ask and bid prices for a cashflow x as

$$a_{\pi,A}(x) := \inf_{y \in L} \{\pi(y) | y - x \in A\} = \inf_{y \in R} \{\pi(y) | y - x \in A\} \tag{1}$$

and

$$b_{\pi,A}(x) := -\, a_{\pi,A}(-x) = \sup_{y \in R} \{-\pi(y) | x + y \in A\}. \tag{2}$$

We should interpret $a_{\pi,A}(x)$ as an unattained infimum selling price for x. Receiving any amount more than $a_{\pi,A}(x)$ while taking on the cashflow $-x$, we will be able to hedge acceptably and retain some profit. Getting exactly $a_{\pi,A}(x)$ would result at best in indifference.

Using L_+ as an acceptance set, we get the no-arbitrage bounds a_{π,L_+} and b_{π,L_+} for pricing and hedging in incomplete markets. If $L_+ \subseteq A$, then $b_{\pi,A} \geq b_{\pi,L_+}$ and $a_{\pi,A} \leq a_{\pi,L_+}$, so we get a bid-ask spread no less tight than the no-arbitrage bounds. If there is no arbitrage in market prices, then for all $y \in R$, $\pi(y) = a_{\pi,L_+}(y)$.

Proposition 2.1. *The ask $a_{\pi,A}$ is monotone. If $0 \in A$, then $a_{\pi,A} \leq \pi$. If A and π are convex, then $a_{\pi,A}$ is convex. If A and π are positively homogeneous, then $a_{\pi,A}$ is positively homogeneous; if moreover $0 \in A$, then $a_{\pi,A}$ is absolutely homogeneous.*

Proof. Monotonicity of $a_{\pi,A}$ follows from monotonicity of A: if $x_2 \geq x_1$, then $x_2+A \subseteq x_1+A$, so the infimum in $a_{\pi,A}(x_2)$ is taken over a smaller set. If $0 \in A$, then $x - x \in A$, so $a_{\pi,A}(x) \leq \pi(x)$.

Consider $x_1, x_2 \in L$ and $y_1, y_2 \in R$ such that $y_1 - x_1, y_2 - x_2 \in A$, that is, y_1 and y_2 are feasible in computing $a_{\pi,A}(x_1)$ and $a_{\pi,A}(x_2)$ respectively. If A is convex, for $\gamma \in [0,1]$, $\gamma(y_1 - x_1) + (1-\gamma)(y_2 - x_2) \in A$. Because

$$\gamma(y_1 - x_1) + (1-\gamma)(y_2 - x_2) = (\gamma y_1 + (1-\gamma)y_2) - (\gamma x_1 + (1-\gamma)x_2),$$

this shows that $\gamma y_1 + (1-\gamma)y_2$ is feasible in computing $a_{\pi,A}(\gamma x_1 + (1-\gamma)x_2)$. If π is convex, then $\pi(\gamma y_1 + (1-\gamma)y_2) \leq \gamma\pi(y_1) + (1-\gamma)\pi(y_2)$, so

$$a_{\pi,A}(\gamma x_1 + (1-\gamma)x_2) \leq \gamma a_{\pi,A}(x_1) + (1-\gamma)a_{\pi,A}(x_2),$$

and $a_{\pi,A}$ is convex.

Consider $x \in L$ and $y \in R$ such that $y - x \in A$. If A is positively homogeneous, then for $\lambda > 0$, $\lambda(y - x) \in A$. If π is positively homogeneous, then $\pi(\lambda y) = \lambda\pi(y)$. So $a_{\pi,A}(\lambda x) \leq \lambda a_{\pi,A}(x)$. But $x = (1/\lambda)(\lambda x)$, so $a_{\pi,A}(x) \leq (1/\lambda)a_{\pi,A}(\lambda x)$. Therefore $a_{\pi,A}(\lambda x) = \lambda a_{\pi,A}(x)$. Because $\pi(0) = 0$, if moreover $0 \in A$, $a_{\pi,A}(0) = 0$. □

We now formulate conditions under which the policy of selling a cashflow x for any price more than an ask $a(x)$ does not backfire by giving away a cashout or a near-arbitrage. These conditions are generalizations of NC(π, A) and NNA(π, A), which dealt only with transactions taking place at an infimum price of zero: see Proposition 5.3 for more about this relationship.

- NC(π, a): For any $x \in L$, $a(x) + a_{\pi,L_+}(-x) \geq 0$.
- NNA(π, a): For any $x \in L$ and $z \in L_+ \setminus \{0\}$, $a(x) + a_{\pi,L_+}(z - x) > 0$.

When NC$(\pi, a_{\pi,A})$ holds, $a_{\pi,A}$ does not give away a cashout: we can rewrite NC$(\pi, a_{\pi,A})$ as $a_{\pi,A}(x) \geq b_{\pi,L_+}(x)$, which shows that our counterparty must pay more than the lower no-arbitrage bound for x. On the other hand, suppose NC$(\pi, a_{\pi,A})$ fails, i.e. $d{:=}b_{\pi,L_+}(x) - a_{\pi,A}(x) > 0$. For any $\epsilon > 0$, there exists $y_\epsilon \in R$ such that $x + y_\epsilon \in L_+$ and $\pi(y_\epsilon) \leq \epsilon - b_{\pi,L_+}(x)$. Our counterparty could buy x from us for price $p{:=}a_{\pi,A}(x) + d/3$, choose $\epsilon = d/3$, and buy $y_{d/3}$ on the market. This strategy has cost $\pi(y_{d/3}) + p \leq (d/3 - b_{\pi,L_+}(x)) + (b_{\pi,L_+}(x) - d + d/3) = -d/3 < 0$, so our counterparty would get a cashout: a negative cost now with no future risk from $x + y_\epsilon \geq 0$.

Likewise, we can rewrite NNA$(\pi, a_{\pi,A})$ as $a_{\pi,A}(x) > b_{\pi,L_+}(x - z)$. Suppose this fails. For any $\epsilon > 0$, there exists $y_\epsilon \in R$ such that $x - z + y_\epsilon \in L_+$ and $\pi(y_\epsilon) \leq \epsilon - b_{\pi,L_+}(x - z)$. For any $\delta > 0$, our counterparty could buy x from us for price $p{:=}a_{\pi,A}(x) + \delta/2$ and buy $y_{\delta/2}$ on the market. This strategy has cost $\pi(y_{\delta/2}) + p \leq (\delta/2 - b_{\pi,L_+}(x - z)) + (a_{\pi,A}(x) + \delta/2) \leq \delta$ and results in the cashflow $x + y_{\delta/2} \geq z > 0$. So our counterparty can get as least as much as the fixed, desirable cashflow $z > 0$ for any positive price δ, no matter how small. This would not be giving away an arbitrage, but it would be arbitrarily close to doing so.

Remark 2.4. One might consider demanding more, for instance, that one does not give away a cashout in the course of selling several cashflows $x_1, \ldots, x_n$:

$$\sum_{i=1}^{n} a_{\pi,A}(x_i) \geq b_{\pi,L_+}\left(-\sum_{i=1}^{n} x_i\right).$$

This relates to Walley's (Wal91) criterion of "avoiding sure loss." However, it appears financially inappropriate for two reasons. First, one's acceptance set should change after a trade. Suppose that before the trade, one possessed the cashflow v and had the acceptance set A for changes. This corresponds to an acceptance set $v + A$ for cashflows. If the trade does not change one's beliefs or preferences, then after selling x and acquiring the hedge y, one's position is $v + y - x$, and the new acceptance set for changes should be $A + x - y$. Second, the act of acquiring the hedge y may have an effect on market prices, due to limited liquidity. (See Çetin et al. (ÇJP) for an approach to understanding and modeling liquidity costs.) To ignore the effect of one's trades on market prices is tantamount to assuming that the market pricing function π is subadditive, as it would be possible to acquire $y = \sum_{i=1}^{n} y_i$ for no more than $\sum_{i=1}^{n} \pi(y_i)$ by making n purchases in rapid succession. These considerations suggest that we may focus on a single pricing decision.

3. Finite-Cost Hedging and Continuity

Some later results involve the hypothesis that the ask price of any cashflow be finite. Like the mathematical conditions discussed so far, such as monotonicity and convexity, this has a financial meaning and is not merely a technical condition. It can be verified without actually computing the ask by analyzing the relationship between the acceptance set A and the market pricing function π.

Definition 3.1 (Full Domain)**.** *Full domain* for a function f means $\operatorname{dom} f = L$, i.e. $\forall x \in L,\ f(x) < \infty$.

What $\operatorname{dom} a_{\pi,A} = L$ says is that for all $x \in L$, there exists $y \in R$ such that $y - x \in A$. Because L is linear, $-x$ is always in L too, so this is equivalent to saying that every cashflow becomes acceptable after hedging at finite cost. This condition could fail, in which case we would need either a different approach than the present for establishing fundamental theorems, or to respecify the problem. One could attempt to price only cashflows that can be acceptably hedged, that is, restrict L to be $\operatorname{dom} a_{\pi,A}$, or one could enrich A to include some hedging residuals of the troublesome elements of L.

The following examples, which illustrate these points, have $L = L^0(\mathbb{R}, \mathcal{B}, \mathbf{P})$, with $\mathcal{B}$ the Borel sigma-algebra on $\mathbb{R}$, and $\mathbf{P}$ a probability measure. This is a space of random variables, interpreted as contingent claims. It makes sense to say L_+ is the set of $\mathbf{P}$-almost surely nonnegative contingent claims. First we observe that L_+ has empty interior under any vector topology $\mathcal{T}$.

Example 3.1 (Empty interior of L^0_+). Any $x \in L_+$ has a finite essential infimum. Consider the non-null event $E = \{\omega \in \mathbb{R} | x(\omega) < \inf x + 1\}$ that it takes a value within 1 of its essential infimum. There is some other random variable x_E that is essentially unbounded below on E. Then L_+ is not radial at x: for any $\delta > 0$, $x + \delta x_E$ is not almost surely bounded below, so it is not in L_+. Therefore x is not in the $\mathcal{T}$-interior of L_+, because any $\mathcal{T}$-open set is radially open (JK00, Prop. 17).

In the following example, $a_{\pi,A}$ does not have full domain, and we consider a way of restricting the space L of cashflows to be priced in order to give $a_{\pi,A}$ full domain.

Example 3.2 (Restricting aims). Let A be any acceptance set containing only contingent claims that are almost surely bounded below by some $K \leq 0$. That is, $-K$ is a maximum acceptable loss, or risk capital. Suppose the only marketed instrument is a riskless bond whose payoff is 1, and its unit price is 1 for transactions of any size. Then the marketed subset $R = \mathbb{R}$ is the linear subspace of constants, and π is effectively the identity. If x is not almost surely bounded below, then it can not be acceptably hedged. For any $c \in R$, $x + c$ is still not almost surely bounded below, so $a_{\pi,A}(-x) = \infty$. We could restrict L to be the linear subspace of almost surely bounded contingent claims, $L^\infty(\mathbb{R}, \mathcal{B}, \mathbf{P})$. If x is almost surely bounded by K, then $x + K \in L_+ \subseteq A$, so x can be hedged acceptably at a cost of K.

The final example considers expanding the acceptance set so that $a_{\pi,A}$ has full domain. The new acceptance set need not be convex: this is an example of how risk-seeking behavior may arise.

Example 3.3 (Limited liability). The setting is the same as in Example 3.2. Now suppose that the decision-maker enjoys limited liability and suffers the same consequences whenever the contingent claim pays off less than K. This might be a trader who can at worst lose his job, or a proprietor of a business empire who can at worst see the group's bank go bankrupt. Let u be a utility function on $\mathbb{R}$, unbounded above. In the absence of risk management, the acceptance set A might include any contingent claim x such that $\mathbf{E}^{\mathbf{P}}[u(x)\mathbf{1}\{x \geq K\}] + u(K)\mathbf{P}[x < K] \geq 0$. This is an expected utility calculation, accounting for a fixed loss in the case of ruin. (Because $u(x)\mathbf{1}\{x \geq K\}$ is almost surely bounded below, the expectation exists, although it might be ∞.) Now every $x \in L$ can be acceptably hedged at finite cost, as follows. Pick $c, d \in \mathbb{R}$ such that $\mathbf{P}[x \geq c] > 0$ and $u(d) \geq -u(K)\mathbf{P}[x < c]/\mathbf{P}[x \geq c]$, and let $z = x - c + d$. Then

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{z \geq K\}] + u(K)\mathbf{P}[z < K] \\
&\quad = \mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{z \geq d\}] + \mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{K \leq z < d\}] + u(K)\mathbf{P}[z < K] \\
&\quad \geq u(d)\mathbf{P}[z \geq d] + u(K)\mathbf{P}[K \leq z < d] + u(K)\mathbf{P}[z < K] \\
&\quad = u(d)\mathbf{P}[x \geq c] + u(K)\mathbf{P}[x < c]
\end{aligned}$$

because $\{z \geq d\} = \{x \geq c\}$ by definition of z. By definition of d, this quantity is nonnegative, so $z \in A$, which shows that x can be acceptably hedged for the finite cost $d - c$.

In using duality theory in Section 4, we will also be concerned with continuity of ask prices with respect to some locally convex vector topology $\mathcal{T}$ on L.

Definition 3.2 (Semi-Continuity)**.** A function f is *lower (upper) semi-continuous* when, equivalently,

- For any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, $f(x) \leq \liminf_{x_n \to x} f(x_n)$, respectively $f(x) \geq \limsup_{x_n \to x} f(x_n)$.

- For any $\alpha \in \mathbb{R}$, the set $\{x|f(x) \leq \alpha\}$ is closed, respectively $\{x|f(x) \geq \alpha\}$ is closed.
- For any $\alpha \in \mathbb{R}$, the set $\{x|f(x) > \alpha\}$ is open, respectively $\{x|f(x) < \alpha\}$ is open.

Together, the two semi-continuities imply ordinary continuity. Because $\mathcal{T}$ is a vector topology, lower semi-continuity of $a_{\pi,A}$ is equivalent to upper semi-continuity of $b_{\pi,A}$, as follows. Lower semi-continuity of $a_{\pi,A}$ is openness of $\{x|a_{\pi,A}(x) > \alpha\} = \{x| - b_{\pi,A}(-x) > \alpha\} = -\{x|b_{\pi,A}(x) < -\alpha\}$ for all α, which is equivalent to openness of $\{x|b_{\pi,A}(x) < \alpha\}$ for all α.

Remark 3.1. The Fatou property for risk measures discussed by Delbaen (Del02) is lower semi-continuity with respect to the topology of bounded convergence in probability. See also Example 4.1.

It turns out that in the sublinear case, finite-cost hedging is a sufficient condition for the existence of a topology with respect to which the ask price is continuous. This will help us in our analysis of duality, where we will want to choose some such topology in order to look at an appropriate dual space of continuous linear functionals, because we can be sure that one exists. Here we prove that the lc-topology, the finest locally convex vector topology, makes the ask continuous.

Proposition 3.1. *If π is sublinear, A is a cone, and* $\operatorname{dom} a_{\pi,A} = L$*, then $a_{\pi,A}$ is lc-continuous.*

Proof. From Proposition 2.1, it follows that $a_{\pi,A}$ is sublinear.

First, we show that $a_{\pi,A}$ is upper semi-continuous with respect to the lc-topology. This means showing that the set $a^{-1}_{\pi,A}([-\infty, \alpha)) = \{x|a_{\pi,A}(x) < \alpha\}$ is lc-open for all $\alpha \in \mathbb{R}$. Any convex, radially open set is lc-open (JK00, Lem. 10(iv)). Because $a_{\pi,A}$ is convex, $a^{-1}_{\pi,A}([-\infty, \alpha))$ is convex. It remains to show that it is radial at all its points. Consider x such that $a_{\pi,A}(x) < \alpha$ and any $u \in L$. If $a_{\pi,A}(u) \leq 0$, then for any $\gamma \geq 0$,

$$a_{\pi,A}(x + \gamma u) \leq a_{\pi,A}(x) + \gamma a_{\pi,A}(u) \leq a_{\pi,A}(x) < \alpha,$$

where the first inequality follows from sublinearity of $a_{\pi,A}$. If $a_{\pi,A}(u) > 0$, pick a positive $\delta < (\alpha - a_{\pi,A}(x))/a_{\pi,A}(u)$, which is positive and finite. Then for any $\gamma \in [0, \delta]$,

$$a_{\pi,A}(x + \gamma u) \leq a_{\pi,A}(x) + \gamma a_{\pi,A}(u) \leq a_{\pi,A}(x) + \delta a_{\pi,A}(u) < \alpha.$$

Whether $a_{\pi,A}(u)$ is positive or not, $x + \gamma u$ is in $a^{-1}_{\pi,A}([-\infty, \alpha))$ for all sufficiently small nonnegative γ, so it is radially open.

Finally, we show that $a_{\pi,A}$ is lower semi-continuous with respect to the lc-topology. This means showing that the set $a^{-1}_{\pi,A}([-\infty, \alpha])$ is lc-closed for all $\alpha \in \mathbb{R}$. Any convex, radially closed set with nonempty radial interior is lc-closed (JK00, Prop. 19). Because $a_{\pi,A}$ is convex, $a^{-1}_{\pi,A}([-\infty, \alpha])$ is convex. It contains $a^{-1}_{\pi,A}([-\infty, \alpha))$, which has just been shown to be radially open, and is nonempty by the following Lemma 3.1. Therefore it has nonempty radial interior, and it remains to show that it is radially closed, or equivalently, that $a^{-1}_{\pi,A}((\alpha, \infty])$ is radially open. Consider any point w at which $a^{-1}_{\pi,A}((\alpha, \infty])$ is not radial. There exists $u \in L$ such that for all $\delta > 0$, there exists $\gamma \in [0, \delta]$ such that $a_{\pi,A}(w + \gamma u) \leq \alpha$. By definition of the ask, the sale of $w + \gamma u$ can be hedged acceptably for any cost exceeding α: for all $\epsilon > 0$, there exists y_ϵ such that $\pi(y_\epsilon) \leq \alpha + \epsilon$ and $y_\epsilon - (w + \gamma u) \in A$. Because $\operatorname{dom} a_{\pi,A} = L$, there exists y_u such that $\pi(y_u) < \infty$ and $y_u + u \in A$. Because A is a cone, the combination of acceptable cashflows is acceptable:

$$(y_\epsilon - (w + \gamma u)) + \gamma(y_u + u) = y_\epsilon - w + \gamma y_u \in A.$$

Because π is sublinear,

$$\pi(y_\epsilon + \gamma y_u) \leq \pi(y_\epsilon) + \gamma\pi(y_u) \leq \alpha + \epsilon + \gamma\pi(y_u).$$

So $a_{\pi,A}(w)$ is less than or equal to this quantity, for arbitarily small positive ϵ and γ. Because $\pi(y_u) < \infty$, this proves $a_{\pi,A}(w) \leq \alpha$, i.e. $w \notin a^{-1}_{\pi,A}((\alpha,\infty])$. Therefore $a^{-1}_{\pi,A}((\alpha,\infty])$ is radial at all its points. $\square$

Lemma 3.1. *If π is positively homogeneous and $0 \in A$, then for all $\alpha \in \mathbb{R}$, there exists x such that $a_{\pi,A}(x) < \alpha$.*

Proof. It suffices to prove this for $\alpha < 0$. By Assumption 2.2, there exists x_0 such that $\pi(x_0) < 0$. Choose $\lambda > \alpha/\pi(x_0)$, which is positive. Then by positive homogeneity, $\pi(\lambda x_0) = \lambda\pi(x_0) < \alpha$. Because $0 \in A$, $a_{\pi,A} \leq \pi$, by Proposition 2.1. $\square$

4. Duality

In this section, we establish a framework for dualization and find a dual representation for the ask $a_{\pi,A}$ and bid $b_{\pi,A}$. This dual representation is of computational interest and is an ingredient in the fundamental theorems of Section 6.

We say $(L, \mathcal{T})$ and $(L', \mathcal{T}')$ are *paired spaces* when $\mathcal{T}$ and $\mathcal{T}'$ are locally convex vector topologies and there is a bilinear form $\langle\cdot,\cdot\rangle : L \times L' \to \mathbb{R}$ such that $\{\langle\cdot, x'\rangle | x' \in L'\}$ is the set of continuous linear functionals on L, and vice versa. For this, $\forall x' \in L', \langle x, x'\rangle = 0$ must imply $x = 0$, and vice versa.

Remark 4.1. The largest space L' for which this can be done is $L^\times$, the algebraic dual of L, consisting of all linear functions on L, in which case L must be equipped with the lc-topology, the finest in which it is locally convex.

Standing Assumption 4.1. $(L, \mathcal{T})$ and $(L', \mathcal{T}')$ are paired spaces.

Example 4.1 (Two pairings). The space of bounded random variables $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, under the strong topology $\mathcal{T}_\infty$ of the $\|\cdot\|_\infty$-norm, pairs with $\mathrm{ba}(\Omega, \mathcal{F}, \mathbf{P})$, the space of finitely additive measures absolutely continuous with respect to $\mathbf{P}$. However, we might prefer to pair it with $\mathrm{ca}(\Omega, \mathcal{F}, \mathbf{P})$, the space of σ-additive measures absolutely continuous with respect to $\mathbf{P}$. To do so requires a coarser topology on $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, with fewer open sets and more convergence, in order to support fewer continuous linear functionals, i.e. pair with a smaller space. This coarser topology turns out to be the topology of bounded convergence in probability. This can be verified directly from the definition of the topology induced on $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ by $\mathrm{ca}(\Omega, \mathcal{F}, \mathbf{P})$ (DS58)[V.3.2]. It can also be seen from results in (Del02) and (FS02b), where the Fatou property for a coherent or convex risk measure, which is lower semi-continuity with respect to bounded convergence in probability, is shown to be equivalent to existence of a dual representation of the risk measure in terms of σ-additive probability measures.

Jaschke and Küchler (JK00, Cor. 9) have a version of the first fundamental theorem of asset pricing which involves the condition that $A - M$ be a closed cone, so that the bipolar theorem applies to it. They suggest finding conditions for closedness or a way to alter the set M so that $A - M$ would be closed. Instead, we shift focus to a different set

$$B := \{x | b_{\pi,A}(x) \geq 0\} = \{x | a_{\pi,A}(-x) \leq 0\} \tag{3}$$

which turns out to be the closure in question, under some conditions.

The right and left *polar cones* of a set $B \subseteq L$ and of a set $B' \subseteq L'$ are respectively

$$B^* := \{x' \in L' | \forall x \in B, \langle x, x' \rangle \geq 0\} \quad \text{and} \quad {}^*B' := \{x \in L | \forall x' \in B', \langle x, x' \rangle \geq 0\}.$$

The bipolar theorem implies that $B = {}^*(B^*)$ if B is a closed cone.

Proposition 4.1. *If π is sublinear, A is a cone, and $a_{\pi,A}$ is continuous, then* $\mathrm{cl}(A - M) = B = {}^*(B^*)$.

Proof. It follows from Proposition 2.1 that B is a cone. If $a_{\pi,A}$ is lower semi-continuous, then B is closed. Once B is a closed cone, the bipolar theorem (JK00, Thm. 20) applies. It follows from the definition of B that it contains $A - M$. Therefore it suffices to show that the radial interior of B is a subset of $A - M$ to establish that B is the radial closure of $A - M$. Consider any x in the radial interior of B. For all $u \in L$, there exists $\delta > 0$ such that $x + \delta u \in B$, i.e. for all $\epsilon > 0$, there is a y such that $\pi(y) \leq \epsilon$ and $x + \delta u + y \in A$. Choose u such that $\pi(u) < 0$. Then $\pi(\delta u + y) \leq \delta\pi(u) + \epsilon$ by subadditivity. This is negative for small enough positive ϵ. Therefore $x \in A - M$. This establishes that B is the radial closure of $A - M$. By Lemma 3.1 and upper semi-continuity of $a_{\pi,A}$, it has nonempty interior. When a convex set has nonempty interior, its closure equals its radial closure (JK00, Prop. 18). □

Remark 4.2. Radial closure is the condition given by Föllmer and Schied (FS02a) for an acceptance set that generates a convex risk measure to equal the acceptance set generated by that convex risk measure.

The translation invariance property of Jaschke and Küchler's numeraire allows them to prove a fundamental theorem of asset pricing as a direct consequence of the comparison of $A - M$ and ${}^*((A - M)^*)$. The connection does not seem so direct here. Instead, we apply duality theory to the ask $a_{\pi,A}$ in the usual way for minimizations, resulting in Theorem 4.1.

Define the penalty function Ψ on L' by

$$\Psi(x') := \sup_{x \in A}(-\langle x, x' \rangle) + \sup_{y \in R}(\langle y, x' \rangle - \pi(y)). \tag{4}$$

The first term has an interpretation as the extent to which x' disagrees about the desirability of cashflows in A. It is (in a more abstract setting) the minimal penalty function in the convex risk measure representation theorems of (FS02a), up to change of sign. Likewise, the second term measures the disagreement between x' and market prices, which π specifies. Now we can find a dual representation for the ask and bid, in the same spirit as the representation theorems for coherent and convex risk measures. This is the dual ingredient in the fundamental theorems.

Theorem 4.1. *For all $x \in L$,*

$$b_{\pi,A}(x) \leq \inf_{x' \in L'} (\langle x, x' \rangle + \Psi(x')) \tag{5}$$

$$a_{\pi,A}(x) \geq \sup_{x' \in L'} (\langle x, x' \rangle - \Psi(x')). \tag{6}$$

If moreover $a_{\pi,A}$ is lower semi-continuous, and π and A are convex, then equality holds.

Proof. The statements about $b_{\pi,A}$ and $a_{\pi,A}$ are equivalent. We focus on $a_{\pi,A}$ because its primal problem is a minimization, which is more usually studied in the convex optimization literature. The primal value is $\inf_{y \in L}\{\pi(y) | y - x \in A\}$. Our framework for dualization is the function $F : L \times L \to (-\infty, \infty]$ given by

$$F(y,u) = \begin{cases} \pi(y) & \text{if } y - (x+u) \in A \\ +\infty & \text{otherwise} \end{cases}$$

where u has the interpretation of a perturbation to x, the cashflow to be priced. The optimal value function is

$$\phi(u) = \inf_{y \in L} F(y,u) = \inf_{y \in L}\{\pi(y) | y - (x+u) \in A\} = a_{\pi,A}(x+u).$$

The Lagrangian $K : L \times L' \to [-\infty, \infty]$ is given by $K(y, x') = \inf_{u \in L}(F(y,u) + \langle u, x' \rangle)$ and the dual objective by $g(x') = \inf_{y \in L} K(y, x')$, so we get

$$g(x') = \inf_{y,u \in L}(F(y,u) + \langle u, x' \rangle) = \inf_{y,u \in L}\{\pi(y) + \langle u, x' \rangle | y - (x+u) \in A\}.$$

The dual value is $\sup_{x' \in L'} g(x') = \sup_{x' \in L'} g(-x')$, so we can exclude from this maximization those values of x' such that $g(-x') = -\infty$. We substitute $z = y - (x+u)$ so the constraint in the minimization that yields $g(-x')$ is $z \in A$. The objective is

$$\pi(y) - \langle u, x' \rangle = \pi(y) - \langle y - x - z, x' \rangle = \langle x, x' \rangle + \langle z, x' \rangle + (\pi(y) - \langle y, x' \rangle).$$

Therefore

$$\begin{aligned} g(-x') &= \langle x, x' \rangle + \inf_{z \in A} \langle z, x' \rangle + \inf_{y \in L} (\pi(y) - \langle y, x' \rangle) \\ &= \langle x, x' \rangle - \sup_{z \in A}(-\langle z, x' \rangle) - \sup_{y \in L} (\langle y, x' \rangle - \pi(y)), \end{aligned}$$

which is the supremand in formula (6). Duality theory asserts that the primal value is greater than or equal to the dual value, justifying the inequality in (6). If A and π are convex, then F is convex. When F is convex, the dual value is $\liminf_{u \to 0} \phi(u)$ (Roc74, Thm. 7). The primal value is $\phi(0)$, so lower semi-continuity of $a_{\pi,A}$ (hence of ϕ) implies no duality gap. ☐

When equality holds, $-b_{\pi,A}$ is a convex risk measure. Föllmer and Schied (FS02a) defined a convex risk measure to have $\rho(1) = -1$ for mathematical convenience. If one were to adopt instead the definition of Artzner et al. (ADEH99), that the risk measure should merely be additive with respect to some numeraire $\mathbf{1}$ of unit price, then the following proposition would hold with the hypothesis that $\pi(c\mathbf{1}) = c$ for all $c \in \mathbb{R}$.

Proposition 4.2. *If $\pi(c) = c$ for all $c \in \mathbb{R}$, then $-\inf_{x' \in L'}(\langle \cdot, x' \rangle + \Psi(x'))$ is a convex risk measure.*

Proof. As in the proof of Thm. 5 of (FS02a), $f_{x'}(x) := \langle x, x' \rangle + \Psi(x')$ is concave, monotone, and constant-additive for each x', and these properties are preserved by taking the infimum. If $\langle \cdot, x' \rangle$ is monotone, $f_{x'}$ is monotone. If not, there exists $y \in L_+$ such that $\langle y, x' \rangle < 0$. By Assumption 2.3, there exists $z \in A$ such that for all $\lambda \geq 0$, $z + \lambda y \in A$. Therefore $\sup_{x \in A}(-\langle x, x' \rangle) \geq \sup_{\lambda \geq 0}(-\langle z + \lambda y, x' \rangle) = \sup_{\lambda \geq 0}(-\langle z, x' \rangle - \lambda \langle y, x' \rangle) = \infty$. So $f_{x'}(x) = \Psi(x') = \infty$, which is monotone anyway. Similarly, if $\langle c, x' \rangle = c$ for all $c \in \mathbb{R}$, we have constant-additivity, and if not, $\sup_{y \in R}(\pi(y) - \langle y, x' \rangle \geq \sup_{c \in \mathbb{R}}(c - \langle c, x' \rangle) = \infty$, and we get constant-additivity anyway, because $\infty + c = \infty$. $\square$

5. Sublinearity and Cones

In this section, we relate the case where π is sublinear and A is a convex cone, described in Propositions 3.1 and 4.1, to the more general case, where π and A need not have these properties. We rely on some definitions and notation relating sets to cones and functions to sublinear functions. For any set C, let $C_\vee$ be the smallest cone containing C. For any function f, let conv f be given by

$$(\text{conv } f)(x) := \inf \left\{ \sum_{i=1}^{n} \lambda_i f(x_i) \middle| \sum_{i=1}^{n} \lambda_i x_i = x, \sum_{i=1}^{n} \lambda_i = 1, x_i \in \text{dom } f, \lambda_i \geq 0 \right\}.$$

This is the greatest convex function dominated by f. Let ah f be given by

$$(\text{ah } f)(x) := \inf\{\lambda f(x/\lambda) | \lambda > 0\}$$

for $x \neq 0$, and $(\text{ah } f)(0) := 0$. If f is convex, $f(0) < \infty$, and $f \neq \infty$, this is the greatest absolutely homogeneous function dominated by f; see (Roc70§5). Let $f_\vee :=$conv ah f, which is given by

$$f_\vee(x) = \inf \left\{ \sum_{i=1}^{n} \lambda_i f(x_i) \middle| \sum_{i=1}^{n} \lambda_i x_i = x, x_i \in \text{dom } f, \lambda_i \geq 0 \right\}.$$

This is the greatest sublinear function dominated by f. In what follows, we may write down only the interesting constraints in this minimization.

The greatest sublinear function dominated by the ask $a_{\pi,A}$ is the ask generated from the acceptance set $A_\vee$ and market pricing function $\pi_\vee$.

Proposition 5.1. $(a_{\pi,A})_\vee = a_{\pi_\vee, A_\vee}$.

Proof. Making the substitutions $y = \sum_{i=1}^n \lambda_i y_i$ and $z_i = y_i - x_i$,

$$\begin{aligned}
(\text{conv ah } a_{\pi,A})(x) &= \inf \left\{ \sum_{i=1}^{n} \lambda_i a_{\pi,A}(x_i) \middle| \sum_{i=1}^{n} \lambda_i x_i = x \right\} \\
&= \inf \left\{ \sum_{i=1}^{n} \lambda_i \pi(y_i) \middle| y_i - x_i \in A, \sum_{i=1}^{n} \lambda_i x_i = x \right\} \\
&= \inf \left\{ \pi_\vee(y) \middle| z_i \in A, \sum_{i=1}^{n} \lambda_i z_i = y - x \right\} \\
&= \inf\{\pi_\vee(y) | y - x \in A_\vee\}.
\end{aligned}$$

□

Define the *marginal ask* and bid by

$$\tilde{a}_{\pi,A}(x) := \liminf_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) \quad \text{and} \quad \tilde{b}_{\pi,A}(x) := \limsup_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda). \tag{7}$$

These are the most favorable prices that a counterparty who wants to execute a small trade can come close to attaining. The following proposition gives conditions under which these also equal the prices generated from the acceptance set $A_\vee$ and market pricing function $\pi_\vee$.

Lemma 5.1. *If f is convex and $f(0) \le 0$, then $(ah\ f)(x) = \lim_{\lambda\to\infty} \lambda f(x/\lambda)$.*

Proof. By convexity, for $\lambda_1 \ge \lambda_2 > 0$,

$$f(x/\lambda_1) \le \left(1 - \frac{\lambda_2}{\lambda_1}\right) f(0) + \left(\frac{\lambda_2}{\lambda_1}\right) f(x/\lambda_2),$$

so $\lambda_1 f(x/\lambda_1) \le (\lambda_1 - \lambda_2) f(0) + \lambda_2 f(x/\lambda_2) \ge \lambda_2 f(x/\lambda_2)$. Hence $\lambda f(x/\lambda)$ is nondecreasing as a function of $\lambda > 0$. Thus the infimum in the definition of (ah f) is a limit. □

Proposition 5.2. *If A is convex and contains 0, and π is convex, then $\tilde{a}_{\pi,A} = a_{\pi_\vee,A_\vee} = \lim_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda)$ and $\tilde{b}_{\pi,A} = b_{\pi_\vee,A_\vee} = \lim_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda)$.*

Proof. By Proposition 2.1, $a_{\pi,A}$ is convex and has $a_{\pi,A}(0) \le 0$. By Lemma 5.1, the limit exists, so

$$\tilde{a}_{\pi,A}(x) = \liminf_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) = \lim_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) = (\text{ah } a_{\pi,A})(x) = (a_{\pi,A})_\vee(x) = a_{\pi_\vee,A_\vee}(x),$$

because $a_{\pi,A}$ is already convex, and using Proposition 5.1. Then $b_{\pi_\vee,A_\vee}(x) = -a_{\pi_\vee,A_\vee}(-x) = -\lim_{\lambda\to\infty} -\lambda b_{\pi,A}(x/\lambda) = \lim_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda)$. □

Next we prove a result, relating conditions on π and A to conditions on $\pi_\vee$ and $A_\vee$, that is the primal ingredient in the fundamental theorems. In a sense, we are a considering a fictitious market in which $\pi_\vee$ gives the prices and $A_\vee$ is our acceptance set. This enables us to connect the case actually under consideration with the sublinear case, in which Propositions 3.1 and 4.1 will apply.

Let $\tilde{M} := \{x | \pi_\vee(x) \le 0\} - L_+$ and $\tilde{C} := \{x | \pi_\vee(-x) < 0\}$ be the set of cashflows you could respectively have for free and sell for cash now if $\pi_\vee$ gave market prices. Likewise let $\tilde{B} := \{x | a_{\pi_\vee,A_\vee}(-x) \le 0\}$. These are not necessarily the same as $M_\vee$, $C_\vee$, and $B_\vee$.

Lemma 5.2. *The following are monotone cones: $A_\vee \subseteq A_\vee - \tilde{M} \subseteq \tilde{B}$.*

Proof. Because $\pi_\vee(0) = 0$, $0 \in \tilde{M}$, so $A_\vee \subseteq A_\vee - \tilde{M}$. If $x = z - y$ where $z \in A_\vee$ and $\pi_\vee(y) \le 0$, then $x + y \in A_\vee$, so $b_{\pi_\vee,A_\vee}(x) \ge -\pi_\vee(y) \ge 0$, and $x \in \tilde{B}$. This shows $A_\vee - \tilde{M} \subseteq \tilde{B}$. By construction, $A_\vee$ and $\tilde{M}$ are cones, which makes $A_\vee - \tilde{M}$ a cone. By Proposition 2.1, $a_{\pi_\vee,A_\vee}$ is convex and absolutely homogeneous, so $\tilde{B}$ is a cone. By Assumption 2.3, A is nonempty and monotone, so any cone containing A contains L_+. Any cone K has the property $K + K \subseteq K$, so $L_+ \subseteq K$ implies $K + L_+ \subseteq K$, i.e. K is monotone. □

Proposition 5.3. *Among the conditions*

(1) *NC*(π, A)*: $(M - A) \cap C = \emptyset$*

(2) *NC*$(\pi, a_{\pi,A})$*:* $a_{\pi,A}(x) + a_{\pi,L_+}(-x) \geq 0$
(3) *NC*$(\pi_\vee, a_{\pi_\vee,A_\vee})$*:* $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(-x) \geq 0$
(4) *NC*$(\pi_\vee, A_\vee)$*:* $(\tilde{M} - A_\vee) \cap \tilde{C} = \emptyset$

the following implications hold: (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1)*. If A is convex and contains 0, and π is convex, then all the conditions are equivalent.*

Proof. (4) $\Rightarrow$ (3): Suppose $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(-x) < 0$. There exist y_1, y_2 such that $y_1 - x \in A_\vee$, $y_2 + x \in L_+$, and $\pi_\vee(y_1) + \pi_\vee(y_2) < 0$. Then by monotonicity of $A_\vee$, the sum $y_1 + y_2 \in A_\vee$, while by subadditivity of $\pi_\vee$, $\pi_\vee(y_1 + y_2) < 0$. So $A_\vee \cap (-\tilde{C})$ is nonempty, and by Lemma 5.2, this is enough.

(3) $\Rightarrow$ (4): Same as (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (2): Because $\pi_\vee \leq \pi$ and $A \subseteq A_\vee$, $a_{\pi,A} \geq a_{\pi_\vee,A_\vee}$ and $a_{\pi,L_+} \geq a_{\pi_\vee,L_+}$.

(2) $\Rightarrow$ (1): Suppose there exists x in the set $(M - A) \cap C$, so $\pi(-x) < 0$ and $x = y - z$ where $z \in A$ and $\pi(y) \leq 0$. Then $a_{\pi,L_+}(-x) \leq \pi(-x) < 0$, while $a_{\pi,A}(x) \leq \pi(y) \leq 0$ because $y - x = z \in A$. So $a_{\pi,A}(x) + a_{\pi,L_+}(-x) < 0$.

(1) $\Rightarrow$ (4): Here assume A is convex and contains 0, and π is convex. Suppose there exists $-x \in (\tilde{M} - A_\vee) \cap \tilde{C}$. That is, $\pi_\vee(x) < 0$ and $x = z - y$ where $z \in A_\vee$ and $\pi_\vee(y) \leq 0$. By subadditivity, $\pi_\vee(z) \leq \pi_\vee(x) + \pi_\vee(y) < 0$. So there exists $z \in A_\vee \cap (-\tilde{C})$. For some $\lambda_1 > 0$, $[0, \lambda_1 z] \subset A$. For some $\lambda_2 > 0$, π is negative on the line segment $(0, \lambda_2 z]$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$, so $(0, \lambda z] \subset A \cap (-C)$. Therefore $A \cap (-C)$ is nonempty, containing λz for some $\lambda > 0$. By Lemma 5.2, this is enough. $\square$

This result offers some guidance about choosing an acceptance set A. If π is convex and we choose a convex (risk-averse) acceptance set A that contains 0 and satisfies $(M - A) \cap C = \emptyset$, then we can be sure of satisfying the desideratum NC$(\pi, a_{\pi,A})$.

Having established this connection between results for (π, A) and for $(\pi_\vee, A_\vee)$, we can now revisit the dual problem (6). The dual-feasible set is $D' {:=} \{x' | \Psi(x') < \infty\}$. For use in the fundamental theorems, we consider the dual-feasible set when market prices are given by $\pi_\vee$ and the acceptance set is $A_\vee$. Define $\tilde{\Psi}$ as the penalty function Ψ in equation (4) with these substitutions, and $\tilde{D}' {:=} \{x' | \tilde{\Psi}(x') < \infty\}$. Call an element of $\tilde{D}'$ a *consistent pricing kernel*: item (4) below shows that it is consistent with the acceptance set and market prices. We now collect some properties of these objects.

Proposition 5.4. *Given the preceding definitions,*

(1) *The following are equivalent:* $x' \in A^*$, $\sup_{x \in A}(-\langle x, x' \rangle) \leq 0$, $x' \in (A_\vee)^*$, *and* $\sup_{x \in A_\vee}(-\langle x, x' \rangle) = 0$.
(2) *The following are equivalent:* $\langle \cdot, x' \rangle \leq \pi$, $\sup_{y \in R}(\langle y, x' \rangle - \pi(y)) = 0$, $\langle \cdot, x' \rangle \leq \pi_\vee$, *and* $\sup_{y \in R}(\langle y, x' \rangle - \pi_\vee(y)) = 0$.
(3) *The function $\tilde{\Psi}$ takes values in $\{0, \infty\}$.*
(4) *The set* $\tilde{D}' = \{x' | \tilde{\Psi}(x') = 0\} = \{x' | \Psi(x') = 0\} = (A_\vee)^* \cap \{x' | \langle \cdot, x' \rangle \leq \pi_\vee\} = A^* \cap \{x' | \langle \cdot, x' \rangle \leq \pi\}$*, that is, x' is a consistent pricing kernel if and only if it is dominated by π and nonnegative on A.*

Proof. (1) Recall that the definition of A^* is $\{x' | \forall x \in A,\ \langle x, x'\rangle \geq 0\}$, which shows the equivalence of $x' \in A^*$ and $\sup_{x\in A}(-\langle x, x'\rangle) \leq 0$. The equivalence of $x' \in (A_\vee)^*$ and $\sup_{x\in A_\vee}(-\langle x, x'\rangle) = 0$ is similar, with the added observation that this supremum is nonnegative because $0 \in A_\vee$. From $A \subseteq A_\vee$ it follows that $(A_\vee)^* \subseteq A^*$. Now consider $x' \in A^*$. Each $x \in A_\vee$ is a nonnegative linear combination of elements of A, at each of which $\langle \cdot, x'\rangle$ is nonnegative. Therefore $\langle \cdot, x'\rangle$ is nonnegative at x, so $x' \in (A_\vee)^*$. This shows $A^* = (A_\vee)^*$.

(2) Because $\pi_\vee(0) = \pi(0) = 0$, $\langle \cdot, x'\rangle \leq \pi$ and $\sup_{y\in R}(\langle y, x'\rangle - \pi(y)) = 0$ are equivalent, and likewise $\langle \cdot, x'\rangle \leq \pi_\vee$ and $\sup_{y\in R}(\langle y, x'\rangle - \pi_\vee(y)) = 0$ are equivalent. From $\pi_\vee \leq \pi$ it follows that $\langle \cdot, x'\rangle \leq \pi_\vee$ implies $\langle \cdot, x'\rangle \leq \pi$. Now suppose $\langle \cdot, x'\rangle \leq \pi$. For any nonnegative linear combination $\sum_{i=1}^n \lambda_i x_i = x$, $\langle x, x'\rangle = \sum_{i=1}^n \lambda_i \langle x_i, x'\rangle \leq \sum_{i=1}^n \lambda_i \pi(x_i)$. So $\pi_\vee(x)$ is the greatest lower bound of a set bounded below by $\langle x, x'\rangle$, therefore $\langle \cdot, x'\rangle \leq \pi_\vee$.

(3) Both terms in $\tilde{\Psi}$ are nonnegative and absolutely homogeneous.

(4) From part (3), $\tilde{D}' = \{x' | \tilde{\Psi}(x') = 0\}$. Then parts (1) and (2) imply the conclusion. □

Remark 5.1. If π is linear and x' is a consistent pricing kernel, then $\langle \cdot, x'\rangle$ is a linear extension of π.

6. Fundamental Theorems

Now we get two versions of the first fundamental theorem of asset pricing, one each for $\mathrm{NC}(\pi, a_{\pi,A})$ and $\mathrm{NNA}(\pi, a_{\pi,A})$. The 0th version does not quite deserve the name, because it relates only to absence of cashouts, not absence of arbitrage. In interpreting the hypothesis of lower semi-continuity for $a_{\pi_\vee,A_\vee}$, recall that by Proposition 3.1, $\operatorname{dom} a_{\pi_\vee,A_\vee} = L$ is sufficient for the existence of a locally convex topology in which $a_{\pi_\vee,A_\vee}$ is continuous. Moreover, because $a_{\pi_\vee,A_\vee} \leq a_{\pi,A}$, $\operatorname{dom} a_{\pi_\vee,A_\vee} = L$ is weaker than $\operatorname{dom} a_{\pi,A} = L$, the more natural hypothesis of finite-cost hedging. See also Proposition 7.2 for a simple sufficient condition for continuity of $a_{\pi_\vee,A_\vee}$ in the strong topology of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 6.1 (0th FTAP)**.** *The existence of a consistent pricing kernel implies $\mathrm{NC}(\pi, a_{\pi,A})$. If moreover A is convex and contains 0, π is convex, and $a_{\pi_\vee,A_\vee}$ is lower semi-continuous, then the converse holds.*

Proof. By Proposition 5.4, a consistent pricing kernel satisfies $\tilde{\Psi}(x') = 0$. Then Theorem 4.1 implies $b_{\pi_\vee,A_\vee} \leq \inf_{x'\in\tilde{D}'}\langle \cdot, x'\rangle \leq \langle \cdot, x'\rangle \leq \pi_\vee$. Therefore $\pi_\vee(x) < 0$ implies $x \notin \tilde{B}$. By Lemma 5.2, $A_\vee - \tilde{M}$ is a subset of $\tilde{B}$, so this implies that $A_\vee - \tilde{M}$ and $\{x|\pi_\vee(x) < 0\} = -\tilde{C}$ are disjoint. This in turn implies $\mathrm{NC}(\pi, a_{\pi,A})$, by Proposition 5.3.

Given the extra hypotheses, $\mathrm{NC}(\pi, a_{\pi,A})$ implies that for all $x \in L$, $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(-x) \geq 0$, by Proposition 5.3. Because $L_+ \subseteq A_\vee$, $a_{\pi_\vee,A_\vee}(0) \leq a_{\pi_\vee,L_+}(0) \leq 0$, so $a_{\pi_\vee,A_\vee}(0) = 0$. From Theorem 4.1 and Proposition 5.4 it follows that $a_{\pi_\vee,A_\vee}(0) = \sup_{x'\in\tilde{D}'}\langle 0, x'\rangle = \sup_{x'\in\tilde{D}'} 0$. Therefore $\tilde{D}' \neq \emptyset$, i.e. a consistent pricing kernel exists. □

The following theorem deserves to be called a first fundamental theorem of asset pricing, because $\mathrm{NNA}(\pi, a_{\pi,A})$ rules out giving away arbitrages. By a strictly monotone x', we mean one for which $\langle \cdot, x'\rangle$ is strictly monotone, i.e. for all $x \in L_+ \setminus \{0\}$, $\langle x, x'\rangle > 0$. The theorem shows that existence of a strictly monotone consistent pricing kernel is sufficient, but establishes its necessity only under hypotheses not only of convexity but also homogeneity, or in relation to the marginal ask.

Theorem 6.2 (1st FTAP). *The existence of a strictly monotone consistent pricing kernel implies NNA($\pi, a_{\pi,A}$). If A is convex and contains 0, π is convex, and $a_{\pi_\vee,A_\vee}$ is lower semi-continuous, then $\forall x \in L_+ \setminus \{0\}, \tilde{a}_{\pi,A}(x) > 0$ is equivalent to the existence of a strictly monotone consistent pricing kernel. If moreover A is a convex cone and π is sublinear, then NNA($\pi, a_{\pi,A}$) is equivalent to the existence of a strictly monotone consistent pricing kernel.*

Proof. Suppose NNA($\pi, a_{\pi,A}$) fails, so there are x and $z > 0$ such that $a_{\pi,A}(x)+a_{\pi,L_+}(z-x) \leq 0$. Then $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(z - x) \leq 0$, and equivalently, $b_{\pi_\vee,A_\vee}(-x) \geq a_{\pi_\vee,L_+}(z - x)$. By definition of the bid $b_{\pi_\vee,A_\vee}$ and ask $a_{\pi_\vee,L_+}$, for any $\epsilon > 0$, there are y_a, y_b such that $y_b-x \in A_\vee$, $y_a + x - z \in L_+$, and $\pi_\vee(y_b) \leq \epsilon - \pi_\vee(y_a)$. Because $A_\vee$ is monotone, we can add to get $y_a + y_b - z \in A_\vee$. By subadditivity, $\pi_\vee(y_a + y_b) \leq \epsilon$. As this can be done for all positive ϵ, $b_{\pi_\vee,A_\vee}(-z) \geq 0$, so $-z \in \tilde{B}$, and $\tilde{B} \cap (L_- \setminus \{0\}) \neq \emptyset$. By the following Lemma 6.1, this implies there is no strictly monotone consistent pricing kernel.

Now assume A is convex and contains 0, π is convex, and $a_{\pi_\vee,A_\vee}$ is lower semi-continuous. By Proposition 5.2, $\tilde{a}_{\pi,A} = a_{\pi_\vee,A_\vee}$. Therefore $\forall x \in L_+ \setminus \{0\}, \tilde{a}_{\pi,A}(x) > 0$ is equivalent to $\forall x \in L_+\setminus\{0\}, a_{\pi_\vee,A_\vee}(x) > 0$, which is in turn equivalent to $\tilde{B}\cap(L_-\setminus\{0\}) = \emptyset$. By Lemma 6.1, this is equivalent to the existence of a strictly monotone consistent pricing kernel.

Now assume A is a convex cone and π is sublinear. Then $A = A_\vee$ and $\pi = \pi_\vee$, so $a_{\pi,A} = a_{\pi_\vee,A_\vee}$. Suppose there is no strictly monotone consistent pricing kernel. By Lemma 6.1, this implies there exists $-z \in \tilde{B} \cap (L_- \setminus \{0\})$. Then $b_{\pi_\vee,A_\vee}(-z) \geq 0$, so $a_{\pi_\vee,A_\vee}(z) \leq 0$, and $a_{\pi_\vee,A_\vee}(z)+a_{\pi_\vee,L_+}(z-z) \leq 0$. That is, $a_{\pi,A}(z)+a_{\pi,L_+}(z-z) \leq 0$, violating NNA($\pi, a_{\pi,A}$). □

Lemma 6.1. *The existence of a strictly monotone consistent pricing kernel implies $\tilde{B} \cap (L_- \setminus \{0\}) = \emptyset$. If $a_{\pi_\vee,A_\vee}$ is lower semi-continuous, then the converse holds.*

Proof. Suppose x' is a strictly monotone consistent pricing kernel. From Theorem 4.1 we get $b_{\pi_\vee,A_\vee}(x) \leq \langle x, x'\rangle$. For $x < 0$, this implies $b_{\pi_\vee,A_\vee}(x) < 0$, thus $x \notin \tilde{B}$. The converse is an application of the same "exhaustion" argument that underpins the Halmos-Savage theorem, as explained by Delbaen in a proof of a similar result (Del02, Thm. 3.5). Consider the class of sets $\mathcal{C}:=\{C \subseteq L_+|\exists x' \in \tilde{D}' \ni \forall x \in C, \langle x, x'\rangle > 0\}$. From Proposition 5.4(4), we can see that $\tilde{D}'$ is convex and contains only nonnegative elements. Therefore class $\mathcal{C}$ is stable under countable unions: take a sequence $\{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$, and let $x'_n \in \tilde{D}'$ be such that for all $x \in C_n$, $\langle x, x'_n\rangle > 0$. Define $x':=\sum_{n=1}^{\infty} 2^{-n}x'_n \in \tilde{D}'$. Then any $x \in \cup_{n\in\mathbb{N}}C_n$ is in $C_j \subseteq L_+$ for some j, so $\langle x, x'\rangle = \langle x, \sum_{n\neq j} 2^{-n}x'_n\rangle + \langle x, 2^{-j}x'_j\rangle \geq 2^{-j}\langle x, x'_j\rangle > 0$. From stability under countable unions, it follows by Zorn's lemma that $\mathcal{C}$ has a maximal element. It is given in the converse that for any $x > 0$, $b_{\pi_\vee,A_\vee}(-x) < 0$, i.e. $a_{\pi_\vee,A_\vee}(x) > 0$. By Theorem 4.1, given lower semi-continuity, $a_{\pi_\vee,A_\vee}(x) = \sup_{x'\in\tilde{D}'}\langle x, x'\rangle$. Therefore for any $x > 0$, there is an $x' \in \tilde{D}'$ such that $\langle x, x'\rangle > 0$, i.e. $\{x\} \in \mathcal{C}$. Thus the only possible maximal element of $\mathcal{C}$ is $L_+ \setminus \{0\}$, and $L_+ \setminus \{0\} \in \mathcal{C}$ implies the existence of $x' \in \tilde{D}'$ such that for all $x > 0$, $\langle x, x'\rangle > 0$, which is a strictly monotone consistent pricing kernel. □

The following example shows that $\mathrm{NNA}(\pi, a_{\pi,A})$ does not imply $\mathrm{NNA}(\pi_\vee, a_{\pi_\vee,A_\vee})$, which is equivalent to the existence of a strictly monotone consistent pricing kernel. This accounts for the difficulty in framing a partial converse.

Example 6.1 (Trouble with non-closed cones). Let $L = \mathbb{R}^2$, the space of contingent claims in a two-state, one-period economy, $\mathcal{T}$ be the Euclidean norm topology, and the acceptance set $A = L_+ \cup \{x|x_2 \geq x_1^2\}$, which is closed, convex, monotone, and contains 0. Then $A_\vee = L_+ \cup \{x|x_2 > 0\}$, as follows. First we show $L_+ \cup \{x|x_2 > 0\} \subseteq A_\vee$. It is clear that $L_+ \subseteq A \subseteq A_\vee$. For the other points x, for which $x_1 < 0$ and $x_2 > 0$, define $\lambda{:=}x_1^2/x_2 \in (0,\infty)$, so $x_2/\lambda = (x_1/\lambda)^2$. Such a point is thus a positive multiple of an element of A, hence in $A_\vee$. Next we show $A_\vee \subseteq L_+ \cup \{x|x_2 > 0\}$, equivalently, $(L_+ \cup \{x|x_2 > 0\})^\complement \subseteq A_\vee^\complement$. The set $(L_+ \cup \{x|x_2 > 0\})^\complement = \{x|x_2 < 0\} \cup \{x|x_1 < 0, x_2 = 0\}$, which is a positively homogeneous set disjoint from A, which is convex. Therefore it is also disjoint from $A_\vee$. The new conic acceptance set $A_\vee$ is still disjoint with $L_- \setminus \{0\}$, but its closure is not, which causes a problem. Let M be the embedding of $\mathbb{R}$, namely $\{x|x_2 = x_1\}$, and $\pi(c) = c = \pi_\vee(c)$. Consider a contingent claim $z = (d, 0)$ with $d > 0$, which is in $L_+ \setminus \{0\}$. Letting $x = z$, $\mathrm{NNA}(\pi_\vee, a_{\pi_\vee,A_\vee})$ is violated, because $a_{\pi_\vee,A_\vee}(x) = 0$ and $a_{\pi_\vee,L_+}(z - x) = 0$. On the other hand, $\mathrm{NNA}(\pi, a_{\pi,A})$ is not violated here, because $a_{\pi,A}(x) = d + (1 - \sqrt{1 + 4d})/2 > 0$ and $a_{\pi,L_+}(z-x) = 0$. A pricing kernel is a point $x' \in \mathbb{R}^2$, using the usual Euclidean inner product $\langle x, x'\rangle = x_1x_1' + x_2x_2'$. A consistent pricing kernel must have $\langle x, x'\rangle \geq 0$ for all $x \in A$, and in particular for $(-d, d^2)$ where $d > 0$. But $d(dx_2' - x_1') = -dx_1' + d^2x_2' \geq 0$ for all $d > 0$ implies $x_1' = 0$, so a consistent pricing kernel can not be strictly monotone.

Next we have a kind of second fundamental theorem of asset pricing, relating uniqueness of pricing to uniqueness of consistent pricing kernel.

Theorem 6.3 (2nd FTAP)**.** *First, if $b_{\pi,A} = a_{\pi,A}$, then D' and $\tilde{D}'$ contain at most one element. If moreover $a_{\pi,A}$ is lower semi-continuous, A is convex and contains 0, and π is convex, then $D' = \tilde{D}'$ is a singleton. Second, if there is a unique consistent pricing kernel x' and A is convex and contains 0, π is convex, and $a_{\pi_\vee,A_\vee}$ is lower semi-continuous, then the marginal bid and ask are equal: $\tilde{b}_{\pi,A} = \tilde{a}_{\pi,A} = \langle\cdot, x'\rangle$. If moreover π is sublinear, A is a cone, and $a_{\pi,A}$ is lower semi-continuous, then $b_{\pi,A} = a_{\pi,A} = \langle\cdot, x'\rangle$.*

Proof. First, given $b_{\pi,A} = a_{\pi,A}$, consider $x_1', x_2' \in D'$. The inequalities in expressions (5) and (6) imply

$$\begin{aligned}\max\{\langle x, x_1'\rangle - \Psi(x_1'), \langle x, x_2'\rangle - \Psi(x_2')\} &\leq a_{\pi,A}(x)\\ &= b_{\pi,A}(x) \leq \min\{\langle x, x_1'\rangle + \Psi(x_1'), \langle x, x_2'\rangle + \Psi(x_2')\}.\end{aligned}$$

Now suppose $x_1' \neq x_2'$. Then there exists $\tilde{x}$ such that the difference $d = \langle\tilde{x}, x_1' - x_2'\rangle \neq 0$. We know $\Psi(x_1') + \Psi(x_2') < \infty$ by definition of D'. So we can pick a real number $\lambda > (\Psi(x_1') + \Psi(x_2'))/d$, and let $x = \lambda\tilde{x}$. Now

$$\langle x, x_1'\rangle - \Psi(x_1') > (\langle x, x_2'\rangle + \Psi(x_1') + \Psi(x_2')) - \Psi(x_1') = \langle x, x_2'\rangle + \Psi(x_2').$$

This contradicts the above inequality, so $x_1' = x_2'$, i.e. D' contains at most one element. From Proposition 5.4, it follows that $\tilde{D}' \subseteq D'$. Under the additional hypotheses, equality holds in

Theorem 4.1. This implies $\langle \cdot, x' \rangle - \Psi(x') = a_{\pi,A} = b_{\pi,A} = \langle \cdot, x' \rangle + \Psi(x')$, so $\Psi(x') = 0$, i.e. $x' \in \tilde{D}'$.

Second, given $\tilde{D}' = \{x'\}$ and the initial hypotheses, from Theorem 4.1 we get $b_{\pi_\vee,A_\vee} = \langle \cdot, x' \rangle = a_{\pi_\vee,A_\vee}$, and from Proposition 5.2, $a_{\pi_\vee,A_\vee} = \tilde{a}_{\pi,A}$ and $b_{\pi_\vee,A_\vee} = \tilde{b}_{\pi,A}$. Given the further hypotheses, by Proposition 2.1, the bid and ask are already sublinear, so $b_{\pi,A} = (b_{\pi,A})_\vee$ and $a_{\pi,A} = (a_{\pi,A})_\vee$. By Proposition 5.1, $(b_{\pi,A})_\vee = b_{\pi_\vee,A_\vee}$ and $(a_{\pi,A})_\vee = a_{\pi_\vee,A_\vee}$. □

There are other approaches to framing a second fundamental theorem of asset pricing. Carr et al. (CGM01) relate uniqueness of a certain pricing kernel to a notion of "acceptable completeness." An acceptably complete market is one in which all cashflows can be hedged so that they are barely acceptable, i.e. one is indifferent between the hedged cashflow and 0. Mathematically speaking, the barely acceptable hedged cashflow is on the boundary of the closed acceptance set. Jarrow and Madan (JM99) relate uniqueness of a signed equivalent local martingale measure to completeness in the traditional sense of exact replicability of contingent claims.

7. Bounded Random Variables

To make matters more concrete, we examine the case where $L = L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, the space of $\mathbf{P}$-equivalence classes of bounded $\mathcal{F}$-measurable random variables (functions) on Ω. Here we can draw connections to the established theories of exact functionals, imprecise probabilities, and convex risk measures. In this setting, we can identify simple sufficient conditions for finite cost hedging and continuity of $a_{\pi_\vee,A_\vee}$.

Our primal space is L^∞ with the strong topology of the $\|\cdot\|_\infty$-norm, under which it is a Banach space. Its positive orthant $L^\infty_+ = \{x \in L^\infty | x \geq 0\}$ with the usual partial ordering generated by the essential infimum: $x_1 \leq x_2$ when $\operatorname{ess\,inf}(x_2 - x_1) \geq 0$.

Our dual space L' is $\mathrm{ba} = \mathrm{ba}(\Omega, \mathcal{F}, \mathbf{P})$, the Banach space of bounded, finitely additive, signed measures μ defined on the σ-algebra $\mathcal{F}$ and absolutely continuous with respect to $\mathbf{P}$. Then $\langle \cdot, \mu \rangle$ will be written $I_\mu(\cdot)$ to denote a (Radon) integral, and $\{I_\mu | \mu \in \mathrm{ba}\}$ is the strong dual of bounded linear functionals on L. For the pairing, it has the weak* topology. The positive orthant ba_+ contains those measures μ such that $\mu(E) \geq 0$ for every event E in $\mathcal{F}$; equivalently, those monotone linear functionals I_μ such that $x_1 \leq x_2 \Rightarrow I_\mu(x_1) \leq I_\mu(x_2)$. We also have a norm on the dual given by $\|\mu\| = \|I_\mu\| = \sup_{\|x\|_\infty \leq 1} |I_\mu(x)|$, which is $I_\mu(1) = \mu(\Omega)$ when $\mu \in \mathrm{ba}_+$.

It can be convenient to assume that one can buy or sell unlimited amounts of a riskless bond with payoff 1. This assumption is equivalent to $\mathbb{R} \subseteq R$. We get by with a weaker hypothesis, that every contingent claim is dominated by a marketed claim. This is equivalent to $\mathbb{R} \subseteq R - L_+$, the availability of a marketed claim dominating any number of riskless bonds.

Proposition 7.1. *If* $\mathbb{R} \subseteq R - L_+$ *then* $\operatorname{dom} a_{\pi,A} = L$.

Proof. By Assumption 2.3, there exists z_0 such that $z \geq z_0$ implies $z \in A$. The hypothesis implies that for any $x \in L$, there exists $y \in R$ such that $y \geq \operatorname{ess\,sup} x + \operatorname{ess\,sup} z_0$, so $y - x \geq z_0$ and thus is in A. Therefore $a_{\pi,A}(x) \leq \pi(y) < \infty$. □

To prove continuity of $a_{\pi_\vee,A_\vee}$, we use the hypothesis that any contingent claim would be dominated by a marketed claim if prices were given by $\pi_\vee$. Letting $\tilde{R}$ be the effective domain of $\pi_\vee$, this is equivalent to $1 \in \tilde{R} - L_+$, a weaker hypothesis than $\mathbb{R} \subseteq R - L_+$.

Proposition 7.2. *If $1 \in \tilde{R} - L_+$, then $a_{\pi_\vee,A_\vee}$ is continuous.*

Proof. Consider any $\alpha \in \mathbb{R}$ and $x \in L$ such that $a_{\pi_\vee,A_\vee}(x) > \alpha$. There exists $\gamma > \alpha$ such that $\pi_\vee(y) < \gamma$ implies $y - x \notin A_\vee$. There exists $y_1 \in \tilde{R}$ such that $y_1 \geq 1$. Pick $\beta \in (\alpha, \gamma)$ and $\delta{:=}(\gamma - \beta)/|\pi_\vee(y_1)| \in (0, \infty]$. Consider any y with $\pi_\vee(y) < \beta$. By subadditivity, $\pi_\vee(y + \delta y_1) \leq \pi_\vee(y) + \delta\pi_\vee(y_1) < \gamma$, so $y + \delta y_1 - x \notin A_\vee$. For any u in the δ-ball at x, $u \geq x - \delta \geq x - \delta y_1$, so by monotonicity, $y - u \leq y + \delta y_1 - x$ is also not in $A_\vee$. Therefore $a_{\pi_\vee,A_\vee}(u) \geq \beta > \alpha$, and $\{x | a_{\pi,A}(x) > \alpha\}$ is open. Thus $a_{\pi_\vee,A_\vee}$ is lower semi-continuous.

Now consider any $\alpha \in \mathbb{R}$ and $x \in L$ such that $a_{\pi_\vee,A_\vee}(x) < \alpha$. There exists $y \in \tilde{R}$ such that $\pi_\vee(y) < \alpha$ and $y - x \in A_\vee$. Again, there exists $y_1 \in \tilde{R}$ such that $y_1 \geq 1$. Let $\delta{:=}(\alpha - \pi_\vee(y))/(2\max\{1, \pi_\vee(y_1)\})$. For any u in the δ-ball at x, $u \leq x + \delta$, so $\delta y_1 + y - u \geq y - x + \delta(y_1 - 1) \geq y - x$, and hence by monotonicity, $\delta y_1 + y - u \in A_\vee$. By subadditivity, $\pi_\vee(\delta y_1 + y) \leq \delta\pi_\vee(y_1) + \pi_\vee(y) \leq (\alpha - \pi_\vee(y))/2 + \pi_\vee(y) < \alpha$. Therefore $a_{\pi_\vee,A_\vee}(u) < \alpha$, and $\{x | a_{\pi,A}(x) < \alpha\}$ is open. Thus $a_{\pi_\vee,A_\vee}$ is upper semi-continuous. $\square$

We now draw connections with the theory of exact functionals, for which see Maaß (Maa02). An absolutely homogeneous, superadditive, real-valued functional is called *superlinear*. In particular, it is concave. A monotone, superlinear functional is called *supermodular*. A constant additive, supermodular functional is called *exact*. Constant additivity of a functional Γ is $\Gamma(x + c) = \Gamma(x) + \Gamma(c)$ when $c \in \mathbb{R}$.

The theory of exact functionals centers on operators similar to the convexity and absolute homogeneity operators defined in Section 5, but with the opposite conventions, of concavity. For instance, one considers the least monotone functional dominating Γ. To get a supermodular functional, one applies successively the operators for positive homogeneity, superadditivity, and monotonicity. Because the application of each operator does not spoil the properties of the previous, the resulting functional given by

$$\Gamma_\wedge(x){:=}(\leq \text{ sa ah } \Gamma)(x) = \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(x_i) \middle| \sum_{i=1}^n \lambda_i x_i \leq x, n \in \mathbb{N}, \forall i\, \lambda_i \geq 0, x_i \in \text{dom } \Gamma \right\}$$

is indeed monotone, superadditive, and absolutely homogeneous. Therefore it is supermodular as long as it is real-valued. Define the "norm" $|\Gamma|{:=}\Gamma_\wedge(1)$. It is nonnegative by monotonicity and homogeneity, which implies $\Gamma_\wedge(0) = 0$. It is a pseudonorm on a linear space of exactifiable functionals on which it is finite-valued (Maa02, Prop. 2). When $|\Gamma|$ is finite, $\Gamma_\wedge$ is real-valued, and may be called the *natural supermodularification* of Γ. When $|\Gamma| < \infty$, we may call Γ *supermodularifiable.*

The *natural exactification* of an exactifiable functional Γ is $\Gamma_\bullet{:=}(\leq \text{ ca sa ah } \Gamma)$ given by

$$\Gamma_\bullet(x) = \sup\left\{ \sum_{i=1}^n \lambda_i \Gamma(x_i) + c|\Gamma| \middle| \sum_{i=1}^n \lambda_i x_i + c \leq x, c \in \mathbb{R}, n \in \mathbb{N}, \forall i\, \lambda_i \geq 0, x_i \in \text{dom } \Gamma \right\}.$$

When $|\Gamma| < \infty$, $\Gamma_\bullet$ is exact, in particular, it is real-valued (Maa02, Thm. 2), and Γ is called *exactifiable*; this is the same thing as supermodularifiability. Thus our concern with $A_\vee$, the cone generated by the acceptance set A, and with $\pi_\vee$, the greatest sublinear function dominated by the market pricing function π, appears entirely analogous to the process of exactification.

The natural exactification of an exactifiable functional Γ is the least exact functional extending Γ and having the same norm as Γ (Maa02, Prop. 4). When Γ is exact, $\Gamma_\bullet$ is called its *natural extension*, and $|\Gamma|$ coincides with the norm $\|\Gamma\|$ ordinarily given to linear operators. This paper analyzes unnatural extensions $a_{\pi,A}$ and $b_{\pi,A}$ of market prices.

Very similar mathematical objects have been studied under different names. An exact functional with unit norm is a *coherent lower prevision*, and an exactifiable functional with unit norm is a *lower prevision avoiding sure loss* (Wal91). When Γ is an exact functional, $-\Gamma$ is a *coherent risk measure* (ADEH99). This makes $-\Gamma$ absolutely homogeneous, subadditive, constant additive, and anti-monotone. A *convex risk measure* is a convex, constant additive, anti-monotone functional (FS02a).

PART II: ROBUST UTILITY MAXIMIZATION

8. Unanimity in Expected Utility: Discussion

Föllmer and Schied (FS02b) point out that one way of getting a convex risk measure is to use the decision framework of Gilboa and Schmeidler (GS89). One may assign to a bounded contingent claim $x \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ the "robust" utility

$$U(x) = \inf_{\mathbf{Q}\in\mathcal{Q}} E_{\mathbf{Q}}[u(x)], \tag{8}$$

where u is a utility function (a strictly increasing, concave function), and $\mathcal{Q}$ is a nonempty set of probability measures. This approach is robust in the sense that it takes account of our lack of knowledge about the probabilistic behavior of asset prices.

We assume that the probability measures in $\mathcal{Q}$ are absolutely continuous with respect to a reference probability measure $\mathbf{P}$. There is no reference probability measure in (FS02b); we reintroduce it here to facilitate the analysis of no-arbitrage conditions. When the state space Ω is uncountable, it seems easier to do this when L_+ is the set of almost everywhere nonnegative elements, rather than those that are everywhere nonnegative.

Choosing x to maximize $U(x)$ is known as maxmin expected utility, as opposed to the ordinary maximization of expected utility based on just one probability measure. Here the contingent claims are being given the interpretation of random wealths, not of changes in wealth. One can also take as the acceptance set the set of random wealths that provide at least u_0 robust expected utils. Then

$$\rho(x) := \inf_{c\in\mathbb{R}} \{c | U(cy_0 + x) \geq u_0\},$$

the number of shares of a numeraire y_0 that must be added to x to produce a random wealth providing at least u_0 robust expected utils, is a convex risk measure (FS02b§4).

This result is of normative significance, because it shows how to construct a risk measure with good properties, starting from the economic primitives of beliefs and preferences over nonstochastic outcomes. It has the virtue of taking into account ambiguity of beliefs by allowing for the inclusion of multiple probability distributions. It would also be easy to allow for multiple utility functions. This is what we want for a robust framework for pricing and hedging in incomplete markets that will be more informative than no-arbitrage theory.

However, maxmin expected utility is not quite the right framework for that task. It ranks random wealths, while we want to rank random changes in wealth. In other words, maxmin expected utility is suitable for the portfolio optimization problem of choosing a random wealth given an initial endowment: starting with x_0 dollars and no other commitments at time 0, we could maximize $U(y)$ subject to the budget constraint $\pi(y) = x_0$. It is not directly suitable for the problem of derivative security pricing, in which we start with a pre-existing random wealth v, which is in general nonzero. Then we want to set a price $b_{\pi,A}(x)$ for a change in wealth x such that buying x for less than $b_{\pi,A}(x)$ and net costlessly hedging will produce a random wealth $v + x - y$ at least as desirable as v.

We suggest G^v defined by

$$G^v(x) := \inf_{i\in I} g_i^v(x) \tag{9}$$

$$g_i^v(x) := \mathbf{E}_{\mathbf{P}_i}[u_i(v + x) - u_i(v)] \tag{10}$$

as the function generating an acceptance set

$$A(G^v){:=}\{x|G^v(x) \geq 0\}. \tag{11}$$

Here I is a nonempty index set, and for each i we have an *outlook* $(\mathbf{P}_i, u_i)$ consisting of a subjective probability measure and a utility function. Again, all $\mathbf{P}_i$ are absolutely continuous with respect to the reference probability measure $\mathbf{P}$. It is important that the evaluation $G^v(x)$ depend on the current portfolio $v \in L$. Thus the acceptable claims are precisely those that result in no diminution of expected utility under any outlook: there is a criterion of unanimous (i.e. Pareto) improvement. Consequently the hedging residual $x_2 - x_1$ is acceptable when it improves expected utility under every outlook.

One may envision the decision-making process as a requirement of unanimity among investment partners with different outlooks. This is different from the maxmin criterion, for which x_2 is preferred to x_1 when the worst-case expected utility of x_2 exceeds that of x_1. However, the expected utility of $x_2 - x_1$ may be negative under some outlooks. With maxmin expected utility, the decision-making process involves a Rawlsian (Raw71) criterion: the investment society prefers x_2 to x_1 because it enhances the subjective expected utility of the least satisfied member, although it may harm others. In this sense, the unanimity criterion is more conservative.

Another interpretation is that a single decision-maker entertains several outlooks, because of imperfect knowledge of the objective probability measure and possibly of utility as well. Then the unanimity criterion is also more conservative in the sense of robustness to the inclusion or exclusion of outlooks. Using statistical jargon loosely, suppose we are testing the null hypothesis that the status quo is the best opportunity available against the alternative hypothesis that it would be preferable to buy another zero-cost contingent claim x, i.e. that $G^v(x) > 0$ or $U(v+x) > U(v)$. Suppose there are n outlooks, based on measures $\mathbf{P}_1, \ldots, \mathbf{P}_n$ and a common utility function u, and we are wondering whether or not to adopt another outlook based on $\mathbf{P}_{n+1}$. Type I error is buying x when it would not have been bought, based on the right set of outlooks, and Type II error is not buying it when it would have.

Under the maxmin scheme, either wrongly including or excluding $\mathbf{P}_{n+1}$ can induce either Type I or Type II error. Suppose $\mathbf{P}_{n+1}$ is wrongly included, while the correct decision (based on $\mathbf{P}_1, \ldots, \mathbf{P}_n$) is not to buy. If $\mathbf{P}_{n+1}$ is the worst-case measure for v and $E_{\mathbf{P}_{n+1}}[u(v)] < U(v+x)$, then the decision is reversed. In the same way, the erroneous inclusion of $\mathbf{P}_{n+1}$ can reverse a correct decision to buy when $\mathbf{P}_{n+1}$ is the worst-case measure for $v+x$ and $\mathbf{E}_{\mathbf{P}_{n+1}}[u(v+x)] < U(v)$. In a similar way, wrongly excluding $\mathbf{P}_{n+1}$ when it is the worst-case measure for $v+x$ can lead to Type II error, while wrongly excluding it when it is worst for v can lead to Type I error.

Under the unanimity scheme, only erroneous exclusion can create Type II error, when one wrongly excludes the only measures that make x look bad. If any of the original n outlooks views the purchase of x unfavorably, it will not be bought, regardless of whether other outlooks are wrongly included. Type I error occurs when one wrongly includes an outlook under which x, which would otherwise have been bought, looks bad. Thus we can be sure that we are erring on the side of caution by including many outlooks. Of course, unanimity is a way of privileging the status quo, but in the business of derivative securities trading, this should be interpreted as prudence, not as injustice. The status quo is safe, there is a good business in doing deals that are known from experience to be better than doing nothing, and there is danger in doing deals that one is not sure are beneficial.

9. Unanimity in Expected Utility: Analysis

In this section, we analyze the properties of the acceptance set and bid-ask prices resulting from unanimity in expected utility. First, we compare it to convex and coherent risk measures. Then we relate it to the valuation measure concept of Carr et al. (CGM01), and apply the fundamental theorems to offer a recipe for constructing arbitrage-free, economically meaningful prices in incomplete markets.

The acceptance set based on unanimity in expected utility has some nice financial properties that nothing (except in trivial cases) based on a coherent risk measure could have. First, the acceptance set of a coherent risk measure is a cone, which implies that acceptable claims are acceptable at unlimited scale. This dangerous behavior does not appear when using the unanimity methodology.

Proposition 9.1. *If any u_i' is unbounded, then no risky claim is acceptable at unlimited scale: for all $x \in A(G^v) \setminus L_+$, there exists λ_x such that $\lambda \geq \lambda_x$ implies $\lambda x \notin A(G^v)$.*

Proof. Consider $x \in A(G^v) \setminus L_+$. There exists $\epsilon > 0$ such that the event $E{:=}\{\omega | x(\omega) \leq -\epsilon\}$ has reference probability $\mathbf{P}[E] > 0$. For any $\lambda > 0$,

$$\begin{aligned} g_i^v(\lambda x) &\leq g_i^v(\lambda((\sup x)\mathbf{1}_{E^\complement} - \epsilon \mathbf{1}_E)) \\ &= \mathbf{E}_{\mathbf{P}_i}[\mathbf{1}_{E^\complement} u_i(v + \lambda \sup x) + \mathbf{1}_E u_i(v - \lambda\epsilon) - u_i(v)] \\ &\leq \mathbf{P}_i(E^\complement) u_i(\sup v + \lambda \sup x) + \mathbf{P}_i(E) u_i(\sup v - \lambda\epsilon) - u_i(\inf v). \end{aligned}$$

Differentiating with respect to λ yields

$$\mathbf{P}_i(E^\complement)(\sup x) u_i'(\sup v + \lambda \sup x) - \mathbf{P}_i(E)\epsilon u_i'(\sup v - \lambda\epsilon).$$

Because u_i' is unbounded, i.e. $\lim_{x\to-\infty} u_i'(x) = \infty$, this expression is less than a strictly negative number for large enough λ. Therefore its antiderivative, which dominates $g_i^v(\lambda x)$, is negative for large enough λ. □

Second, the acceptance set of a coherent risk measure has the form $\{x|\inf_{i\in I}\mathbf{E}_{\mathbf{P}_i}[x]\geq 0\}$. If I is finitely generated (which allows $\{\mathbf{Q}_i\}_{i\in I}$ to be a polyhedron), as seems necessary in practice, while L is infinite-dimensional, then we expect that $\cap_{i\in I}\ker(\mathbf{E}_{\mathbf{P}_i})$ will contain claims other than 0. Then such a claim x and $-x$ will both be acceptable, risky claims. Such behavior seems hard to justify economically, and it it does not occur when using the unanimity methodology.

Proposition 9.2. *If any u_i' is strictly decreasing, then $x, -x \in A(G^v)$ implies $x = 0$.*

Proof. Given $x, -x \in A(G^v)$, we have $\mathbf{E}_{\mathbf{P}_i}[u_i(v+x)-u_i(v)] \geq 0$ and $\mathbf{E}_{\mathbf{P}_i}[u_i(v-x)-u_i(v)] \geq 0$. So $\mathbf{E}_{\mathbf{P}_i}[u_i(v)] \leq \mathbf{E}_{\mathbf{P}_i}[u_i(v+x)]/2+\mathbf{E}_{\mathbf{P}_i}[u_i(v-x)]/2$. Because u_i' is strictly decreasing, $u_i(v) \leq u_i(v+x)/2+u_i(v-x)/2$ with strict inequality when $x(\omega)\neq 0$. Therefore $\{\omega|x(\omega)\neq 0\}$ is a $\mathbf{P}$-null set, and $x = 0$. □

We also consider uniformity conditions on the utility functions and the measures. We may want to require that $\inf_{i\in I} u_i'(x) > 0$ or $\sup_{i\in I} u_i'(x) < \infty$ for all x, or $\inf_{i\in I}(d\mathbf{P}_i/d\mathbf{P})(\omega) > 0$ or $\sup_{i\in I}(d\mathbf{P}_i/d\mathbf{P})(\omega) < \infty$ for all ω. These uniformity conditions mean that the collection of outlooks does not reflect utter ignorance about marginal utility and the probabilities of events. When there are a finite number $|I|$ of outlooks, these uniformity conditions hold. On the other hand, it is still permissible and indeed may be desirable that for each i we should have u_i' and $d\mathbf{P}_i/d\mathbf{P}$ be unbounded on $\mathbb{R}$ and Ω respectively.

Example 9.1 (Unboundedness on $\mathbb{R}$ and Ω). Logarithmic utility has unbounded marginal utility. We regard this function as $\ln : \mathbb{R} \to [-\infty, \infty)$, so expected utility is defined for all bounded contingent claims, although equal to $-\infty$ for contingent claims that are not almost surely nonnegative. The Radon-Nikodym derivative between the risk-neutral and statistical probability measures in the Black-Scholes model is $\exp(-(\|\lambda\|^2/2)T-\lambda W_T)$, which is unbounded on Ω.

The following proposition shows that this scheme produces rational, risk-averse acceptance behavior. If the inputs do not reflect complete ignorance, then G^v detects strict dominance of claims, and accepts some risky claim.

Proposition 9.3. *The functions g_i^v and G^v are concave and monotone, and the associated acceptance sets $A(g_i^v)$ and $A(G^v)$ are convex and rational. Further, g_i^v is strictly monotone. If $\forall x, \inf_{i\in I} u_i'(x) > 0$ and $\forall\omega, \inf_{i\in I}(d\mathbf{P}_i/d\mathbf{P})(\omega) > 0$, then G^v is also strictly monotone. If moreover $\mathbf{P}$ is nontrivial and $\forall x, \sup_{i\in I} u_i'(x) < \infty$, then $A(G^v)$ is strictly larger than L_+.*

Proof. For $\gamma \in [0, 1]$, write $v + \gamma x_1 + (1-\gamma)x_2 = \gamma(v + x_1) + (1-\gamma)(v + x_2)$. By concavity of u_i,

$$u_i(v + \gamma x_1 + (1-\gamma)x_2) \geq \gamma u_i(v + x_1) + (1-\gamma)u_i(v + x_2).$$

Therefore $g_i^v(\gamma x_1 + (1-\gamma)x_2) \geq \gamma g_i^v(x_1) + (1-\gamma)g_i^v(x_2)$, so g_i^v is concave. Because the infimum of concave functions is concave, G^v is concave. This shows the acceptance sets are convex. Because $G^v(0) = g_i^v(0) = 0$, the acceptance sets contain 0.

Consider $x_1 < x_2$. By concavity, $u_i(v+x_2)-u_i(v+x_1) \geq u_i'(\sup(v+x_2))(x_2-x_1)$. Because u_i is strictly increasing and the random variables are bounded, $m_i := u_i'(\sup(v+x_2)) > 0$. There exist $\epsilon > 0$ and $\delta_i > 0$ such that $\mathbf{P}_i(x_2 - x_1 \geq \epsilon) = \delta_i$. Therefore

$$g_i^v(x_2) - g_i^v(x_1) = \mathbf{E}_{\mathbf{P}_i}[u_i(v+x_2) - u_i(v+x_1)] \geq m_i \delta_i \epsilon > 0.$$

Thus g_i^v is strictly monotone. From the hypotheses of uniform positivity, it would follow that $\inf_{i \in I} m_i$ and $\inf_{i \in I} \delta_i$ are both positive, making G^v also strictly monotone.

From strict monotonicity of g_i^v we get two consequences. First, it implies G^v is monotone:

$$x_1 \leq x_2 \Rightarrow g_i^v(x_1) \leq g_i^v(x_2) \Rightarrow G^v(x_1) \leq G^v(x_2).$$

Monotonicity implies the acceptance sets are monotone. Second, for $x < 0$, we have $g_i^v(x) < g_i^v(0) = 0$, and thus $G^v(x) = \inf_{i \in I} g_i^v(x) < 0$. This makes the acceptance sets disjoint with $L_- \setminus \{0\}$.

If $\mathbf{P}$ is nontrivial, there exists an event $E \in \mathcal{F}$ such that $\mathbf{P}(E) \in (0,1)$. From the uniformity of the Radon-Nikodym derivative, it follows that $p = \inf_{i \in I} \mathbf{P}_i(E) > 0$. From the uniformity hypotheses on marginal utility, $\ell = \inf_{i \in I} u_i'(\sup v + 1) > 0$ and $m = \sup_{i \in I} u_i'(\inf v - 1) < \infty$. Consider $x_\epsilon = \mathbf{1}_E - \epsilon \mathbf{1}_{E^\complement}$ for $\epsilon \in (0,1]$.

$$\begin{aligned} g_i^v(x_\epsilon) &= \mathbf{E}_{\mathbf{P}_i}[\mathbf{1}_E(u_i(v+1) - u_i(v))] + \mathbf{E}_{\mathbf{P}_i}[\mathbf{1}_{E^\complement}(u_i(v-\epsilon) - u_i(v))] \\ &\geq \mathbf{P}_i(E) u_i'(\sup v + 1) - \epsilon \mathbf{P}_i(E^\complement) u_i'(\inf v - 1) \\ &\geq p\ell - \epsilon m. \end{aligned}$$

This is nonnegative for some positive ϵ, so $G^v(x_\epsilon) = \inf_{i \in I} g_i^v(x_\epsilon) \geq 0$. Then x_ϵ is in $A(G^v)$ but not in L_+, because it is negative on the non-null event $E^\complement$. ☐

Remark 9.1. Due to diminution of marginal utility with increasing wealth, G^v need not be constant additive, so $-G^v$ need not be a convex risk measure. However, because $A(G^v)$ is a nonempty, convex, monotone set, it can be used to define a convex risk measure (FS02a, Prop. 4).

Next we see that the fundamental theorems apply. Because v is bounded, $u_i'(v)$ is positive and bounded, so $\mathbf{E}_{\mathbf{P}_i}[u_i'(v)] \in (0, \infty)$. In analogy to Carr et al. (CGM01), we can define a *valuation measure* $\mathbf{Q}_i^v$ by

$$\frac{d\mathbf{Q}_i^v}{d\mathbf{P}_i} := \frac{u_i'(v)}{\mathbf{E}_{\mathbf{P}_i}[u_i'(v)]}. \tag{12}$$

The marginal utility u_i' is strictly positive, so any absolute continuity properties are preserved. Because of the normalization, $\mathbf{Q}_i^v$ is indeed a probability measure. Define $\tilde{g}_i^v$ and $\tilde{G}^v$ by

$$\tilde{g}_i^v(x) := -(\text{ah } -g_i^v)(x) \quad \text{and} \quad \tilde{G}^v(x) := \inf_{i \in I} \tilde{g}_i^v(x). \tag{13}$$

Proposition 9.4. $g_i^v(x) \leq \tilde{g}_i^v(x) = \mathbf{E}_{\mathbf{P}_i}[u_i'(v)] \mathbf{E}_{\mathbf{Q}_i^v}[x]$.

Proof. By Proposition 9.3 and Lemma 5.1,

$$g_i^v(x) \leq -(\text{ah } -g_i^v)(x) = \lim_{\lambda \to \infty} \lambda g_i^v(x/\lambda) = \lim_{\lambda \to \infty} \mathbf{E}_{\mathbf{P}_i}[\lambda(u_i(v + x/\lambda) - u_i(v))],$$

and the integrand is nondecreasing in λ. By the monotone convergence theorem, this equals

$$\mathbf{E}_{\mathbf{P}_i}\left[\lim_{\lambda\to\infty}\lambda(u_i(v+x/\lambda)-u_i(v))\right]=\mathbf{E}_{\mathbf{P}_i}[u_i'(v)x]=\mathbf{E}_{\mathbf{P}_i}[u_i'(v)]\mathbf{E}_{\mathbf{Q}_i^v}[x].$$

□

The following preparatory proposition gives a more useful representation for consistent pricing kernels. We saw in Proposition 5.4(4) that consistent pricing kernels are dominated by market prices π and in $A^*=(A_\vee)^*$. In this setting where $A=A(G^v)$, we may be able to focus on a more easily analyzed set, $A(\tilde{G}^v)^*$. The fundamental theorems of Carr et al. (CGM01) involve existence and uniqueness of a "representative state pricing function," which is a strictly positive linear combination of all the valuation measures, that is, it gives strictly positive weight to each one. It also agrees with the market prices, which are assumed linear in that paper. Here we will call a nonnegative linear combination $\mu=\sum_{i\in I}\lambda_i\mathbf{Q}_i^v$, where $\lambda_i\geq 0$, a *representative pricing kernel*; we do not call it a state pricing function because the state space Ω is not finite here. The reason for the difference of positive versus nonnegative coefficients is that the criterion of "no strictly acceptable opportunities" in (CGM01) is, in our notation, $\{x|\exists i\in I\ni g_i^v(x)>0\}\cap A(G^v)\cap M=\emptyset$: one can not get for free a claim that appears at least as good as the status quo under all outlooks, and strictly better under at least one. This differs from our conditions involving arbitrages or cashouts.

Proposition 9.5. *The set $A(\tilde{G}^v)$ is a closed cone containing $A(G^v)_\vee$, and $A(\tilde{G}^v)^*$ is the set of representative pricing kernels. If $\forall x,\inf_{i\in I}u_i'(x)>0$, then $A(\tilde{G}^v)=\mathrm{cl}(A(G^v)_\vee)$, and $A(G^v)_\vee^*=A(\tilde{G}^v)^*$.*

Proof. First we show $A(\tilde{G}^v)$ is closed. The complement of $A(\tilde{G}^v)$ is $\{x|\exists i\ni\tilde{g}_i^v(x)<0\}$. For such x and i, by Proposition 9.4, $\mathbf{E}_{\mathbf{Q}_i^v}[x]<0$. Pick $\delta\in(0,-\mathbf{E}_{\mathbf{Q}_i^v}[x])$. Then the δ-ball at x is contained in the complement of $A(\tilde{G}^v)$.

Next, because $A(\tilde{G}^v)=\{x|\forall i\in I,\mathbf{E}_{\mathbf{Q}_i^v}(x)\geq 0\}$, it is a cone. As a closed cone of this form, its polar cone $A(\tilde{G}^v)^*$ is the cone generated by $\{\mathbf{Q}_i^v\}_{i\in I}$ (Roc70§14), i.e. the set of representative pricing kernels. From the inequality in Proposition 9.4, $A(G^v)\subseteq A(\tilde{G}^v)$. But $A(G^v)_\vee$ is the smallest cone containing $A(G^v)$, so $A(G^v)_\vee\subseteq A(\tilde{G}^v)$.

Now supposing $\forall x,\inf_{i\in I}u_i'(x)>0$, we show $A(\tilde{G}^v)\subseteq\mathrm{cl}(A(G^v)_\vee)$, which implies $A(\tilde{G}^v)=\mathrm{cl}(A(G^v)_\vee)$. Consider $x\in A(\tilde{G}^v)$, i.e. $\tilde{g}_i^v(x)\geq 0$ for all i. For any $\delta>0$ and i,

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda g_i^v((x+\delta)/\lambda) &= \tilde{g}_i^v(x+\delta)\\ &= \tilde{g}_i^v(x)+\tilde{g}_i^v(\delta)\\ &\geq 0+\mathbf{E}_{\mathbf{P}_i}[u_i'(v)]\mathbf{E}_{\mathbf{Q}_i^v}[\delta]\\ &\geq \delta\inf_{i\in I}u_i'(\sup v)>0.\end{aligned}$$

Therefore, for λ sufficiently large, $g_i^v((x+\delta)/\lambda)$ is positive for all i, i.e. $(x+\delta)/\lambda\in A(G^v)$. This implies $x+\delta\in A(G^v)_\vee$. As this is true for all $\delta>0$, x is an accumulation point of $A(G^v)_\vee$, i.e. $x\in\mathrm{cl}(A(G^v)_\vee)$.

Finally, because $A(G^v)_\vee \subseteq A(\tilde{G}^v)$, $A(\tilde{G}^v)^* \subseteq A(G^v)^*_\vee$. It remains to show that $A(G^v)^*_\vee \subseteq A(\tilde{G}^v)^*$. Consider $\mathbf{Q} \in A(G^v)^*_\vee$ and $x \in A(\tilde{G}^v)$. As before, for all $\delta > 0$, $x + \delta \in A(G^v)_\vee$, so $\mathbf{E_Q}[x+\delta] \geq 0$. This implies $\mathbf{E_Q}[x] \geq 0$. ☐

As mentioned in the preceding proof, a representative pricing kernel $\mathbf{Q} \in A(\tilde{G}^v)^* \subseteq A(G^v)^*_\vee$, so if $\mathbf{E_Q} \leq \pi$, then $\mathbf{Q}$ is also consistent. Now the 0th fundamental theorem tells us how to choose the outlooks so that the pricing scheme avoids giving away cashouts: make sure that there is a consistent representative pricing kernel.

Proposition 9.6. *The existence of a consistent representative pricing kernel is sufficient for NC$(\pi, a_{\pi,A(G^v)})$. If $1 \subseteq \tilde{R} - L_+$, π is convex, and $\forall x, \inf_{i\in I} u_i'(x) > 0$, the converse holds.*

Proof. By Theorem 6.1, the existence of a consistent representative pricing kernel is sufficient for $\mathrm{NC}(\pi, a_{\pi,A(G^v)})$. By Proposition 9.3, $A(G^v)$ is convex and contains 0. Now assume the extra hypotheses. By Proposition 7.2, $a_{\pi_\vee,A(G^v)_\vee}$ is continuous. Then Theorem 6.1 asserts that the existence of a consistent pricing kernel is necessary for $\mathrm{NC}(\pi, a_{\pi,A(G^v)})$. By Proposition 9.5, $A(\tilde{G}^v)^* = A(G^v)^*_\vee$, so the consistent pricing kernel is representative. ☐

To avoid arbitrages, the key is to have a consistent representative pricing kernel that is equivalent to the reference probability measure $\mathbf{P}$, i.e. is strictly monotone.

Proposition 9.7. *The existence of an equivalent consistent representative pricing kernel is sufficient for NNA$(\pi, a_{\pi,A(G^v)})$. If $1 \subseteq \tilde{R} - L_+$, π is convex, and $\forall x \in L_+ \setminus \{0\}, \tilde{a}_{\pi,A(G^v)}(x) > 0$, then it is equivalent to $\forall x \in L_+ \setminus \{0\}, \tilde{a}_{\pi,A(G^v)}(x) > 0$.*

Proof. Similar to the previous proof, but relying on Theorem 6.2. ☐

10. Calibration and Computation

We now consider how to construct a set of outlooks so that an equivalent consistent representative pricing kernel exists, and how to compute bid and ask prices from this set. We control the current random wealth v, the subjective measures $\mathbf{P}_i$, and the utility functions u_i. Together, these determine the valuation measures $\mathbf{Q}_i^v$. The set of representative pricing kernels is the cone generated by $\{\mathbf{Q}_i^v\}_{i\in I}$, and we want this set to contain a measure $\mathbf{Q}^v$ that is equivalent to the reference probability measure $\mathbf{P}$ and consistent with market prices: $\mathbf{E}_{\mathbf{Q}^v} \leq \pi$.

One expects that statistical analysis of data including historical price processes would yield a subjective measure. Estimation could be done within a model always yielding measures equivalent to $\mathbf{P}$. Introspection or the use of an elicitation procedure would yield a utility function. The presence of multiple outlooks models the imprecision in these estimates. It seems plausible to argue that the outlooks should be chosen thus, to reflect only beliefs and preferences, and not to be consistent with market prices. If so, the only way to ensure consistency is to change one's trading strategy in order to alter the random wealth v. Certainly this is what a portfolio optimizer would do: take good deals until one's appetite for them is exhausted due to risk aversion, or up to the scale at which they disappear due to imperfect liquidity, i.e. one's impact on market prices.

Another approach, perhaps more appropriate to derivatives traders, is to set a policy of believing that all marketed claims are unattractive when one holds a reference position v_0, such as 0 or an investment in a money market account. Because marketed claims are typically available at very small scale, it seems ineffectual to attempt to accomplish this through choice of the utility functions: finite risk-aversion will not prevent risky investment while accepting risky hedging residuals. Instead, one could let market prices, possibly in conjunction with historical data, determine the choice of measures $\mathbf{P}_i$ in the outlooks. For instance, one might perform statistical estimation from historical data subject to the constraint that an equivalent consistent representative pricing kernel $\mathbf{Q}^{v_0}$ exist.

Regarding computation, it seems desirable to have a finite number $|I|$ of outlooks. This is also attractive for theoretical reasons: it ensures that the uniformity conditions on marginal utility and Radon-Nikodym derivative hold. With a finite number of outlooks, in a one-period economy with finitely many marketed securities, linear programming makes efficient computation possible, as explained by Augustin (Aug02). Given the difficulty of computing optimal continuous-time hedging strategies even in a complete market, it is not to be expected that such a simple solution exists outside the one-period case. However, the stochastic dynamic programming method for approximate computation of hedging strategies in incomplete markets of Bertsimas et al. (BKL01) is encouraging.

11. Conclusions and Directions

This paper offers a methodology for using imprecise beliefs and preferences to determine arbitrage-free prices and corresponding hedging strategies in incomplete markets. It is intended to be more robust than expected utility maximization while providing price bounds more useful than no-arbitrage. The results include extension of Jaschke and Küchler's (JK00) first fundamental theorem of asset pricing, and a generalization (of a slight modification) of the system of Carr et al. (CGM01), with additional economic grounding.

Several tasks remain before this methodology can become practical. We require efficient and reliable algorithms for computation and calibration. It would even seem that further attention should be paid to the philosophical basis for calibration of a scheme that involves beliefs, preferences, and hedging costs to data including current market prices of derivatives as well as price histories. For instance, our suggestion of performing statistical estimation from historical data, subject to a no-arbitrage constraint imposed by current market prices of derivatives, is not wholly satisfactory. One is tempted to argue that derivatives prices reflect aggregate beliefs about the future evolution of underlying price processes, and that this information should be incorporated into one's outlooks, not merely used as a constraint. On the other hand, market prices reflect not only participants' beliefs, but also their preferences given their portfolios, which complicates matters.

It would also be of interest to extend this framework, which at present is quite restricted with respect to risk and ambiguity attitudes. The representation of a coherent or convex risk measure as an infimum expresses extreme ambiguity aversion: a worst-case analysis among all plausible measures, taking no account of perspectives that make a risk seem more attractive (Aug02). Furthermore, the sharpness of the distinction between plausible and implausible measures (for instance, those that are or are not consistent pricing kernels) is displeasing. These objections may point the way to fruitful generalizations.

We have also given necessary conditions for this pricing scheme to avoid arbitrage only under some convexity hypotheses. Given the nature of convex optimization, this seems difficult to avoid. However, it might be worthwhile to consider the possibility of risk-seeking behavior (nonconvex acceptance sets) on the part of agents, such as derivatives traders, who participate economically in their trading gains to a greater extent than in their losses. An understanding of the incentives faced by such agents, and the objectionable behavior that may result, could lead to improved risk control systems. With regard to convexity of market prices, it would be desirable to know whether and to what extent market prices are nonconvex, and what impact that has.

Moreover, the use of the market price function π obscures potentially important and challenging aspects of market modelling. A key issue not addressed in the present paper is the relationship between admissible, self-financing continuous-time trading strategies and the market price of attainable contingent claims. This gives the paper a one-period flavor that is typical of recent research on coherent and convex risk measures, but does not do justice to the richness of continuous-time finance, particularly not to the fundamental theorems of asset pricing described by Delbaen and Schachermayer (1999). To demonstrate the value of this methodology requires some examples in which it produces practical bid-ask intervals for interesting unattainable claims in realistic incomplete markets with continuous-time trading.

References

[ADEH99] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath, *Coherent measures of risk*, Mathematical Finance **9** (1999), 203–228.

[Aug02] Thomas Augustin, *Expected utility within a generalized concept of probability—a comprehensive framework for decision making under ambiguity*, Statistical Papers **43** (2002), 5–22.

[BKL01] Dimitris Bertsimas, Leonid Kogan, and Andrew W. Lo, *Hedging derivative securities and incomplete markets: an ϵ-arbitrage approach*, Operations Research **49** (2001), 372–397.

[CGM01] Peter Carr, Hélyette Geman, and Dilip B. Madan, *Pricing and hedging in incomplete markets*, Journal of Financial Economics **62** (2001), 131–167.

[ČH01] Aleš Černý and Stewart Hodges, *The theory of good-deal pricing in financial markets*, Mathematical Finance—Bachelier Congress 2000 (Berlin) (Hélyette Geman, Dilip Madan, Stanley Pliska, and Ton Vorst, eds.), Springer-Verlag, 2001, pp. 175–202.

[ÇJP] Umut Çetin, Robert Jarrow, and Philip Protter, *Liquidity risk and arbitrage pricing theory*, working paper, Cornell University, Ithaca, New York.

[CSR00] John H. Cochrane and Jesús Saá-Requejo, *Beyond arbitrage: good-deal asset price bounds in incomplete markets*, Journal of Political Economy **108** (2000), no. 11, 79–119.

[Del02] Freddy Delbaen, *Coherent risk measures on general probability spaces*, Advances in Finance and Stochastics (Klaus Sandmann and Philipp J. Schönbucher, eds.), Springer-Verlag, Berlin, 2002.

[DS58] Nelson Dunford and Jacob T. Schwartz, *Linear operators*, vol. 1, Pure and Applied Mathematics, no. 7, Wiley-Interscience, New York, 1958.

[DS99] Freddy Delbaen and Walter Schachermayer, *Non-arbitrage and the fundamental theorem of asset pricing: Summary of main results*, Introduction to Mathematical Finance (Providence, Rhode Island) (David C. Heath and Glen Swindle, eds.), Proceedings of Symposia in Applied Mathematics, no. 57, American Mathematical Society, 1999, pp. 49–58.

[Fri00a] Marco Frittelli, *Introduction to a theory of value coherent with the no-arbitrage principle*, Finance and Stochastics **4** (2000), 275–297.

[Fri00b] Marco Frittelli, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Mathematical Finance **10** (2000), 39–52.

[FS02a] Hans Föllmer and Alexander Schied, *Convex measures of risk and trading constraints*, Finance and Stochastics **6** (2002), no. 4, 429–447.

[FS02b] Hans Föllmer and Alexander Schied, *Robust preferences and convex measures of risk*, Advances in Finance and Stochastics (Klaus Sandmann and Philipp J. Schönbucher, eds.), Springer-Verlag, Berlin, 2002.

[FS02c] Hans Föllmer and Alexander Schied, *Stochastic finance: An introduction in discrete time*, Studies in Mathematics, no. 27, De Gruyter, Berlin, 2002.

[GS89] Itzhak Gilboa and David Schmeidler, *Maxmin expected utility with non-unique prior*, Journal of Mathematical Economics **18** (1989), 141–153.

[JK00] Stefan Jaschke and Uwe Küchler, *Coherent risk measures, valuation bounds, and* (μ, ρ)*-portfolio optimization*, available at `<http://www.jaschke.net/papers/crm.pdf>`, July 2000, version 1.39.

[JM99] Robert Jarrow and Dilip B. Madan, *Hedging contingent claims on semimartingales*, Finance and Stochastics **3** (1999), 111–134.

[Maa02] Sebastian Maaß, *Exact functionals and their cores*, Statistical Papers **43** (2002), 75–93.

[Raw71] John Rawls, *A theory of justice*, Harvard University Press, Cambridge, Massachusetts, 1971.

[Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton University Press, Princeton, New Jersey, 1970.

[Roc74] R. Tyrrell Rockafellar, *Conjugate duality and optimization*, Regional Conference Series in Applied Mathematics, no. 16, Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[Roo02] Berend Roorda, *Martingale characterizations of coherent acceptability measures*, working paper, FELab, University of Twente, Enschede, the Netherlands, July 2002.

[Sch96] Martin Schweizer, *Approximation pricing and the variance-optimal martingale measure*, Annals of Probability **24** (1996), 206–236.

[Wal91] Peter Walley, *Statistical reasoning with imprecise probabilities*, Chapman and Hall, New York, 1991.

Jeremy Staum, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY 14853

E-mail address: `staum@orie.cornell.edu`