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# FUNDAMENTAL THEOREMS OF ASSET PRICING FOR GOOD DEAL BOUNDS

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We prove fundamental theorems of asset pricing for good deal bounds in incomplete markets. These theorems relate arbitrage-freedom and uniqueness of prices for over-the-counter derivatives to existence and uniqueness of a pricing kernel that is consistent with market prices and the acceptance set of good deals. They are proved using duality of convex optimization in locally convex linear topological spaces. The concepts investigated are closely related to convex and coherent risk measures, exact functionals, and coherent lower previsions in the theory of imprecise probabilities.

KEY WORDS: asset pricing, coherent risk measure, convex risk measure, equivalent martingale measure, fundamental theorem, good deal bounds, imprecise probabilities, incomplete markets.

## 1. INTRODUCTION

The problem of pricing and hedging in incomplete markets demands a synthesis of the approaches of mathematical finance and economics: How does one hedge risks and establish preferences over residual, unhedgeable risks, and what implications does this have for pricing risks? At the same time, one must take account of the cost of hedging, as determined by current market prices, and of beliefs about future market prices and of fundamental preferences, which do not derive solely from current or historical market prices. In mathematical terms, the problem of pricing in incomplete markets is the problem of extending a function that gives the prices of marketed cashflows to a larger space of cashflows. The cashflows in the larger space but not the smaller marketed space are potential over-the-counter securities. The extension should have economic justification and be suitable for implementation by financial decision makers. The problem is important because the incorporation of features such as price jumps, transaction costs, and illiquidity into a model often yields incompleteness.

One approach to this problem arises from the consideration of equivalent martingale measures (EMMs) in no-arbitrage pricing theory. Under some conditions, market prices equal expected discounted terminal values, with the expectation taken under an EMM

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(Delbaen and Schachermayer 1999). In incomplete markets, there can be many EMMs, and one may propose criteria for selecting one. Expectation under this most-favored EMM is then the chosen extension of the market price function. Two criteria that have attracted extensive attention are minimization of hedging residual variance (Schweizer 1996) and minimization of the relative entropy between the EMM and a subjective probability measure (Frittelli 2000b).

Another approach arises from consideration of the lower and upper no-arbitrage bounds for prices of nonredundant contingent claims. It analyzes tighter *good deal bounds*, which arise from the exclusion not only of arbitrages, but a larger *acceptance set* of *good deals* (Cochrane and Saá-Requejo 2000). Recently, research in this area has taken inspiration from the work by Artzner et al. (1999) on coherent risk measures. It explicitly aims at creating a theory that occupies an intermediate position between no-arbitrage theory and expected utility theory, being more useful than the former and more robust than the latter. Recent papers include Carr, Geman, and Madan (2001), Černý and Hodges (2001), Jaschke and Küchler (2000), and Roorda (2002). An investigation that similarly seeks to interpolate between no-arbitrage and expected utility theories, although not explicitly treating price bounds, is Frittelli (2000a). See Carr et al. and Černý and Hodges for further discussion of the relative merits and disadvantages of no-arbitrage and expected utility theories, as well as references to previous work along the same lines by economists not drawing on the coherent risk measure concept. The present paper continues the approach inspired by coherent risk measures, but is not restricted to the coherent case.

This paper extends the results of Jaschke and Küchler (2000). We drop some assumptions of coherence and resolve some difficulties surrounding the converse in the fundamental theorem of asset pricing. The main tool is the duality theory of optimization in locally convex linear topological spaces. We use as a recurring example the case of bounded random variables, which several authors have treated in different contexts: Delbaen (2002) and Föllmer and Schied (2002a, 2002b, 2002c) on coherent and convex risk measures, Maaß (2002) on exact functionals, and Walley (1991) on imprecise probabilities. In Section 2 we formulate no-good-deal price bounds and conditions for them to avoid arbitrage. Section 3 covers financial interpretations of the mathematical hypotheses needed for subsequent results. Sections 4 and 5 develop respectively the dual and primal results required for proving versions of the fundamental theorems of asset pricing, which occupy Section 6. In Section 7 we focus on the important special case of bounded random variables. We conclude and discuss directions for future research in Section 8.

## 2. ACCEPTANCE AND PRICING

Let $L$ be a linear space of cashflows for which we desire to establish bid and ask prices. We will derive results for pricing where $L$ is endowed with a locally convex topology and paired with a dual space (see Section 4). The reader may consult the appendix of Jaschke and Küchler (2000) for an introduction to locally convex linear topological spaces and their duality theory.

Let $L_+ \subset L$ be the subset of nonnegative cashflows. We assume that it is a cone, meaning a convex, absolutely homogeneous set, where "absolutely homogeneous" means positively homogeneous and containing 0. We do not assume that it has any "nice" topological properties, such as closedness or nonempty interior; Example 3.1 in Section 3 is an example where it has empty interior.

STANDING ASSUMPTION 2.1. *$L$ is a linear space and $L_+ \subseteq L$ is a cone.*

We model the market with a function $\pi : L \mapsto (-\infty, \infty]$ giving market prices. It has the interpretation of an ask price; that is, if you purchase $x$, you must pay $\pi(x)$, whereas if you sell $x$, you receive $\pi(-x)$. Naturally, the 0 cashflow costs 0. To avoid some trivial cases, we assume there exist cashflows of both positive and negative price.

STANDING ASSUMPTION 2.2. *There is a market ask pricing function $\pi : L \to (-\infty, \infty]$ taking both positive and negative values, and with $\pi(0) = 0$.*

The *effective domain* of $\pi$ is the subset $R \subseteq L$ of cashflows that are marketed, or replicable; elsewhere $\pi$ takes the value $\infty$. Our goal is to establish bid and ask prices for over-the-counter securities providing cashflows in $L \backslash R$.

Important special cases are those in which $\pi$ is linear on a linear effective domain $R$, sublinear (convex and absolutely homogeneous), or convex. Linearity corresponds to frictionless markets. Sublinearity allows for proportional transaction costs—that is, a fixed bid-ask spread for any transaction size. Convexity is consistent with more general transaction costs, trading constraints, and liquidity effects.

The set of cashflows you can have for free is $M := \{x \mid \pi(x) \leq 0\} - L_+$; the set of valuable and riskless cashflows is $L_+ \backslash \{0\}$ and the set of cashflows you can sell for cash now is $C := \{x \mid \pi(-x) < 0\}$. An element of $M \cap (L_+ \backslash \{0\})$ is an *arbitrage*. An element of $M \cap C$ will be called a *cashout*. It is sometimes also called an arbitrage, but the distinction between these concepts is important enough here to warrant different names. Let a *near-arbitrage* be an element of $\mathrm{cl}(M) \cap (L_+ \backslash \{0\})$. The financial significance of being "near" depends on the topology in which the closure is taken.

REMARK 2.1. If the topology is the strong topology of the $\|\cdot\|_\infty$-norm, a near-arbitrage is known as a *free lunch with vanishing risk* (see Delbaen and Schachermayer 1999).

Let $A \subseteq L$ denote an *acceptance set*—that is, the set of cashflows that one is willing to accept without compensation. Say a set $A$ is *monotone* when $A + L_+ \subseteq A$. We will assume the acceptance set is monotone; this represents a modicum of financial rationality.

STANDING ASSUMPTION 2.3. *$A$ is nonempty and monotone.*

For purposes of derivative security pricing, one interprets $x \in L$ as a change in wealth, so 0 is the status quo. Then it makes sense to have $0 \in A$. (From the perspective of portfolio optimization, $x \in L$ is a wealth, and 0 might very well not be acceptable.) Monotonicity and $0 \in A$ imply $L_+ \subseteq A$. Pure losses should be unacceptable: $A \cap (L_- \backslash \{0\}) = \emptyset$. Monotonicity, acceptability of the status quo, and unacceptability of pure losses are together equivalent to a subset of the axioms for coherent risk measures of Artzner et al. (1999). Another interesting property is convexity of $A$, which corresponds to risk aversion. We do not assume that any of these properties other than monotonicity holds, but they feature as hypotheses of some results.

REMARK 2.2. Both $\pi$ and $A$ may depend on one's current portfolio. For instance, considerations of credit risk suggest that the price received for issuing liabilities in a state of the world depends on one's wealth in that state. A contingent claim's acceptability may depend on whether it hedges or exacerbates risks already present in the portfolio.

When we are willing to accept any claim $x \in A$, from our counterparty's point of view, the set of cashflows to be had for free is $M - A$. We can also describe the set $A - M$ as our hedging-aware acceptance set. We would like this set to satisfy the following conditions:

$$\text{NC}(\pi, A)\colon (M - A) \cap C = \emptyset \qquad \text{(No Cashout)}$$
$$\text{NA}\colon (M - A) \cap L_+ \backslash \{0\} = \emptyset \qquad \text{(No Arbitrage)}$$
$$\text{NNA}(\pi, A)\colon \text{cl}(M - A) \cap L_+ \backslash \{0\} = \emptyset \qquad \text{(No Near–Arbitrage)}$$

REMARK 2.3. Although the concepts are not quite equivalent, the condition NA$(\pi, A)$ relates to (the absence of) Jaschke and Küchler's (2000) good deals of the first kind, and NC$(\pi, A)$ relates to their good deals of the second kind, or good deals simply.

Define our ask and bid prices for a cashflow $x$ as

$$a_{\pi,A}(x) := \inf_{y \in L}\{\pi(y) \mid y - x \in A\} = \inf_{y \in R}\{\pi(y) \mid y - x \in A\} \tag{2.1}$$

and

$$b_{\pi,A}(x) := -a_{\pi,A}(-x) = \sup_{y \in R}\{-\pi(y) \mid x + y \in A\}. \tag{2.2}$$

We should interpret $a_{\pi,A}(x)$ as an unattained infimum selling price for $x$. Receiving any amount more than $a_{\pi,A}(x)$ while taking on the cashflow $-x$, we will be able to hedge acceptably and retain some profit. Getting exactly $a_{\pi,A}(x)$ would result at best in indifference.

Using $L_+$ as an acceptance set, we get the no-arbitrage bounds $a_{\pi,L_+}$ and $b_{\pi,L_+}$ for pricing and hedging in incomplete markets. If $L_+ \subseteq A$, then $b_{\pi,A} \geq b_{\pi,L_+}$ and $a_{\pi,A} \leq a_{\pi,L_+}$, so we get a bid-ask spread no less tight than the no-arbitrage bounds. If there is no arbitrage in market prices, then for all $y \in R$, $\pi(y) = a_{\pi,L_+}(y)$.

PROPOSITION 2.1. *The ask $a_{\pi,A}$ is monotone. If $0 \in A$, then $a_{\pi,A} \leq \pi$. If $A$ and $\pi$ are convex, then $a_{\pi,A}$ is convex. If $A$ and $\pi$ are positively homogeneous, then $a_{\pi,A}$ is positively homogeneous; if moreover $0 \in A$, then $a_{\pi,A}$ is absolutely homogeneous.*

*Proof*. Monotonicity of $a_{\pi,A}$ follows from monotonicity of $A$: If $x_2 \geq x_1$, then $x_2 + A \subseteq x_1 + A$, so the infimum in $a_{\pi,A}(x_2)$ is taken over a smaller set. If $0 \in A$, then $x - x \in A$, so $a_{\pi,A}(x) \leq \pi(x)$.

Consider $x_1, x_2 \in L$ and $y_1, y_2 \in R$ such that $y_1 - x_1, y_2 - x_2 \in A$; that is, $y_1$ and $y_2$ are feasible in computing $a_{\pi,A}(x_1)$ and $a_{\pi,A}(x_2)$ respectively. If $A$ is convex, for $\gamma \in [0, 1]$, $\gamma(y_1 - x_1) + (1 - \gamma)(y_2 - x_2) \in A$. Because

$$\gamma(y_1 - x_1) + (1 - \gamma)(y_2 - x_2) = (\gamma y_1 + (1 - \gamma)y_2) - (\gamma x_1 + (1 - \gamma)x_2),$$

this shows that $\gamma y_1 + (1 - \gamma)y_2$ is feasible in computing $a_{\pi,A}(\gamma x_1 + (1 - \gamma)x_2)$. If $\pi$ is convex, then $\pi(\gamma y_1 + (1 - \gamma)y_2) \leq \gamma\pi(y_1) + (1 - \gamma)\pi(y_2)$, so

$$a_{\pi,A}(\gamma x_1 + (1 - \gamma)x_2) \leq \gamma a_{\pi,A}(x_1) + (1 - \gamma)a_{\pi,A}(x_2)$$

and $a_{\pi,A}$ is convex.

Consider $x \in L$ and $y \in R$ such that $y - x \in A$. If $A$ is positively homogeneous, then for $\lambda > 0$, $\lambda(y - x) \in A$. If $\pi$ is positively homogeneous, then $\pi(\lambda y) = \lambda\pi(y)$. So $a_{\pi,A}(\lambda x) \leq$

$\lambda a_{\pi,A}(x)$. But $x = (1/\lambda)(\lambda x)$, so $a_{\pi,A}(x) \le (1/\lambda)a_{\pi,A}(\lambda x)$. Therefore $a_{\pi,A}(\lambda x) = \lambda a_{\pi,A}(x)$. Because $\pi(0) = 0$, if moreover $0 \in A$, $a_{\pi,A}(0) = 0$. □

We now formulate conditions under which the policy of selling a cashflow $x$ for any price more than an ask $a(x)$ does not backfire by giving away a cashout or a near-arbitrage. These conditions are generalizations of NC($\pi$, $A$) and NNA($\pi$, $A$), which dealt only with transactions taking place at an infimum price of zero; see Proposition 5.3 for more about this relationship.

- NC($\pi$, $a$): For any $x \in L$, $a(x) + a_{\pi,L_+}(-x) \ge 0$.
- NNA($\pi$, $a$): For any $x \in L$ and $z \in L_+ \setminus \{0\}$, $a(x) + a_{\pi,L_+}(z - x) > 0$.

When NC($\pi$, $a_{\pi,A}$) holds, $a_{\pi,A}$ does not give away a cashout; we can rewrite NC($\pi$, $a_{\pi,A}$) as $a_{\pi,A}(x) \ge b_{\pi,L_+(x)}$, which shows that our counterparty must pay more than the lower no-arbitrage bound for $x$. On the other hand, suppose NC($\pi$, $a_{\pi,A}$) fails; that is, $d := b_{\pi,L_+}(x) - a_{\pi,A}(x) > 0$. For any $\epsilon > 0$, there exists $y_\epsilon \in R$ such that $x + y_\epsilon \in L_+$ and $\pi(y_\epsilon) \le \epsilon - b_{\pi,L_+}(x)$. Our counterparty could buy $x$ from us for price $p := a_{\pi,A}(x) + d/3$, choose $\epsilon = d/3$, and buy $y_{d/3}$ on the market. This strategy has cost $\pi(y_{d/3}) + p \le (d/3 - b_{\pi,L_+}(x)) + (b_{\pi,L_+}(x) - d + d/3) = -d/3 < 0$, so our counterparty would get a cashout: a negative cost now with no future risk from $x + y_\epsilon \ge 0$.

Likewise, we can rewrite NNA($\pi$, $a_{\pi,A}$) as $a_{\pi,A}(x) > b_{\pi,L_+}(x - z)$. Suppose this fails. For any $\epsilon > 0$, there exists $y_\epsilon \in R$ such that $x - z + y_\epsilon \in L_+$ and $\pi(y_\epsilon) \le \epsilon - b_{\pi,L_+}(x - z)$. For any $\delta > 0$, our counterparty could buy $x$ from us for price $p := a_{\pi,A}(x) + \delta/2$ and buy $y_{\delta/2}$ on the market. This strategy has cost $\pi(y_{\delta/2}) + p \le (\delta/2 - b_{\pi,L_+}(x - z)) + (a_{\pi,A}(x) + \delta/2) \le \delta$ and results in the cashflow $x + y_{\delta/2} \ge z > 0$. So our counterparty can get as least as much as the fixed, desirable cashflow $z > 0$ for any positive price $\delta$, no matter how small. This would not be giving away an arbitrage but it would be arbitrarily close to doing so.

REMARK 2.4. One might consider demanding more, for instance that one does not give away a cashout in the course of selling several cashflows $x_1, \dots, x_n$:

$$\sum_{i=1}^{n} a_{\pi,A}(x_i) \ge b_{\pi,L_+}\left(-\sum_{i=1}^{n} x_i\right).$$

This relates to Walley's (1991) criterion of "avoiding sure loss." However, it appears financially inappropriate for two reasons. First, one's acceptance set should change after a trade. Suppose that before the trade, one possessed the cashflow $v$ and had the acceptance set $A$ for changes. This corresponds to an acceptance set $v + A$ for cashflows. If the trade does not change one's beliefs or preferences, then after selling $x$ and acquiring the hedge $y$, one's position is $v + y - x$, and the new acceptance set for changes should be $A + x - y$. Second, the act of acquiring the hedge $y$ may have an effect on market prices, due to limited liquidity. (See Çetin, Jarrow, and Protter, 2002, for an approach to understanding and modeling liquidity costs.) To ignore the effect of one's trades on market prices is tantamount to assuming that the market pricing function $\pi$ is subadditive, as it would be possible to acquire $y = \sum_{i=1}^{n} y_i$ for no more than $\sum_{i=1}^{n} \pi(y_i)$ by making $n$ purchases in rapid succession. These considerations suggest that we may focus on a single pricing decision.

## 3. FINITE-COST HEDGING AND CONTINUITY

Some later results involve the hypothesis that the ask price of any cashflow is finite. Like the mathematical conditions discussed so far, such as monotonicity and convexity, this has a financial meaning and is not merely a technical condition. It can be verified without actually computing the ask by analyzing the relationship between the acceptance set $A$ and the market pricing function $\pi$.

DEFINITION 3.1 (Full domain). *Full Domain* for a function $f$ means dom $f = L$; that is, $\forall x \in L, f(x) < \infty$.

What dom $a_{\pi,A} = L$ says is that for all $x \in L$ there exists $y \in R$ such that $y - x \in A$. Because $L$ is linear, $-x$ is always in $L$ too, so this is equivalent to saying that every cashflow becomes acceptable after hedging at finite cost. This condition could fail, in which case we would need either a different approach than the present for establishing fundamental theorems, or to respecify the problem. One could attempt to price only cashflows that can be acceptably hedged—that is, restrict $L$ to be dom $a_{\pi,A}$—or one could enrich $A$ to include some hedging residuals of the troublesome elements of $L$.

The following examples, which illustrate these points, have $L = L^0(\mathbb{R}, \mathcal{B}, \mathbf{P})$, with $\mathcal{B}$ the Borel sigma-algebra on $\mathbb{R}$, and $\mathbf{P}$ a probability measure. This is a space of random variables, interpreted as contingent claims. It makes sense to say $L_+$ is the set of $\mathbf{P}$-almost surely nonnegative contingent claims. First we observe that $L_+$ has empty interior under any vector topology $\mathcal{T}$.

EXAMPLE 3.1 (Empty interior of $L^0_+$). Any $x \in L_+$ has a finite essential infimum. Consider the non-null event $E = \{\omega \in \mathbb{R} \mid x(\omega) < \inf x + 1\}$ that it takes a value within 1 of its essential infimum. There is some other random variable $x_E$ that is essentially unbounded below on $E$. Then $L_+$ is not radial at $x$: for any $\delta > 0$, $x + \delta x_E$ is not almost surely bounded below, so it is not in $L_+$. Therefore $x$ is not in the $\mathcal{T}$-interior of $L_+$ because any $\mathcal{T}$-open set is radially open (Jaschke and Küchler 2000, Prop. 17).

In the following example, $a_{\pi,A}$ does not have full domain, and we consider a way of restricting the space $L$ of cashflows to be priced in order to give $a_{\pi,A}$ full domain.

EXAMPLE 3.2 (Restricting aims). Let $A$ be any acceptance set containing only contingent claims that are almost surely bounded below by some $K \leq 0$. That is, $-K$ is a maximum acceptable loss, or risk capital. Suppose the only marketed instrument is a riskless bond whose payoff is 1, and its unit price is 1 for transactions of any size. Then the marketed subset $R = \mathbb{R}$ is the linear subspace of constants, and $\pi$ is effectively the identity. If $x$ is not almost surely bounded below, then it cannot be acceptably hedged. For any $c \in R$, $x + c$ is still not almost surely bounded below, so $a_{\pi,A}(-x) = \infty$. We could restrict $L$ to be the linear subspace of almost surely bounded contingent claims, $L^\infty(\mathbb{R}, \mathcal{B}, \mathbf{P})$. If $x$ is almost surely bounded by $K$, then $x + K \in L_+ \subseteq A$, so $x$ can be hedged acceptably at a cost of $K$.

The final example considers expanding the acceptance set so that $a_{\pi,A}$ has full domain. The new acceptance set need not be convex: this is an example of how risk-seeking behavior may arise.

EXAMPLE 3.3 (Limited liability). The setting is the same as in Example 3.2. Now suppose that the decisionmaker enjoys limited liability and suffers the same consequences whenever the contingent claim pays off less than $K$. This might be a trader who can at worst lose his job, or a proprietor of a business empire who can at worst see the group's bank go bankrupt. Let $u$ be a utility function on $\mathbb{R}$, unbounded above. In the absence of risk management, the acceptance set $A$ might include any contingent claim $x$ such that $\mathbf{E}^{\mathbf{P}}[u(x)\mathbf{1}\{x \geq K\}] + u(K)\mathbf{P}[x < K] \geq 0$. This is an expected utility calculation, accounting for a fixed loss in the case of ruin. (Because $u(x)\mathbf{1}\{x \geq K\}$ is almost surely bounded below, the expectation exists, although it might be $\infty$.) Now every $x \in L$ can be acceptably hedged at finite cost, as follows. Pick $c, d \in \mathbb{R}$ such that $\mathbf{P}[x \geq c] > 0$ and $u(d) \geq -u(K)\mathbf{P}[x < c]/\mathbf{P}[x \geq c]$, and let $z = x - c + d$. Then

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{z \geq K\}] + u(K)\mathbf{P}[z < K] \\
&\quad = \mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{z \geq d\}] + \mathbf{E}^{\mathbf{P}}[u(z)\mathbf{1}\{K \leq z < d\}] + u(K)\mathbf{P}[z < K] \\
&\quad \geq u(d)\mathbf{P}[z \geq d] + u(K)\mathbf{P}[K \leq z < d] + u(K)\mathbf{P}[z < K] \\
&\quad = u(d)\mathbf{P}[x \geq c] + u(K)\mathbf{P}[x < c]
\end{aligned}$$

because $\{z \geq d\} = \{x \geq c\}$ by definition of $z$. By definition of $d$, this quantity is nonnegative, so $z \in A$, which shows that $x$ can be acceptably hedged for the finite cost $d - c$.

In using duality theory in Section 4, we will also be concerned with continuity of ask prices with respect to some locally convex vector topology $\mathcal{T}$ on $L$.

DEFINITION 3.2 (Semicontinuity). A function $f$ is *lower (upper) semicontinuous* when, equivalently,

- for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x$, $f(x) \leq \liminf_{x_n \to x} f(x_n)$, respectively $f(x) \geq \limsup_{x_n \to x} f(x_n)$
- for any $\alpha \in \mathbb{R}$, the set $\{x \mid f(x) \leq \alpha\}$ is closed, respectively $\{x \mid f(x) \geq \alpha\}$ is closed
- for any $\alpha \in \mathbb{R}$, the set $\{x \mid f(x) > \alpha\}$ is open, respectively $\{x \mid f(x) < \alpha\}$ is open.

Together, the two semicontinuities imply ordinary continuity. Because $\mathcal{T}$ is a vector topology, lower semicontinuity of $a_{\pi,A}$ is equivalent to upper semicontinuity of $b_{\pi,A}$, as follows. Lower semicontinuity of $a_{\pi,A}$ is openness of $\{x \mid a_{\pi,A}(x) > \alpha\} = \{x \mid -b_{\pi,A}(-x) > \alpha\} = -\{x \mid b_{\pi,A}(x) < -\alpha\}$ for all $\alpha$, which is equivalent to openness of $\{x \mid b_{\pi,A}(x) < \alpha\}$ for all $\alpha$.

REMARK 3.1. The Fatou property for risk measures discussed by Delbaen (2002) is lower semicontinuity with respect to the topology of bounded convergence in probability. See also Example 4.1.

It turns out that in the sublinear case, finite-cost hedging is a sufficient condition for the existence of a topology with respect to which the ask price is continuous. This will help us in our analysis of duality, where we will want to choose some such topology in order to look at an appropriate dual space of continuous linear functionals, because we can be sure that one exists. Here we prove that the lc-topology, the finest locally convex vector topology, makes the ask continuous.

PROPOSITION 3.1. *If $\pi$ is sublinear, $A$ is a cone, and* dom $a_{\pi,A} = L$, *then $a_{\pi,A}$ is lc-continuous.*

*Proof*. From Proposition 2.1, it follows that $a_{\pi,A}$ is sublinear.

First, we show that $a_{\pi,A}$ is upper semicontinuous with respect to the lc-topology. This means showing that the set $a_{\pi,A}^{-1}([-\infty, \alpha)) = \{x \mid a_{\pi,A}(x) < \alpha\}$ is lc-open for all $\alpha \in \mathbb{R}$. Any convex, radially open set is lc-open (Jaschke and Küchler 2000, Lem. 10(iv)). Because $a_{\pi,A}$ is convex, $a_{\pi,A}^{-1}([-\infty, \alpha))$ is convex. It remains to show that it is radial at all its points. Consider $x$ such that $a_{\pi,A}(x) < \alpha$ and any $u \in L$. If $a_{\pi,A}(u) \leq 0$, then, for any $\gamma \geq 0$,

$$a_{\pi,A}(x + \gamma u) \leq a_{\pi,A}(x) + \gamma a_{\pi,A}(u) \leq a_{\pi,A}(x) < \alpha,$$

where the first inequality follows from sublinearity of $a_{\pi,A}$. If $a_{\pi,A}(u) > 0$, pick a positive $\delta < (\alpha - a_{\pi,A}(x))/a_{\pi,A}(u)$, which is positive and finite. Then, for any $\gamma \in [0, \delta]$,

$$a_{\pi,A}(x + \gamma u) \leq a_{\pi,A}(x) + \gamma a_{\pi,A}(u) \leq a_{\pi,A}(x) + \delta a_{\pi,A}(u) < \alpha.$$

Whether $a_{\pi,A}(u)$ is positive or not, $x + \gamma u$ is in $a_{\pi,A}^{-1}([-\infty, \alpha))$ for all sufficiently small nonnegative $\gamma$, so it is radially open.

Finally, we show that $a_{\pi,A}$ is lower semicontinuous with respect to the lc-topology. This means showing that the set $a_{\pi,A}^{-1}([-\infty, \alpha])$ is lc-closed for all $\alpha \in \mathbb{R}$. Any convex, radially closed set with nonempty radial interior is lc-closed (Jaschke and Küchler 2000, Prop. 19). Because $a_{\pi,A}$ is convex, $a_{\pi,A}^{-1}([-\infty, \alpha])$ is convex. It contains $a_{\pi,A}^{-1}([-\infty, \alpha))$, which has just been shown to be radially open, and is nonempty by the following Lemma 3.1. Therefore it has nonempty radial interior, and it remains to show that it is radially closed or, equivalently, that $a_{\pi,A}^{-1}((\alpha, \infty])$ is radially open. Consider any point $w$ at which $a_{\pi,A}^{-1}((\alpha, \infty])$ is not radial. There exists $u \in L$ such that, for all $\delta > 0$, there exists $\gamma \in [0, \delta]$ such that $a_{\pi,A}(w + \gamma u) \leq \alpha$. By definition of the ask, the sale of $w + \gamma u$ can be hedged acceptably for any cost exceeding $\alpha$: for all $\epsilon > 0$, there exists $y_\epsilon$ such that $\pi(y_\epsilon) \leq \alpha + \epsilon$ and $y_\epsilon - (w + \gamma u) \in A$. Because dom $a_{\pi,A} = L$, there exists $y_u$ such that $\pi(y_u) < \infty$ and $y_u + u \in A$. Because $A$ is a cone, the combination of acceptable cashflows is acceptable:

$$(y_\epsilon - (w + \gamma u)) + \gamma(y_u + u) = y_\epsilon - w + \gamma y_u \in A.$$

Because $\pi$ is sublinear,

$$\pi(y_\epsilon + \gamma y_u) \leq \pi(y_\epsilon) + \gamma\pi(y_u) \leq \alpha + \epsilon + \gamma\pi(y_u).$$

So $a_{\pi,A}(w)$ is less than or equal to this quantity, for arbitarily small positive $\epsilon$ and $\gamma$. Because $\pi(y_u) < \infty$, this proves $a_{\pi,A}(w) \leq \alpha$, i.e. $w \notin a_{\pi,A}^{-1}((\alpha, \infty])$. Therefore $a_{\pi,A}^{-1}((\alpha, \infty])$ is radial at all its points. □

LEMMA 3.1. *If $\pi$ is positively homogeneous and $0 \in A$, then for all $\alpha \in \mathbb{R}$ there exists $x$ such that $a_{\pi,A}(x) < \alpha$.*

*Proof*. It suffices to prove this for $\alpha < 0$. By Assumption 2.2, there exists $x_0$ such that $\pi(x_0) < 0$. Choose $\lambda > \alpha/\pi(x_0)$, which is positive. Then by positive homogeneity, $\pi(\lambda x_0) = \lambda\pi(x_0) < \alpha$. Because $0 \in A$, $a_{\pi,A} \leq \pi$, by Proposition 2.1. □

## 4. DUALITY

In this section, we establish a framework for dualization and find a dual representation for the ask $a_{\pi,A}$ and bid $b_{\pi,A}$. This dual representation is of computational interest and is an ingredient in the fundamental theorems of Section 6.

We say $(L, \mathcal{T})$ and $(L', \mathcal{T}')$ are *paired spaces* when $\mathcal{T}$ and $\mathcal{T}'$ are locally convex vector topologies and there is a bilinear form $\langle\cdot,\cdot\rangle : L \times L' \to \mathbb{R}$ such that $\{\langle\cdot, x'\rangle \mid x' \in L'\}$ is the set of continuous linear functionals on $L$, and vice versa. For this, $\forall x' \in L'$, $\langle x, x'\rangle = 0$ must imply $x = 0$, and vice versa.

REMARK 4.1. The largest space $L'$ for which this can be done is $L^{\times}$, the algebraic dual of $L$, consisting of all linear functions on $L$, in which case $L$ must be equipped with the lc-topology, the finest in which it is locally convex.

STANDING ASSUMPTION 4.1. *$(L, \mathcal{T})$ and $(L', \mathcal{T}')$ are paired spaces.*

EXAMPLE 4.1 (Two pairings). The space of bounded random variables $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, under the strong topology $\mathcal{T}_\infty$ of the $\|\cdot\|_\infty$-norm, pairs with $\mathrm{ba}(\Omega, \mathcal{F}, \mathbf{P})$, the space of finitely additive measures absolutely continuous with respect to $\mathbf{P}$. However, we might prefer to pair it with $\mathrm{ca}(\Omega, \mathcal{F}, \mathbf{P})$, the space of $\sigma$-additive measures absolutely continuous with respect to $\mathbf{P}$. To do so requires a coarser topology on $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, with fewer open sets and more convergence, in order to support fewer continuous linear functionals—that is, pair with a smaller space. This coarser topology turns out to be the topology of bounded convergence in probability. This can be verified directly from the definition of the topology induced on $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ by $\mathrm{ca}(\Omega, \mathcal{F}, \mathbf{P})$ (Dunford and Schwartz 1958, V.3.2). It can also be seen from results in Delbaen (2002) and Föllmer and Schied (2002b), where the Fatou property for a coherent or convex risk measure, which is lower semicontinuity with respect to bounded convergence in probability, is shown to be equivalent to existence of a dual representation of the risk measure in terms of $\sigma$-additive probability measures.

Jaschke and Küchler (2000, Cor. 9) have a version of the first fundamental theorem of asset pricing which involves the condition that $A - M$ be a closed cone, so that the bipolar theorem applies to it. They suggest finding conditions for closedness or a way to alter the set $M$ so that $A - M$ would be closed. Instead, we shift focus to a different set,

$$B := \{x \mid b_{\pi,A}(x) \geq 0\} = \{x \mid a_{\pi,A}(-x) \leq 0\}, \tag{4.1}$$

which turns out to be the closure in question, under some conditions.

The right and left *polar cones* of a set $B \subseteq L$ and of a set $B' \subseteq L'$ are, respectively,

$$B^* := \{x' \in L' \mid \forall x \in B, \langle x, x'\rangle \geq 0\} \quad \text{and} \quad {}^*B' := \{x \in L \mid \forall x' \in B', \langle x, x'\rangle \geq 0\}.$$

The bipolar theorem implies that $B = {}^*(B^*)$ if $B$ is a closed cone.

PROPOSITION 4.1. *If $\pi$ is sublinear, $A$ is a cone, and $a_{\pi,A}$ is continuous, then* $\mathrm{cl}(A - M) = B = {}^*(B^*)$.

*Proof*. It follows from Proposition 2.1 that $B$ is a cone. If $a_{\pi,A}$ is lower semicontinuous, then $B$ is closed. Once $B$ is a closed cone, the bipolar theorem (Jaschke and Küchler 2000, Thm. 20) applies. It follows from the definition of $B$ that it contains $A - M$. Therefore it suffices to show that the radial interior of $B$ is a subset of $A - M$ to establish that $B$ is the radial closure of $A - M$. Consider any $x$ in the radial interior of $B$. For all $u \in L$, there exists $\delta > 0$ such that $x + \delta u \in B$; that is, for all $\epsilon > 0$, there is a $y$ such that $\pi(y) \leq \epsilon$ and $x + \delta u + y \in A$. Choose $u$ such that $\pi(u) < 0$. Then $\pi(\delta u + y) \leq \delta\pi(u) + \epsilon$ by subadditivity. This is negative for small enough positive $\epsilon$. Therefore, $x \in A - M$. This establishes that $B$ is the radial closure of $A - M$. By Lemma 3.1 and upper semicontinuity of $a_{\pi,A}$, it has nonempty interior. When a convex set has nonempty interior, its closure equals its radial closure (Jaschke and Küchler 2000, Prop. 18). □

REMARK 4.2. Radial closure is the condition given by Föllmer and Schied (2002a) for an acceptance set that generates a convex risk measure to equal the acceptance set generated by that convex risk measure.

The translation invariance property of Jaschke and Küchler's numéraire allows them to prove a fundamental theorem of asset pricing as a direct consequence of the comparison of $A - M$ and $^*((A - M)^*)$. The connection does not seem so direct here. Instead, we apply duality theory to the ask $a_{\pi,A}$ in the usual way for minimizations, resulting in Theorem 4.1.

Define the penalty function $\Psi$ on $L'$ by

$$\Psi(x') := \sup_{x \in A}(-\langle x, x'\rangle) + \sup_{y \in R}(\langle y, x'\rangle - \pi(y)). \tag{4.2}$$

The first term has an interpretation as the extent to which $x'$ disagrees about the desirability of cashflows in $A$. It is (in a more abstract setting) the minimal penalty function in the convex risk measure representation theorems of Föllmer and Schied (2002a), up to change of sign. Likewise, the second term measures the disagreement between $x'$ and market prices, which $\pi$ specifies. Now we can find a dual representation for the ask and bid, in the same spirit as the representation theorems for coherent and convex risk measures. This is the dual ingredient in the fundamental theorems.

THEOREM 4.1. *For all $x \in L$,*

$$b_{\pi,A}(x) \leq \inf_{x' \in L'}(\langle x, x'\rangle + \Psi(x')) \tag{4.3}$$

$$a_{\pi,A}(x) \geq \sup_{x' \in L'}(\langle x, x'\rangle - \Psi(x')). \tag{4.4}$$

*If, moreover, $a_{\pi,A}$ is lower semicontinuous, and $\pi$ and $A$ are convex, then equality holds.*

*Proof*. The statements about $b_{\pi,A}$ and $a_{\pi,A}$ are equivalent. We focus on $a_{\pi,A}$ because its primal problem is a minimization, which is more usually studied in the convex optimization literature. The primal value is $\inf_{y \in L}\{\pi(y) \mid y - x \in A\}$. Our framework for dualization is the function $F : L \times L \to (-\infty, \infty]$ given by

$$F(y, u) = \begin{cases} \pi(y) & \text{if } y - (x + u) \in A \\ +\infty & \text{otherwise} \end{cases}$$

where $u$ has the interpretation of a perturbation to $x$, the cashflow to be priced. The optimal value function is

$$\phi(u) = \inf_{y \in L} F(y, u) = \inf_{y \in L}\{\pi(y) \mid y - (x + u) \in A\} = a_{\pi,A}(x + u).$$

The Lagrangian $K : L \times L' \to [-\infty, \infty]$ is given by $K(y, x') = \inf_{u\in L}(F(y, u) + \langle u, x'\rangle)$ and the dual objective by $g(x') = \inf_{y\in L} K(y, x')$, so we get

$$g(x') = \inf_{y,u\in L}(F(y, u) + \langle u, x'\rangle) = \inf_{y,u\in L}\{\pi(y) + \langle u, x'\rangle | y - (x + u) \in A\}.$$

The dual value is $\sup_{x'\in L'} g(x') = \sup_{x'\in L'} g(-x')$, so we can exclude from this maximization those values of $x'$ such that $g(-x') = -\infty$. We substitute $z = y - (x + u)$ so the constraint in the minimization that yields $g(-x')$ is $z \in A$. The objective is

$$\pi(y) - \langle u, x'\rangle = \pi(y) - \langle y - x - z, x'\rangle = \langle x, x'\rangle + \langle z, x'\rangle + (\pi(y) - \langle y, x'\rangle).$$

Therefore,

$$\begin{aligned} g(-x') &= \langle x, x'\rangle + \inf_{z\in A}\langle z, x'\rangle + \inf_{y\in L}(\pi(y) - \langle y, x'\rangle) \\ &= \langle x, x'\rangle - \sup_{z\in A}(-\langle z, x'\rangle) - \sup_{y\in L}(\langle y, x'\rangle - \pi(y)), \end{aligned}$$

which is the supremand in formula (4.4). Duality theory asserts that the primal value is greater than or equal to the dual value, justifying the inequality in (4.4). If $A$ and $\pi$ are convex, then $F$ is convex. When $F$ is convex, the dual value is $\liminf_{u\to 0}\phi(u)$ (Rockafellar 1974, Thm. 7). The primal value is $\phi(0)$, so lower semicontinuity of $a_{\pi,A}$ (hence of $\phi$) implies no duality gap. □

When equality holds, $-b_{\pi,A}$ is a convex risk measure. Föllmer and Schied (2002a) defined a convex risk measure to have $\rho(1) = -1$ for mathematical convenience. If one were to adopt instead the definition of Artzner et al. (1999), that the risk measure should merely be additive with respect to some numéraire $\mathbf{1}$ of unit price, then the following proposition would hold with the hypothesis that $\pi(c\mathbf{1}) = c$ for all $c \in \mathbb{R}$.

PROPOSITION 4.2. *If $\pi(c) = c$ for all $c \in \mathbb{R}$, then $-\inf_{x'\in L'}(\langle\cdot, x'\rangle + \Psi(x'))$ is a convex risk measure.*

*Proof*. As in the proof of Theorem 5 of Föllmer and Schied (2002a), $f_{x'}(x) := \langle x, x'\rangle + \Psi(x')$ is concave, monotone, and constant-additive for each $x'$, and these properties are preserved by taking the infimum. If $\langle\cdot, x'\rangle$ is monotone, $f_{x'}$ is monotone. If not, there exists $y \in L_+$ such that $\langle y, x'\rangle < 0$. By Assumption 2.3, there exists $z \in A$ such that for all $\lambda \geq 0, z + \lambda y \in A$. Therefore $\sup_{x\in A}(-\langle x, x'\rangle) \geq \sup_{\lambda\geq 0}(-\langle z + \lambda y, x'\rangle) = \sup_{\lambda\geq 0}(-\langle z, x'\rangle - \lambda\langle y, x'\rangle) = \infty$. So $f_{x'}(x) = \Psi(x') = \infty$, which is monotone anyway. Similarly, if $\langle c, x'\rangle = c$ for all $c \in \mathbb{R}$, we have constant-additivity, and, if not, $\sup_{y\in R}(\pi(y) - \langle y, x'\rangle \geq \sup_{c\in\mathbb{R}}(c - \langle c, x'\rangle) = \infty$ and we get constant-additivity anyway because $\infty + c = \infty$. □

## 5. SUBLINEARITY AND CONES

In this section, we relate the case where $\pi$ is sublinear and $A$ is a convex cone, described in Propositions 3.1 and 4.1, to the more general case, where $\pi$ and $A$ need not have these properties. We rely on some definitions and notation relating sets to cones and functions to sublinear functions. For any set $C$, let $C_\vee$ be the smallest cone containing $C$. For any function $f$, let conv $f$ be given by

$$(\operatorname{conv} f)(x) := \inf\left\{\sum_{i=1}^{n}\lambda_i f(x_i)\,\middle|\, \sum_{i=1}^{n}\lambda_i x_i = x, \sum_{i=1}^{n}\lambda_i = 1, x_i \in \operatorname{dom} f, \lambda_i \geq 0\right\}.$$

This is the greatest convex function dominated by $f$. Let ah $f$ be given by

$$(\operatorname{ah} f)(x) := \inf\{\lambda f(x/\lambda) \mid \lambda > 0\}$$

for $x \neq 0$, and $(\operatorname{ah} f)(0) := 0$. If $f$ is convex, $f(0) < \infty$, and $f \neq \infty$, this is the greatest absolutely homogeneous function dominated by $f$; see Rockafellar (1970, §5). Let $f_\vee :=$ conv ah $f$, which is given by

$$f_\vee(x) = \inf\left\{\sum_{i=1}^{n}\lambda_i f(x_i)\,\middle|\, \sum_{i=1}^{n}\lambda_i x_i = x, x_i \in \operatorname{dom} f, \lambda_i \geq 0\right\}.$$

This is the greatest sublinear function dominated by $f$. In what follows, we may write down only the interesting constraints in this minimization.

The greatest sublinear function dominated by the ask $a_{\pi,A}$ is the ask generated from the acceptance set $A_\vee$ and market pricing function $\pi_\vee$.

PROPOSITION 5.1. $(a_{\pi,A})_\vee = a_{\pi_\vee, A_\vee}$.

*Proof*. Making the substitutions $y = \sum_{i=1}^{n}\lambda_i y_i$ and $z_i = y_i - x_i$,

$$\begin{aligned}
(\operatorname{conv}\operatorname{ah} a_{\pi,A})(x) &= \inf\left\{\sum_{i=1}^{n}\lambda_i a_{\pi,A}(x_i)\,\middle|\, \sum_{i=1}^{n}\lambda_i x_i = x\right\}\\
&= \inf\left\{\sum_{i=1}^{n}\lambda_i \pi(y_i)\,\middle|\, y_i - x_i \in A, \sum_{i=1}^{n}\lambda_i x_i = x\right\}\\
&= \inf\left\{\pi_\vee(y) \mid z_i \in A, \sum_{i=1}^{n}\lambda_i z_i = y - x\right\}\\
&= \inf\{\pi_\vee(y) \mid y - x \in A_\vee\}.
\end{aligned}$$

□

Define the *marginal ask* and bid by

$$\tilde{a}_{\pi,A}(x) := \liminf_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) \quad \text{and} \quad \tilde{b}_{\pi,A}(x) := \limsup_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda). \tag{5.1}$$

These are the most favorable prices that a counterparty who wants to execute a small trade can come close to attaining. The following proposition gives conditions under which these also equal the prices generated from the acceptance set $A_\vee$ and market pricing function $\pi_\vee$.

LEMMA 5.1. *If $f$ is convex and $f(0) \leq 0$, then* $(\operatorname{ah} f)(x) = \lim_{\lambda\to\infty}\lambda f(x/\lambda)$.

*Proof*. By convexity, for $\lambda_1 \geq \lambda_2 > 0$,

$$f(x/\lambda_1) \leq \left(1 - \frac{\lambda_2}{\lambda_1}\right) f(0) + \left(\frac{\lambda_2}{\lambda_1}\right) f(x/\lambda_2),$$

so $\lambda_1 f(x/\lambda_1) \leq (\lambda_1 - \lambda_2)f(0) + \lambda_2 f(x/\lambda_2) \geq \lambda_2 f(x/\lambda_2)$. Hence $\lambda f(x/\lambda)$ is nondecreasing as a function of $\lambda > 0$. Thus the infimum in the definition of $(\text{ah } f)$ is a limit. $\square$

PROPOSITION 5.2. *If $A$ is convex and contains* 0*, and $\pi$ is convex, then $\tilde{a}_{\pi,A} = a_{\pi_\vee, A_\vee} = \lim_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda)$ and $\tilde{b}_{\pi,A} = b_{\pi_\vee, A_\vee} = \lim_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda)$.*

*Proof.* By Proposition 2.1, $a_{\pi,A}$ is convex and has $a_{\pi,A}(0) \leq 0$. By Lemma 5.1, the limit exists, so

$$\tilde{a}_{\pi,A}(x) = \liminf_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) = \lim_{\lambda\to\infty} \lambda a_{\pi,A}(x/\lambda) = (\text{ah } a_{\pi,A})(x) = (a_{\pi,A})_\vee(x) = a_{\pi_\vee,A_\vee}(x)$$

because $a_{\pi,A}$ is already convex, and using Proposition 5.1. Then $b_{\pi_\vee,A_\vee}(x) = -a_{\pi_\vee,A_\vee}(-x) = -\lim_{\lambda\to\infty} -\lambda b_{\pi,A}(x/\lambda) = \lim_{\lambda\to\infty} \lambda b_{\pi,A}(x/\lambda)$. $\square$

Next we prove a result, relating conditions on $\pi$ and $A$ to conditions on $\pi_\vee$ and $A_\vee$, that is the primal ingredient in the fundamental theorems. In a sense, we are a considering a fictitious market in which $\pi_\vee$ gives the prices and $A_\vee$ is our acceptance set. This enables us to connect the case actually under consideration with the sublinear case, in which Propositions 3.1 and 4.1 will apply.

Let $\tilde{M} := \{x \mid \pi_\vee(x) \leq 0\} - L_+$ and $\tilde{C} := \{x \mid \pi_\vee(-x) < 0\}$ be the set of cashflows you could respectively have for free and sell for cash now if $\pi_\vee$ gave market prices. Likewise let $\tilde{B} := \{x \mid a_{\pi_\vee,A_\vee}(-x) \leq 0\}$. These are not necessarily the same as $M_\vee$, $C_\vee$, and $B_\vee$.

LEMMA 5.2. *The following are monotone cones: $A_\vee \subseteq A_\vee - \tilde{M} \subseteq \tilde{B}$.*

*Proof.* Because $\pi_\vee(0) = 0$, $0 \in \tilde{M}$, so $A_\vee \subseteq A_\vee - \tilde{M}$. If $x = z - y$ where $z \in A_\vee$ and $\pi_\vee(y) \leq 0$, then $x + y \in A_\vee$, so $b_{\pi_\vee,A_\vee}(x) \geq -\pi_\vee(y) \geq 0$, and $x \in \tilde{B}$. This shows $A_\vee - \tilde{M} \subseteq \tilde{B}$. By construction, $A_\vee$ and $\tilde{M}$ are cones, which makes $A_\vee - \tilde{M}$ a cone. By Proposition 2.1, $a_{\pi_\vee,A_\vee}$ is convex and absolutely homogeneous, so $\tilde{B}$ is a cone. By Assumption 2.3, $A$ is nonempty and monotone, so any cone containing $A$ contains $L_+$. Any cone $K$ has the property $K + K \subseteq K$, so $L_+ \subseteq K$ implies $K + L_+ \subseteq K$; that is, $K$ is monotone. $\square$

PROPOSITION 5.3. *Among the conditions*

(i) $NC(\pi, A)$: $(M - A) \cap C = \emptyset$
(ii) $NC(\pi, a_{\pi,A})$: $a_{\pi,A}(x) + a_{\pi,L_+}(-x) \geq 0$
(iii) $NC(\pi_\vee, a_{\pi_\vee,A_\vee})$: $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(-x) \geq 0$
(iv) $NC(\pi_\vee, A_\vee)$: $(\tilde{M} - A_\vee) \cap \tilde{C} = \emptyset$

*the following implications hold:* (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). *If $A$ is convex and contains 0, and $\pi$ is convex, then all the conditions are equivalent.*

*Proof.* (iv) $\Rightarrow$ (iii): Suppose $a_{\pi_\vee,A_\vee}(x) + a_{\pi_\vee,L_+}(-x) < 0$. There exist $y_1, y_2$ such that $y_1 - x \in A_\vee$, $y_2 + x \in L_+$, and $\pi_\vee(y_1) + \pi_\vee(y_2) < 0$. Then, by monotonicity of $A_\vee$, the sum $y_1 + y_2 \in A_\vee$, while by subadditivity of $\pi_\vee$, $\pi_\vee(y_1 + y_2) < 0$. So $A_\vee \cap (-\tilde{C})$ is nonempty, and by Lemma 5.2, this is enough.

(iii) $\Rightarrow$ (iv): Same as (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii): Because $\pi_\vee \leq \pi$ and $A \subseteq A_\vee$, $a_{\pi,A} \geq a_{\pi_\vee,A_\vee}$ and $a_{\pi,L_+} \geq a_{\pi_\vee,L_+}$.

(ii) $\Rightarrow$ (i): Suppose there exists $x$ in the set $(M - A) \cap C$, so $\pi(-x) < 0$ and $x = y - z$ where $z \in A$ and $\pi(y) \leq 0$. Then $a_{\pi, L_+}(-x) \leq \pi(-x) < 0$, while $a_{\pi, A}(x) \leq \pi(y) \leq 0$ because $y - x = z \in A$. So $a_{\pi, A}(x) + a_{\pi, L_+}(-x) < 0$.

(i) $\Rightarrow$ (iv): Here assume $A$ is convex and contains 0, and $\pi$ is convex. Suppose there exists $-x \in (\tilde{M} - A_\vee) \cap \tilde{C}$. That is, $\pi_\vee(x) < 0$ and $x = z - y$ where $z \in A_\vee$ and $\pi_\vee(y) \leq 0$. By subadditivity, $\pi_\vee(z) \leq \pi_\vee(x) + \pi_\vee(y) < 0$. So there exists $z \in A_\vee \cap (-\tilde{C})$. For some $\lambda_1 > 0, [0, \lambda_1 z] \subset A$. For some $\lambda_2 > 0, \pi$ is negative on the line segment $(0, \lambda_2 z]$. Let $\lambda = \min\{\lambda_1, \lambda_2\}$, so $(0, \lambda z] \subset A \cap (-C)$. Therefore $A \cap (-C)$ is nonempty, containing $\lambda z$ for some $\lambda > 0$. By Lemma 5.2, this is enough. $\square$

This result offers some guidance about choosing an acceptance set $A$. If $\pi$ is convex and we choose a convex (risk-averse) acceptance set $A$ that contains 0 and satisfies $(M - A) \cap C = \emptyset$, then we can be sure of satisfying the desideratum $\mathrm{NC}(\pi, a_{\pi,A})$.

Having established this connection between results for $(\pi, A)$ and for $(\pi_\vee, A_\vee)$, we can now revisit the dual problem (4.4). The dual-feasible set is $D' := \{x' \mid \Psi(x') < \infty\}$. For use in the fundamental theorems, we consider the dual-feasible set when market prices are given by $\pi_\vee$ and the acceptance set is $A_\vee$. Define $\tilde{\Psi}$ as the penalty function $\Psi$ in equation (4.2) with these substitutions, and $\tilde{D}' := \{x' \mid \tilde{\Psi}(x') < \infty\}$. Call an element of $\tilde{D}'$ a *consistent pricing kernel*: item (iv) below shows that it is consistent with the acceptance set and market prices. We now collect some properties of these objects.

PROPOSITION 5.4. *Given the preceding definitions,*

(i) *The following are equivalent:* $x' \in A^*$, $\sup_{x \in A}(-\langle x, x' \rangle) \leq 0$, $x' \in (A_\vee)^*$, *and* $\sup_{x \in A_\vee}(-\langle x, x' \rangle) = 0$.
(ii) *The following are equivalent:* $\langle \cdot, x' \rangle \leq \pi$, $\sup_{y \in R}(\langle y, x' \rangle - \pi(y)) = 0$, $\langle \cdot, x' \rangle \leq \pi_\vee$, *and* $\sup_{y \in R}(\langle y, x' \rangle - \pi_\vee(y)) = 0$.
(iii) *The function* $\tilde{\Psi}$ *takes values in* $\{0, \infty\}$.
(iv) *The set* $\tilde{D}' = \{x' \mid \tilde{\Psi}(x') = 0\} = \{x' \mid \Psi(x') = 0\} = (A_\vee)^* \cap \{x' \mid \langle \cdot, x' \rangle \leq \pi_\vee\} = A^* \cap \{x' \mid \langle \cdot, x' \rangle \leq \pi\}$, *that is, $x'$ is a consistent pricing kernel if and only if it is dominated by $\pi$ and nonnegative on $A$.*

*Proof*. (i) Recall that the definition of $A^*$ is $\{x' \mid \forall x \in A, \langle x, x' \rangle \geq 0\}$, which shows the equivalence of $x' \in A^*$ and $\sup_{x \in A}(-\langle x, x' \rangle) \leq 0$. The equivalence of $x' \in (A_\vee)^*$ and $\sup_{x \in A_\vee}(-\langle x, x' \rangle) = 0$ is similar, with the added observation that this supremum is nonnegative because $0 \in A_\vee$. From $A \subseteq A_\vee$ it follows that $(A_\vee)^* \subseteq A^*$. Now consider $x' \in A^*$. Each $x \in A_\vee$ is a nonnegative linear combination of elements of $A$, at each of which $\langle \cdot, x' \rangle$ is nonnegative. Therefore $\langle \cdot, x' \rangle$ is nonnegative at $x$, so $x' \in (A_\vee)^*$. This shows $A^* = (A_\vee)^*$.

(ii) Because $\pi_\vee(0) = \pi(0) = 0$, $\langle \cdot, x' \rangle \leq \pi$ and $\sup_{y \in R}(\langle y, x' \rangle - \pi(y)) = 0$ are equivalent, and likewise $\langle \cdot, x' \rangle \leq \pi_\vee$ and $\sup_{y \in R}(\langle y, x' \rangle - \pi_\vee(y)) = 0$ are equivalent. From $\pi_\vee \leq \pi$ it follows that $\langle \cdot, x' \rangle \leq \pi_\vee$ implies $\langle \cdot, x' \rangle \leq \pi$. Now suppose $\langle \cdot, x' \rangle \leq \pi$. For any nonnegative linear combination $\sum_{i=1}^n \lambda_i x_i = x$, $\langle x, x' \rangle = \sum_{i=1}^n \lambda_i \langle x_i, x' \rangle \leq \sum_{i=1}^n \lambda_i \pi(x_i)$. So $\pi_\vee(x)$ is the greatest lower bound of a set bounded below by $\langle x, x' \rangle$; therefore $\langle \cdot, x' \rangle \leq \pi_\vee$.

(iii) Both terms in $\tilde{\Psi}$ are nonnegative and absolutely homogeneous.

(iv) From part (iii), $\tilde{D}' = \{x' \mid \tilde{\Psi}(x') = 0\}$. Then parts (i) and (ii) imply the conclusion. $\square$

REMARK 5.1. If $\pi$ is linear and $x'$ is a consistent pricing kernel, then $\langle\cdot, x'\rangle$ is a linear extension of $\pi$.

## 6. FUNDAMENTAL THEOREMS

Now we get two versions of the first fundamental theorem of asset pricing (FTAP), one each for $\mathrm{NC}(\pi, a_{\pi,A})$ and $\mathrm{NNA}(\pi, a_{\pi,A})$. The 0th version does not quite deserve the name, because it relates only to absence of cashouts, not absence of arbitrage. In interpreting the hypothesis of lower semicontinuity for $a_{\pi_\vee, A_\vee}$, recall that by Proposition 3.1 $\operatorname{dom} a_{\pi_\vee, A_\vee} = L$ is sufficient for the existence of a locally convex topology in which $a_{\pi_\vee, A_\vee}$ is continuous. Moreover, because $a_{\pi_\vee, A_\vee} \leq a_{\pi, A}$, $\operatorname{dom} a_{\pi_\vee, A_\vee} = L$ is weaker than $\operatorname{dom} a_{\pi,A} = L$, the more natural hypothesis of finite-cost hedging. See also Proposition 7.2 for a simple sufficient condition for continuity of $a_{\pi_\vee, A_\vee}$ in the strong topology of $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$.

THEOREM 6.1 (0th FTAP). *The existence of a consistent pricing kernel implies* $\mathrm{NC}(\pi, a_{\pi,A})$. *If $A$ is convex and contains 0, $\pi$ is convex, and $a_{\pi_\vee, A_\vee}$ is lower semicontinuous, then the converse holds.*

*Proof*. By Proposition 5.4, a consistent pricing kernel satisfies $\tilde\Psi(x') = 0$. Then Theorem 4.1 implies $b_{\pi_\vee, A_\vee} \leq \inf_{x' \in \tilde D'} \langle\cdot, x'\rangle \leq \langle\cdot, x'\rangle \leq \pi_\vee$. Therefore $\pi_\vee(x) < 0$ implies $x \notin \tilde B$. By Lemma 5.2, $A_\vee - \tilde M$ is a subset of $\tilde B$, so this implies that $A_\vee - \tilde M$ and $\{x \,|\, \pi_\vee(x) < 0\} = -\tilde C$ are disjoint. This in turn implies $\mathrm{NC}(\pi, a_{\pi,A})$, by Proposition 5.3.

Given the extra hypotheses, $\mathrm{NC}(\pi, a_{\pi,A})$ implies that for all $x \in L$, $a_{\pi_\vee, A_\vee}(x) + a_{\pi_\vee, L_+}(-x) \geq 0$, by Proposition 5.3. Because $L_+ \subseteq A_\vee$, $a_{\pi_\vee, A_\vee}(0) \leq a_{\pi_\vee, L_+}(0) \leq 0$, so $a_{\pi_\vee, A_\vee}(0) = 0$. From Theorem 4.1 and Proposition 5.4 it follows that $a_{\pi_\vee, A_\vee}(0) = \sup_{x' \in \tilde D'} \langle 0, x'\rangle = \sup_{x' \in \tilde D'} 0$. Therefore $\tilde D' \neq \emptyset$; that is, a consistent pricing kernel exists. □

The following theorem deserves to be called a first fundamental theorem of asset pricing, because $\mathrm{NNA}(\pi, a_{\pi,A})$ rules out giving away arbitrages. By a strictly monotone $x'$, we mean one for which $\langle\cdot, x'\rangle$ is strictly monotone; in other words, for all $x \in L_+ \backslash \{0\}$, $\langle x, x'\rangle > 0$. The theorem shows that existence of a strictly monotone consistent pricing kernel is sufficient, but establishes its necessity only under hypotheses not only of convexity but also homogeneity, or in relation to the marginal ask.

THEOREM 6.2 (1st FTAP). *The existence of a strictly monotone consistent pricing kernel implies* $\mathrm{NNA}(\pi, a_{\pi,A})$. *If $A$ is convex and contains* 0, *$\pi$ is convex, and $a_{\pi_\vee, A_\vee}$ is lower semicontinuous, then $\forall x \in L_+ \backslash \{0\}$, $\tilde a_{\pi,A}(x) > 0$ is equivalent to the existence of a strictly monotone consistent pricing kernel. If moreover $A$ is a convex cone and $\pi$ is sublinear, then* $\mathrm{NNA}(\pi, a_{\pi,A})$ *is equivalent to the existence of a strictly monotone consistent pricing kernel.*

*Proof*. Suppose $\mathrm{NNA}(\pi, a_{\pi,A})$ fails, so there are $x$ and $z > 0$ such that $a_{\pi,A}(x) + a_{\pi,L_+}(z - x) \leq 0$. Then $a_{\pi_\vee, A_\vee}(x) + a_{\pi_\vee, L_+}(z - x) \leq 0$ and, equivalently, $b_{\pi_\vee, A_\vee}(-x) \geq a_{\pi_\vee, L_+}(z - x)$. By definition of the bid $b_{\pi_\vee, A_\vee}$ and ask $a_{\pi_\vee, L_+}$, for any $\epsilon > 0$, there are $y_a, y_b$ such that $y_b - x \in A_\vee$, $y_a + x - z \in L_+$, and $\pi_\vee(y_b) \leq \epsilon - \pi_\vee(y_a)$. Because $A_\vee$ is monotone, we can add to get $y_a + y_b - z \in A_\vee$. By subadditivity, $\pi_\vee(y_a + y_b) \leq \epsilon$. As this can be done for all positive $\epsilon$, $b_{\pi_\vee, A_\vee}(-z) \geq 0$, so $-z \in \tilde B$, and $\tilde B \cap (L_- \backslash \{0\}) \neq \emptyset$. By the following Lemma 6.1, this implies there is no strictly monotone consistent pricing kernel.

Now assume $A$ is convex and contains 0, $\pi$ is convex, and $a_{\pi_\vee, A_\vee}$ is lower semicontinuous. By Proposition 5.2, $\tilde{a}_{\pi,A} = a_{\pi_\vee, A_\vee}$. Therefore, $\forall x \in L_+\backslash\{0\}$, $\tilde{a}_{\pi,A}(x) > 0$ is equivalent to $\forall x \in L_+\backslash\{0\}$, $a_{\pi_\vee, A_\vee}(x) > 0$, which is in turn equivalent to $\tilde{B} \cap (L_-\backslash\{0\}) = \emptyset$. By Lemma 6.1, this is equivalent to the existence of a strictly monotone consistent pricing kernel.

Now assume $A$ is a convex cone and $\pi$ is sublinear. Then $A = A_\vee$ and $\pi = \pi_\vee$, so $a_{\pi,A} = a_{\pi_\vee, A_\vee}$. Suppose there is no strictly monotone consistent pricing kernel. By Lemma 6.1, this implies there exists $-z \in \tilde{B} \cap (L_-\backslash\{0\})$. Then $b_{\pi_\vee, A_\vee}(-z) \geq 0$, so $a_{\pi_\vee, A_\vee}(z) \leq 0$, and $a_{\pi_\vee, A_\vee}(z) + a_{\pi_\vee, L_+}(z - z) \leq 0$. That is, $a_{\pi,A}(z) + a_{\pi, L_+}(z - z) \leq 0$, violating NNA$(\pi, a_{\pi,A})$ . $\square$

LEMMA 6.1. *The existence of a strictly monotone consistent pricing kernel implies $\tilde{B} \cap (L_-\backslash\{0\}) = \emptyset$. If $a_{\pi_\vee, A_\vee}$ is lower semicontinuous, then the converse holds.*

*Proof*. Suppose $x'$ is a strictly monotone consistent pricing kernel. From Theorem 4.1 we get $b_{\pi_\vee, A_\vee}(x) \leq \langle x, x'\rangle$. For $x < 0$, this implies $b_{\pi_\vee, A_\vee}(x) < 0$; therefore, $x \notin \tilde{B}$. The converse is an application of the same "exhaustion" argument that underpins the Halmos-Savage theorem, as explained by Delbaen in the proof of a similar result (Delbaen 2002, Thm. 3.5). Consider the class of sets $\mathcal{C} := \{C \subseteq L_+ \mid \exists x' \in \tilde{D}' \ni \forall x \in C, \langle x, x'\rangle > 0\}$. From Proposition 5.4 (iv), we can see that $\tilde{D}'$ is convex and contains only nonnegative elements. Therefore class $\mathcal{C}$ is stable under countable unions: take a sequence $\{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$, and let $x'_n \in \tilde{D}'$ be such that for all $x \in C_n$, $\langle x, x'_n\rangle > 0$. Define $x' := \sum_{n=1}^{\infty} 2^{-n} x'_n \in \tilde{D}'$. Then any $x \in \cup_{n\in\mathbb{N}} C_n$ is in $C_j \subseteq L_+$ for some $j$, so $\langle x, x'\rangle = \langle x, \sum_{n\neq j} 2^{-n} x'_n\rangle + \langle x, 2^{-j} x'_j\rangle \geq 2^{-j}\langle x, x'_j\rangle > 0$. From stability under countable unions, it follows by Zorn's lemma that $\mathcal{C}$ has a maximal element. It is given in the converse that for any $x > 0$, $b_{\pi_\vee, A_\vee}(-x) < 0$; that is, $a_{\pi_\vee, A_\vee}(x) > 0$. By Theorem 4.1, given lower semicontinuity, $a_{\pi_\vee, A_\vee}(x) = \sup_{x'\in\tilde{D}'}\langle x, x'\rangle$. Therefore, for any $x > 0$ there is an $x' \in \tilde{D}'$ such that $\langle x, x'\rangle > 0$; that is, $\{x\} \in \mathcal{C}$. Thus the only possible maximal element of $\mathcal{C}$ is $L_+\backslash\{0\}$, and $L_+\backslash\{0\} \in \mathcal{C}$ implies the existence of $x' \in \tilde{D}'$ such that for all $x > 0$, $\langle x, x'\rangle > 0$, which is a strictly monotone consistent pricing kernel. $\square$

The following example shows that NNA$(\pi, a_{\pi,A})$ does not imply NNA$(\pi_\vee, a_{\pi_\vee, A_\vee})$, which is equivalent to the existence of a strictly monotone consistent pricing kernel. This accounts for the difficulty in framing a partial converse.

EXAMPLE 6.1 (Trouble with nonclosed cones). Let $L = \mathbb{R}^2$, the space of contingent claims in a two-state, one-period economy, $\mathcal{T}$ be the Euclidean norm topology, and the acceptance set $A = L_+ \cup \{x \mid x_2 \geq x_1^2\}$, which is closed, convex, monotone, and contains 0. Then $A_\vee = L_+ \cup \{x \mid x_2 > 0\}$, as follows. First we show $L_+ \cup \{x \mid x_2 > 0\} \subseteq A_\vee$. It is clear that $L_+ \subseteq A \subseteq A_\vee$. For the other points $x$, for which $x_1 < 0$ and $x_2 > 0$, define $\lambda := x_1^2/x_2 \in (0, \infty)$, so $x_2/\lambda = (x_1/\lambda)^2$. Such a point is thus a positive multiple of an element of $A$, hence in $A_\vee$. Next we show $A_\vee \subseteq L_+ \cup \{x \mid x_2 > 0\}$, equivalently, $(L_+ \cup \{x \mid x_2 > 0\})^{\mathcal{C}} \subseteq A_\vee^{\mathcal{C}}$. The set $(L_+ \cup \{x \mid x_2 > 0\})^{\mathcal{C}} = \{x \mid x_2 < 0\} \cup \{x \mid x_1 < 0, x_2 = 0\}$, which is a positively homogeneous set disjoint from $A$, which is convex. Therefore it is also disjoint from $A_\vee$. The new conic acceptance set $A_\vee$ is still disjoint with $L_-\backslash\{0\}$, but its closure is not, which causes a problem. Let $M$ be the embedding of $\mathbb{R}$, namely $\{x \mid x_2 = x_1\}$, and $\pi(c) = c = \pi_\vee(c)$. Consider a contingent claim $z = (d, 0)$ with $d > 0$, which is in $L_+\backslash\{0\}$. Letting $x = z$, NNA$(\pi_\vee, a_{\pi_\vee, A_\vee})$ is violated, because $a_{\pi_\vee, A_\vee}(x) = 0$ and $a_{\pi_\vee, L_+}(z - x) = 0$. On the other hand, NNA$(\pi, a_{\pi,A})$ is not violated here because $a_{\pi,A}(x) = d + (1 - \sqrt{1 + 4d})/2 > 0$ and $a_{\pi, L_+}(z - x) = 0$. A pricing kernel is a point $x' \in \mathbb{R}^2$, using the usual

Euclidean inner product $\langle x, x'\rangle = x_1x'_1 + x_2x'_2$. A consistent pricing kernel must have $\langle x, x'\rangle \geq 0$ for all $x \in A$, and in particular for $(-d, d^2)$ where $d > 0$. But $d(dx'_2 - x'_1) = -dx'_1 + d^2x'_2 \geq 0$ for all $d > 0$ implies $x'_1 = 0$, so a consistent pricing kernel cannot be strictly monotone.

Next we have a kind of second fundamental theorem of asset pricing, relating uniqueness of pricing to uniqueness of consistent pricing kernel.

THEOREM 6.3 (2nd FTAP). *First, if $b_{\pi,A} = a_{\pi,A}$, then $D'$ and $\tilde{D}'$ contain at most one element. If moreover $a_{\pi,A}$ is lower semicontinuous, $A$ is convex and contains 0, and $\pi$ is convex, then $D' = \tilde{D}'$ is a singleton. Second, if there is a unique consistent pricing kernel $x'$ and $A$ is convex and contains 0, $\pi$ is convex, and $a_{\pi_\vee, A_\vee}$ is lower semicontinuous, then the marginal bid and ask are equal: $\tilde{b}_{\pi,A} = \tilde{a}_{\pi,A} = \langle \cdot, x'\rangle$. If moreover $\pi$ is sublinear, $A$ is a cone, and $a_{\pi,A}$ is lower semicontinuous, then $b_{\pi,A} = a_{\pi,A} = \langle \cdot, x'\rangle$.*

*Proof*. First, given $b_{\pi,A} = a_{\pi,A}$, consider $x'_1, x'_2 \in D'$. The inequalities in expressions (4.3) and (4.4) imply

$$\max\{\langle x, x'_1\rangle - \Psi(x'_1), \langle x, x'_2\rangle - \Psi(x'_2)\} \leq a_{\pi,A}(x)$$
$$= b_{\pi,A}(x) \leq \min\{\langle x, x'_1\rangle + \Psi(x'_1), \langle x, x'_2\rangle + \Psi(x'_2)\}.$$

Now suppose $x'_1 \neq x'_2$. Then there exists $\tilde{x}$ such that the difference $d = \langle \tilde{x}, x'_1 - x'_2\rangle \neq 0$. We know $\Psi(x'_1) + \Psi(x'_2) < \infty$ by definition of $D'$. So we can pick a real number $\lambda > (\Psi(x'_1) + \Psi(x'_2))/d$, and let $x = \lambda\tilde{x}$. Now

$$\langle x, x'_1\rangle - \Psi(x'_1) > (\langle x, x'_2\rangle + \Psi(x'_1) + \Psi(x'_2)) - \Psi(x'_1) = \langle x, x'_2\rangle + \Psi(x'_2).$$

This contradicts the above inequality, so $x'_1 = x'_2$; that is, $D'$ contains at most one element. From Proposition 5.4 it follows that $\tilde{D}' \subseteq D'$. Under the additional hypotheses, equality holds in Theorem 4.1. This implies $\langle \cdot, x'\rangle - \Psi(x') = a_{\pi,A} = b_{\pi,A} = \langle \cdot, x'\rangle + \Psi(x')$, so $\Psi(x') = 0$; in other words, $x' \in \tilde{D}'$.

Second, given $\tilde{D}' = \{x'\}$ and the initial hypotheses, from Theorem 4.1 we get $b_{\pi_\vee, A_\vee} = \langle \cdot, x'\rangle = a_{\pi_\vee, A_\vee}$, and from Proposition 5.2, $a_{\pi_\vee, A_\vee} = \tilde{a}_{\pi,A}$ and $b_{\pi_\vee, A_\vee} = \tilde{b}_{\pi,A}$. Given the further hypotheses, by Proposition 2.1, the bid and ask are already sublinear, so $b_{\pi,A} = (b_{\pi,A})_\vee$ and $a_{\pi,A} = (a_{\pi,A})_\vee$. By Proposition 5.1, $(b_{\pi,A})_\vee = b_{\pi_\vee, A_\vee}$ and $(a_{\pi,A})_\vee = a_{\pi_\vee, A_\vee}$. □

There are other approaches to framing a second fundamental theorem of asset pricing. Carr et al. (2001) relate uniqueness of a certain pricing kernel to a notion of "acceptable completeness." An acceptably complete market is one in which all cashflows can be hedged so that they are barely acceptable— that is, one is indifferent between the hedged cashflow and 0. Mathematically speaking, the barely acceptable hedged cashflow is on the boundary of the closed acceptance set. Jarrow and Madan (1999) related uniqueness of a signed equivalent local martingale measure to completeness in the traditional sense of exact replicability of contingent claims.

## 7. BOUNDED RANDOM VARIABLES

To make matters more concrete, we examine the case where $L = L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, the space of $\mathbf{P}$-equivalence classes of bounded $\mathcal{F}$-measurable random variables (functions) on $\Omega$. Here we can draw connections to the established theories of exact functionals, imprecise probabilities, and convex risk measures. In this setting, we can identify simple sufficient conditions for finite cost hedging and continuity of $a_{\pi_\vee, A_\vee}$.

Our primal space is $L^\infty$ with the strong topology of the $\|\cdot\|_\infty$-norm, under which it is a Banach space. Its positive orthant $L^\infty_+ = \{x \in L^\infty \mid x \geq 0\}$ with the usual partial ordering generated by the essential infimum: $x_1 \leq x_2$ when $\operatorname{ess\,inf}(x_2 - x_1) \geq 0$.

Our dual space $L'$ is $\mathrm{ba} = \mathrm{ba}(\Omega, \mathcal{F}, \mathbf{P})$, the Banach space of bounded, finitely additive, signed measures $\mu$ defined on the $\sigma$-algebra $\mathcal{F}$ and absolutely continuous with respect to $\mathbf{P}$. Then $\langle \cdot, \mu \rangle$ will be written $I_\mu(\cdot)$ to denote a (Radon) integral, and $\{I_\mu \mid \mu \in \mathrm{ba}\}$ is the strong dual of bounded linear functionals on $L$. For the pairing, it has the weak* topology. The positive orthant $\mathrm{ba}_+$ contains those measures $\mu$ such that $\mu(E) \geq 0$ for every event $E$ in $\mathcal{F}$; equivalently, it can be viewed as containing those monotone linear functionals $I_\mu$ such that $x_1 \leq x_2 \Rightarrow I_\mu(x_1) \leq I_\mu(x_2)$. We also have a norm on the dual given by $\|\mu\| = \|I_\mu\| = \sup_{\|x\|_\infty \leq 1} |I_\mu(x)|$, which is $I_\mu(1) = \mu(\Omega)$ when $\mu \in \mathrm{ba}_+$.

It can be convenient to assume that one can buy or sell unlimited amounts of a riskless bond with payoff 1. This assumption is equivalent to $\mathbb{R} \subseteq R$. We get by with the weaker hypothesis that every contingent claim is dominated by a marketed claim. This is equivalent to $\mathbb{R} \subseteq R - L_+$, the availability of a marketed claim dominating any number of riskless bonds.

PROPOSITION 7.1. *If* $\mathbb{R} \subseteq R - L_+$ *then* $\operatorname{dom} a_{\pi,A} = L$.

*Proof*. By Assumption 2.3, there exists $z_0$ such that $z \geq z_0$ implies $z \in A$. The hypothesis implies that for any $x \in L$, there exists $y \in R$ such that $y \geq \operatorname{ess\,sup} x + \operatorname{ess\,sup} z_0$, so $y - x \geq z_0$ and thus is in $A$. Therefore $a_{\pi,A}(x) \leq \pi(y) < \infty$. □

To prove continuity of $a_{\pi_\vee, A_\vee}$, we use the hypothesis that any contingent claim would be dominated by a marketed claim if prices were given by $\pi_\vee$. Letting $\tilde{R}$ be the effective domain of $\pi_\vee$, this is equivalent to $1 \in \tilde{R} - L_+$, a weaker hypothesis than $\mathbb{R} \subseteq R - L_+$.

PROPOSITION 7.2. *If* $1 \in \tilde{R} - L_+$, *then* $a_{\pi_\vee, A_\vee}$ *is continuous.*

*Proof*. Consider any $\alpha \in \mathbb{R}$ and $x \in L$ such that $a_{\pi_\vee, A_\vee}(x) > \alpha$. There exists $\gamma > \alpha$ such that $\pi_\vee(y) < \gamma$ implies $y - x \notin A_\vee$. There exists $y_1 \in \tilde{R}$ such that $y_1 \geq 1$. Pick $\beta \in (\alpha, \gamma)$ and $\delta := (\gamma - \beta)/|\pi_\vee(y_1)| \in (0, \infty]$. Consider any $y$ with $\pi_\vee(y) < \beta$. By subadditivity, $\pi_\vee(y + \delta y_1) \leq \pi_\vee(y) + \delta \pi_\vee(y_1) < \gamma$, so $y + \delta y_1 - x \notin A_\vee$. For any $u$ in the $\delta$-ball at $x$, $u \geq x - \delta \geq x - \delta y_1$, so by monotonicity, $y - u \leq y + \delta y_1 - x$ is also not in $A_\vee$. Therefore, $a_{\pi_\vee, A_\vee}(u) \geq \beta > \alpha$ and $\{x \mid a_{\pi,A}(x) > \alpha\}$ is open. Thus $a_{\pi_\vee, A_\vee}$ is lower semicontinuous.

Now consider any $\alpha \in \mathbb{R}$ and $x \in L$ such that $a_{\pi_\vee, A_\vee}(x) < \alpha$. There exists $y \in \tilde{R}$ such that $\pi_\vee(y) < \alpha$ and $y - x \in A_\vee$. Again, there exists $y_1 \in \tilde{R}$ such that $y_1 \geq 1$. Let $\delta := (\alpha - \pi_\vee(y))/(2\max\{1, \pi_\vee(y_1)\})$. For any $u$ in the $\delta$-ball at $x$, $u \leq x + \delta$, so $\delta y_1 + y - u \geq y - x + \delta(y_1 - 1) \geq y - x$, and hence, by monotonicity, $\delta y_1 + y - u \in A_\vee$. By subadditivity, $\pi_\vee(\delta y_1 + y) \leq \delta \pi_\vee(y_1) + \pi_\vee(y) \leq (\alpha - \pi_\vee(y))/2 + \pi_\vee(y) < \alpha$. Therefore $a_{\pi_\vee, A_\vee}(u) < \alpha$, and $\{x \mid a_{\pi,A}(x) < \alpha\}$ is open. Thus $a_{\pi_\vee, A_\vee}$ is upper semicontinuous. □

We now draw connections with the theory of exact functionals, for which see Maaß (2002). An absolutely homogeneous, superadditive, real-valued functional is called *superlinear*. In particular, it is concave. A monotone, superlinear functional is called *supermodular*. A constant additive, supermodular functional is called *exact*. Constant additivity of a functional $\Gamma$ is $\Gamma(x + c) = \Gamma(x) + \Gamma(c)$ when $c \in \mathbb{R}$.

The theory of exact functionals centers on operators similar to the convexity and absolute homogeneity operators defined in Section 5, but with the opposite conventions, of concavity. For instance, one considers the least monotone functional dominating $\Gamma$. To get a supermodular functional, one applies successively the operators for positive homogeneity, superadditivity, and monotonicity. Because the application of each operator does not spoil the properties of the previous, the resulting functional given by

$$\begin{aligned}\Gamma_{\wedge}(x) &:= (\leq \text{ sa ah } \Gamma)(x)\\ &= \sup\left\{\sum_{i=1}^{n}\lambda_i\Gamma(x_i)\;\middle|\;\sum_{i=1}^{n}\lambda_i x_i \leq x, n\in\mathbb{N}, \forall i\lambda_i \geq 0, x_i \in \operatorname{dom}\Gamma\right\}\end{aligned}$$

is indeed monotone, superadditive, and absolutely homogeneous. Therefore it is supermodular as long as it is real-valued. Define the "norm" $|\Gamma| := \Gamma_{\wedge}(1)$. It is nonnegative by monotonicity and homogeneity, which implies $\Gamma_{\wedge}(0) = 0$. It is a pseudonorm on a linear space of exactifiable functionals on which it is finite-valued (Maaß 2002, Prop. 2). When $|\Gamma|$ is finite, $\Gamma_{\wedge}$ is real-valued and may be called the *natural supermodularification* of $\Gamma$. When $|\Gamma| < \infty$, we may call $\Gamma$ *supermodularifiable*.

The *natural exactification* of an exactifiable functional $\Gamma$ is $\Gamma_{\bullet} := (\leq \text{ ca sa ah } \Gamma)$ given by

$$\Gamma_{\bullet}(x) = \sup\left\{\sum_{i=1}^{n}\lambda_i\Gamma(x_i) + c\,|\Gamma|\;\middle|\;\sum_{i=1}^{n}\lambda_i x_i + c \leq x, c\in\mathbb{R}, n\in\mathbb{N},\; \forall\, i\lambda_i \geq 0, x_i \in \operatorname{dom}\Gamma\right\}.$$

When $|\Gamma| < \infty$, $\Gamma_{\bullet}$ is exact; in particular, it is real-valued (Maaß 2002, Thm. 2), and $\Gamma$ is called *exactifiable*; this is the same thing as supermodularifiability. Thus our concern with $A_{\vee}$, the cone generated by the acceptance set $A$, and with $\pi_{\vee}$, the greatest sublinear function dominated by the market pricing function $\pi$, appears entirely analogous to the process of exactification.

The natural exactification of an exactifiable functional $\Gamma$ is the least exact functional extending $\Gamma$ and having the same norm as $\Gamma$ (Maaß 2002, Prop. 4). When $\Gamma$ is exact, $\Gamma_{\bullet}$ is called its *natural extension*, and $|\Gamma|$ coincides with the norm $\|\Gamma\|$ ordinarily given to linear operators. This paper analyzes unnatural extensions $a_{\pi,A}$ and $b_{\pi,A}$ of market prices.

Very similar mathematical objects have been studied under different names. An exact functional with unit norm is a *coherent lower prevision*, and an exactifiable functional with unit norm is a *lower prevision avoiding sure loss* (Walley 1991). When $\Gamma$ is an exact functional, $-\Gamma$ is a *coherent risk measure* (Artzner et al. 1999). This makes $-\Gamma$ absolutely homogeneous, subadditive, constant additive, and antimonotone. A *convex risk measure* is a convex, constant additive, antimonotone functional (Föllmer and Schied 2002a).

## 8. CONCLUSIONS AND DIRECTIONS

This paper provides fundamental theorems of asset pricing for good deal bounds. The intention is to enable the use of good deal bounds to establish bid-ask spreads for over-the-counter derivatives in incomplete markets. When no-arbitrage bounds are too wide to use as bid-ask spreads, one may use narrower good deal bounds based on an acceptance set of good deals that incorporates one's beliefs and preferences. At the same time, one wants the bid and ask prices to avoid arbitrage given prevailing market prices. For good deal bounds, this requires some consistency between the acceptance set and market prices. Theorems 6.1 and 6.2 express this consistency in terms of the existence of a consistent pricing kernel, or one that is strictly monotone.

These results supply conditions for choosing an acceptance set: it is safe to use an acceptance set that admits the existence of a strictly monotone consistent pricing kernel. It remains to describe concrete schemes for producing acceptance sets from beliefs and preferences in a way that is consistent with the goals of traders in over-the-counter derivatives. One such attempt appears in Part II of Staum (2002), taking as a point of departure the work of Föllmer and Schied (2002b) on robust preferences and risk measures.

The present paper is a generalization of the results of previous authors on good deal bounds, especially Jaschke and Küchler (2000) and Carr et al. (2001). Although the setting is rather general here, some significant limitations remain.

First, the fundamental theorems' necessary conditions involve some convexity hypotheses. Given the nature of convex optimization, this seems difficult to avoid. However, it might be worthwhile to consider the possibility of risk-seeking behavior (nonconvex acceptance sets) on the part of agents, such as derivatives traders, who participate economically in their trading gains to a greater extent than in their losses (see Example 3.3). An understanding of the incentives faced by such agents, and the objectionable behavior that may result, could lead to improved risk management. It would also be desirable to know whether and to what extent market prices are nonconvex, and what impact that has.

Second, the use of the market price function $\pi$ obscures potentially important and challenging aspects of market modeling. A key issue not addressed in the present paper is the relationship between admissible, self-financing, continuous-time trading strategies and the market price of attainable contingent claims. This gives the paper a one-period flavor that is typical of recent research on coherent and convex risk measures, but does not do justice to the richness of continuous-time finance, particularly not to the fundamental theorems of asset pricing described by Delbaen and Schachermayer (1999). To demonstrate the value of good deal bounds requires some examples in which they produces practical bid-ask spreads for interesting nonmarketed claims in realistic models of incomplete markets with continuous-time trading.

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