Excess Invariance and Shortfall Risk Measures

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Abstract

This paper introduces an axiom of excess invariance for risk measures, meaning insensitivity to the amount by which a portfolio's value exceeds a benchmark. The paper also introduces the class of shortfall risk measures, which are excess-invariant as well as normalized, non-negative, and monotone non-increasing. Shortfall risk measures are suitable for regulatory or risk management applications in which risk is associated with shortfall beneath a benchmark, whereas excess above the benchmark is not important. Non-negativity is incompatible with the usual construction of a cash-additive risk measure from an acceptance set. It is shown how to construct shortfall risk measures with the property of cash additivity subject to positivity, thus retaining the interpretation, valuable in capital adequacy applications, that the risk of an unacceptable portfolio is measured as the amount of cash that must be added to make it acceptable. Representation theorems are provided that characterize shortfall risk measures that are convex and positively homogeneous.

Key words: risk measures, shortfall, excess invariant, non-negative, cash additive, cash invariant, translation invariant, Minkowski functional, support function

1 Introduction

What axioms should a financial risk measure satisfy? The answer depends on the financial application and on what one means by "risk."

Artzner et al. (1999) approach the question from the standpoint of supervision, considering the set of acceptable portfolios as the primary object, and making the risk measure a description of how far from acceptable the portfolio is. They generate a risk measure from an acceptance set by making the risk of a portfolio equal to the amount of cash that must be added to the portfolio (and "invested prudently") to make it acceptable. This is especially appropriate for capital adequacy applications. The opposite of such a risk measure can be interpreted as a measure of acceptability or desirability.

Pflug (2006) and Rockafellar et al. (2006) approach the question from the standpoint of portfolio selection. A basic idea in portfolio selection is to examine the trade-off between risk and reward. Often, reward is measured as mean return, and risk is measured in terms of uncertainty about the difference between the return and its mean, i.e., variability of the returns.

This paper studies a new class of risk measure, which measures how bad the portfolio's downside is.[1] The downside is the portfolio's shortfall compared to a benchmark. For simplicity, let us assume that the benchmark is zero, so that negative portfolio values constitute shortfall and positive portfolio values constitute excess. Risk measures that satisfy the axiom of excess invariance, defined in Section 3, measure only how bad the shortfall is, while ignoring the contribution of excess to desirability and uncertainty. The class of shortfall risk measures is defined in Section 3 as those risk measures that are excess-invariant as well as normalized, non-negative, and monotone non-increasing. Section 2 contains motivations, both application-driven and theoretical, for considering excess invariance and shortfall risk measures.

Shortfall risk measures can be constructed from familiar risk measures by enforcing the requisite properties. For example, enforcing excess invariance on worst conditional expectation (or, similarly, conditional value at risk or expected shortfall—see Examples 2.1 and 3.1) results in a shortfall risk measure, worst conditional mean shortfall, with similar behavior. However, non-negativity and excess-invariance are incompatible (Propositions 2.1 and 3.2) with the cash-additivity axiom which supports an interpretation of the risk measure as a measure of capital adequacy. Section 4 shows how to construct shortfall risk measures with a new property called "cash-additivity subject to positivity" so as to retain the interpretation of the risk measure as the amount of cash that must be added to a portfolio to make it acceptable. The acceptance set associated with an excess-invariant risk measure has a different interpretation from that typically associated with cash-additive risk measures: the meaning of "acceptable" associated with excess invariance is "tolerable," not "desirable."

Convex, positively homogeneous shortfall risk measures are characterized in Section 5 through representation theorems that aid in understanding and constructing such risk measures, and in using them in applications such as optimization and risk attribution.

For simplicity and concreteness, we consider finite-valued risk measures on the space of bounded random variables. That is, a risk measure is a functional $\rho : L^\infty \to \mathbb{R}$, where $L^\infty = L^\infty(\Omega, \mathcal{F}, \boldsymbol{P})$ is the space of $\boldsymbol{P}$-equivalence classes of bounded $\mathcal{F}$-measurable functions on Ω.

Cont et al. (2011) also study risk measures that are excess-invariant, which they call "loss-dependent." Their class of loss-based risk measures has a somewhat different set of axioms than the class of shortfall risk measures. Most importantly, they replace the cash-additivity axiom with a "cash loss" axiom stipulating that the risk measure of a deterministic loss equals the magnitude of that loss.

2 Motivations

Consider the application of regulatory capital adequacy. Suppose that the portfolio value is the net value of a firm, e.g., capital plus paid-in premia minus claims, for an insurer. Zero is the boundary of default, excess belongs to the firm's owners, and shortfall is borne by the firm's creditors. Excess invariance is appropriate if the regulator is concerned only about the losses that a firm imposes on its creditors and not about the wealth of its owners.

Consider the application of portfolio risk management. Surely the portfolio's owner cares about excess, but it may also be beneficial to have a risk measure that is influenced neither negatively nor positively by the portfolio's excess. If the portfolio's excess makes the portfolio very desirable, a coherent risk measure (Artzner et al., 1999) may say the portfolio has negative risk even though it also has a considerable shortfall: see Example 2.1. Although this answer may be interesting

[1] Only the axiomatic study of this class is new. Risk measures that consider only losses have been used for centuries. Artzner et al. (1999, Remark 4.5) mentioned one such risk measure that satisfies all their axioms but one.

from the standpoint of portfolio selection or for some other purpose, it is not the information a risk manager needs. On the other hand, consider an arbitrage opportunity with zero shortfall but with excess that is uncertain. A pure risk functional (Pflug, 2006) would say that the arbitrage opportunity has positive risk. Although this answer may be interesting to a portfolio manager who needs to choose among arbitrage opportunities whose payoffs have different means and variabilities around their means, it is not a suitable answer from the perspective of risk management, which regards an arbitrage opportunity as something with zero risk.

Example 2.1 (worst conditional expectation). Let τ_p be *worst conditional expectation* at level $p \in (0,1)$, a coherent risk measure (Artzner et al., 1999). The worst conditional expectation of $\mathbf{X}$ is $\tau_p(\mathbf{X}) = \sup_{\boldsymbol{Q}\in\mathcal{Q}} \mathrm{E}_{\boldsymbol{Q}}[-\mathbf{X}]$ where $\mathcal{Q} = \{\boldsymbol{P}(\cdot|E) : E \in \mathcal{F}, \boldsymbol{P}(E) > p\}$. Consider the particular example with $p = 5\%$ and $\mathbf{X}$ having three possible outcomes: -30 with probability 1%, 10 with probability 7%, and 20 with probability 92%. The worst conditional expectation of $\mathbf{X}$ is $\tau_p(\mathbf{X}) = -(-30 \times 12.5\% + 10 \times 87.5\% + 20 \times 0\%) = -5$. The negative value of the risk measure has nothing to do with the discreteness of the probability space and the consequent difference between worst conditional expectation and expected shortfall or conditional value at risk (Acerbi and Tasche, 2002). The latter risk measures equal $-(-30 \times 20\% + 10 \times 80\% + 20 \times 0\%) = -2$ in this example.

From a theoretical perspective, this discussion suggests that it would be interesting to study risk measures that have two properties sometimes deemed to be part of the meaning of risk: non-negativity and monotonicity. Non-negativity relates to the interpretation of risk as an object of aversion (to risk-averse decision-makers), e.g., any random outcome entails a non-negative quantity of variability, shortfall, or fear. Pure risk functionals and deviation measures (Rockafellar et al., 2006) are non-negative. Monotonicity (more precisely, the property of being monotone non-increasing) relates to the economic principle that more is better: more future wealth, i.e., more excess and less shortfall, implies less risk. Coherent risk measures and convex measures of risk (Föllmer and Schied, 2002a) have this monotonicity property. Proposition 2.1 shows that risk measures like pure risk functionals and deviation measures are incompatible with monotonicity, whereas risk measures like coherent risk measures and convex measures of risk are incompatible with non-negativity.

Pure risk functionals and deviation measures satisfy a *translation invariance* axiom (Pflug, 2006): $\rho(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X})$ for all $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}$. Coherent risk measures and convex measures of risk satisfy an axiom that has several names and some variants related to the choice of numéraire (see El Karoui and Ravanelli, 2009, and references therein) or to the choice of what Artzner et al. (1999) call the "reference instrument" that constitutes a prudent investment. Let us pass over this issue and assume that cash added to the initial portfolio yields an equal value in all future scenarios, e.g., is invested in a riskless bond with zero interest. We adopt the formulation of El Karoui and Ravanelli (2009) and say that the axiom of *cash additivity* is $\rho(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X}) - \alpha$ for all $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}$. Axioms that are similar or the same have been called "translation invariance" by Artzner et al. (1999) and "cash invariance" by Föllmer and Schied (2004). The same thought has been expressed by saying that $-\rho$ is constant additive (Maaß, 2002) or translation equivariant (Pflug, 2006). Cash additivity and translation invariance are each incompatible with non-negativity or monotonicity.

Proposition 2.1. *There is no risk measure that is both non-negative and cash-additive. If a risk measure is monotone non-increasing and translation-invariant, then it is constant.*

Proof. If ρ is cash-additive, then $\rho(\mathbf{X} + (\rho(\mathbf{X}) + 1)\mathbf{1}) = \rho(\mathbf{X}) - (\rho(\mathbf{X}) + 1) < 0$ for any $\mathbf{X} \in L^\infty$.

Suppose ρ is monotone non-increasing and translation-invariant. For any $\mathbf{X}, \mathbf{X}' \in L^\infty$, $\rho(\mathbf{X}' - \|\mathbf{X}' - \mathbf{X}\|_\infty \mathbf{1}) = \rho(\mathbf{X}')$ by translation invariance. Because $\mathbf{X}' - \|\mathbf{X}' - \mathbf{X}\|_\infty \mathbf{1} \leq \mathbf{X}$, monotonicity implies $\rho(\mathbf{X}' - \|\mathbf{X}' - \mathbf{X}\|_\infty \mathbf{1}) \geq \rho(\mathbf{X})$. Therefore $\rho(\mathbf{X}') \geq \rho(\mathbf{X})$. By reversing the roles of $\mathbf{X}$ and $\mathbf{X}'$, $\rho(\mathbf{X}) \geq \rho(\mathbf{X}')$. Therefore ρ is constant. $\square$

Section 4 shows how non-negativity can be reconciled with the idea of a risk measure as the distance to acceptability, specified as the amount of cash that must be added to a portfolio to make it acceptable. Non-negative risk measures can have this structure, subject to the qualification that they assign zero risk to acceptable portfolios. Such risk measures retain the essential feature, useful in capital adequacy applications, that they measure the risk of unacceptable portfolios as the amount of cash that must be added.

3 Basic Theory

Let $\mathbf{X} \in L^\infty$ represent the future performance of a portfolio compared to a benchmark, its positive part $\mathbf{X}^+$ be the *excess*, and its negative part $\mathbf{X}^-$ be the *shortfall.* In various applications, $\mathbf{X}$ may be the portfolio's return minus the return of a benchmark index, the portfolio's terminal value minus its initial value, or the net value of a firm's assets minus its liabilities. In these cases, respectively, the excess and shortfall are outperformance and underperformance, profit and loss, or future equity value and loss borne by creditors. Excess invariance means that the risk measure involves only the shortfall, not the excess.

Definition 3.1. *Excess invariance* of a risk measure ρ means $\rho(\mathbf{X}) = \rho(\mathbf{X}')$ for all $\mathbf{X}, \mathbf{X}' \in L^\infty$ such that $\mathbf{X}^- = (\mathbf{X}')^-$.

Excess invariance is equivalent to $\rho(\mathbf{X}) = \rho(-\mathbf{X}^-)$ for all $\mathbf{X} \in L^\infty$. Excess invariance places no restriction on the behavior of the risk measure on the negative orthant L^∞_-, but it determines the risk measure's values on all other orthants in terms of its restriction to the negative orthant. Therefore an excess-invariant risk measure can be constructed from another risk measure using only its restriction to the negative orthant. This is a recipe for constructing excess-invariant risk measures with similar behavior to well-known risk measures that are not excess-invariant.

Definition 3.2. For any $\rho : L^\infty \to \mathbb{R}$, its *excess-invariant counterpart* $\tilde{\rho}$ is given by $\tilde{\rho}(\mathbf{X}) = \rho(-\mathbf{X}^-)$ for all $\mathbf{X} \in L^\infty$.

As explained in the introduction, it is of interest to study risk measures that are non-negative and *monotone non-increasing*, meaning $\rho(\mathbf{X}) \geq \rho(\mathbf{X}')$ for all $\mathbf{X}, \mathbf{X}' \in L^\infty$ such that $\mathbf{X} \leq \mathbf{X}'$. Proposition 3.1 shows that non-negativity and monotonicity fit well with excess invariance if the risk measure is also *normalized*, meaning $\rho(\mathbf{0}) = 0$. These four axioms are suitable for a risk measure that measures how bad shortfall is: only shortfall matters, not excess (excess invariance), and shortfall is purely bad (non-negativity); zero shortfall implies no risk (normalization), and more wealth means less shortfall and thus less risk (monotonicity).

Definition 3.3. A *shortfall risk measure* is a risk measure that is normalized, non-negative, monotone non-increasing, and excess-invariant.

Proposition 3.1. *If $\rho : L^\infty \to \mathbb{R}$ is normalized, monotone non-increasing on L^∞_-, and excess-invariant, then it is a shortfall risk measure.*

Proof. Let ρ be a risk measure with the given properties. For any $\mathbf{X} \in L^\infty$, $\mathbf{0} \geq -\mathbf{X}^- \in L^\infty_-$, so $\rho(\mathbf{X}) = \rho(-\mathbf{X}^-) \geq \rho(\mathbf{0}) = 0$. For any $\mathbf{X}' \geq \mathbf{X}$, $-(\mathbf{X}')^- \geq -\mathbf{X}^-$, so $\rho(\mathbf{X}') = \rho(-(\mathbf{X}')^-) \leq \rho(-\mathbf{X}^-) = \rho(\mathbf{X})$. $\square$

Corollary 3.1. *If $\rho : L^\infty \to \mathbb{R}$ is normalized and monotone non-increasing on L^∞_-, then its excess-invariant counterpart is a shortfall risk measure.*

Example 3.1 (worst conditional mean shortfall). In Example 2.1, τ_p is the worst conditional expectation, given by $\tau_p(\mathbf{X}) = \sup_{\boldsymbol{Q}\in\mathcal{Q}} \mathrm{E}_{\boldsymbol{Q}}[-\mathbf{X}]$ where $\mathcal{Q} = \{\boldsymbol{P}(\cdot|E) : E \in \mathcal{F}, \boldsymbol{P}(E) > p\}$. The excess-invariant counterpart $\tilde{\tau}_p$ of τ_p is the *worst conditional mean shortfall*, given by $\tilde{\tau}_p(\mathbf{X}) = \sup_{\boldsymbol{Q}\in\mathcal{Q}} \mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^-]$. It is a shortfall risk measure. Unlike worst conditional expectation, worst conditional mean shortfall assigns positive risk to any $\mathbf{X}$ such that $\mathbf{X} < 0$ with positive probability, and assigns zero risk to any $\mathbf{X}$ such that $\mathbf{X} \geq 0$ with probability one. In the particular example with $p = 5\%$ and three outcomes, -30 with probability 1%, 10 with probability 7%, and 20 with probability 92%, the worst conditional mean shortfall is $\tilde{\tau}_p(\mathbf{X}) = 30\times 12.5\% + 0\times 87.5\% + 0\times 0\% = 3.75$.

Example 3.2 (robust mean shortfall). Let $\mathcal{Q}$ be a set of probability measures, β be a real number, and $\rho_{\mathcal{Q},\beta}$ be the convex measure of risk given by $\rho_{\mathcal{Q},\beta}(\mathbf{X}) = -\beta - \inf\{\mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}] : \boldsymbol{Q} \in \mathcal{Q}\}$ (Föllmer and Schied, 2002a). The excess-invariant counterpart of $\rho_{\mathcal{Q},\beta}$ is given by $\tilde{\rho}_{\mathcal{Q},\beta}(\mathbf{X}) = -\beta + \sup\{\mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^-] : \boldsymbol{Q} \in \mathcal{Q}\}$. For any β, $\tilde{\rho}_{\mathcal{Q},\beta}$ is monotone non-increasing and excess-invariant, but it is not normalized if $\beta \neq 0$ and it fails to be non-negative if $\beta > 0$. Taking $\beta = 0$, $\tilde{\rho}_{\mathcal{Q},0}$ is a shortfall risk measure, which may be called *robust mean shortfall* (cf. Föllmer and Schied, 2002b). If $\mathcal{Q}$ is a singleton, then the risk measure is simply *mean shortfall*.

Example 3.3 (shortfall risk). Let $-\rho$ be expected utility, i.e., $\rho(\mathbf{X}) = \mathrm{E}_{\boldsymbol{P}}[-u(\mathbf{X})]$, where $u : \mathbb{R} \to \mathbb{R}$ is a monotone non-decreasing utility function such that $u(0) = 0$. (Here $u(\mathbf{X})$ represents the composition $u \circ \mathbf{X} : L^\infty \to \mathbb{R}$.) The excess-invariant counterpart of ρ is the shortfall risk measure $\tilde{\rho}$ given by $\tilde{\rho}(\mathbf{X}) = \mathrm{E}_{\boldsymbol{P}}[-u(-\mathbf{X}^-)] = \mathrm{E}_{\boldsymbol{P}}[-\tilde{u}(\mathbf{X})]$ where $\tilde{u}(x) = 0 \wedge u(x)$. It is known as *shortfall risk* (Föllmer and Schied, 2002a).

Excess invariance can be considered as a substitute for translation invariance or cash additivity. It does not specify how the risk measure responds to translation by a constant, i.e., give a relationship between $\rho(\mathbf{X} + \alpha\mathbf{1})$ and $\rho(\mathbf{X})$. However, each of the three axioms is incompatible with the others. Rockafellar et al. (2006, p. 61) observed that the axioms of cash additivity and translation invariance are incompatible. The following proposition shows that excess invariance is incompatible with both of them.

Proposition 3.2. *There is no risk measure that is both excess-invariant and cash-additive. If a risk measure is excess-invariant and translation-invariant then it is constant.*

Proof. Consider any $\mathbf{X} \in L^\infty_+$. If ρ is excess-invariant, then $\rho(\mathbf{X} + \mathbf{1}) = \rho(\mathbf{X}) = \rho(\mathbf{0})$. If ρ is cash-additive, then $\rho(\mathbf{X} + \mathbf{1}) = \rho(\mathbf{X}) - 1 \neq \rho(\mathbf{X})$.

For any $\mathbf{X} \in L^\infty$, $\mathbf{X} + \|\mathbf{X}\|_\infty\mathbf{1} \in L^\infty_+$. If ρ is excess-invariant and translation-invariant, then $\rho(\mathbf{X}) = \rho(\mathbf{X} + \|\mathbf{X}\|_\infty\mathbf{1}) = \rho(-(\mathbf{X} + \|\mathbf{X}\|_\infty\mathbf{1})^-) = \rho(\mathbf{0})$. □

Excess invariance is compatible with the *cash subadditivity* axiom of El Karoui and Ravanelli (2009): $\rho(\mathbf{X} + \alpha\mathbf{1}) \geq \rho(\mathbf{X}) - \alpha$ for all $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}_+$.

Example 3.4. Robust mean shortfall, given by $\tilde{\rho}_{\mathcal{Q},0}(\mathbf{X}) = \sup\{\mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^-] : \boldsymbol{Q} \in \mathcal{Q}\}$, is cash-subadditive. For all $\boldsymbol{Q} \in \mathcal{Q}$, $\mathbf{X} \in L^\infty$, and $\alpha \in \mathbb{R}_+$,

$$\mathrm{E}_{\boldsymbol{Q}}[(\mathbf{X} + \alpha\mathbf{1})^-] = \mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^- - (\alpha\mathbf{1} \wedge \mathbf{X}^-)] \geq \mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^- - \alpha\mathbf{1}] = \mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^-] - \alpha.$$

4 Capital Adequacy

In capital adequacy applications, it is useful to have risk measures that can be interpreted in terms of the amount of cash that must be added to a portfolio for it to become acceptable, as holds for cash-additive risk measures such as convex measures of risk. Section 4.1 discusses the differences between acceptance sets associated with coherent risk measures and with excess-invariant risk measures, including a difference in the interpretation of what it means for a portfolio to be "acceptable." Section 4.2 shows how to construct shortfall risk measures that have a property that gives them an interpretation in terms of capital adequacy, although the property is weaker than cash additivity.

4.1 Acceptance Sets

A useful perspective arises from considering convex measures of risk as arising from an acceptance set $\mathcal{A} \subseteq L^\infty$ via

$$\rho_{\mathcal{A}}(\mathbf{X}) = \inf\{\alpha \in \mathbb{R} : \mathbf{X} + \alpha\mathbf{1} \in \mathcal{A}\} \tag{4.1}$$

(Föllmer and Schied, 2002a). In the literature on convex measures of risk, it is typical to work with acceptance sets that satisfy the following condition, which originates with Artzner et al. (1999).

Definition 4.1. A subset $\mathcal{X} \subseteq L^\infty$ is *monotone increasing* if $\mathbf{X} \in \mathcal{X}$ and $\mathbf{X}' \geq \mathbf{X}$ imply $\mathbf{X}' \in \mathcal{X}$; $\mathcal{X}$ is *monotone decreasing* if $\mathbf{X} \in \mathcal{X}$ and $\mathbf{X}' \leq \mathbf{X}$ imply $\mathbf{X}' \in \mathcal{X}$.

Condition 4.1. The acceptance set $\mathcal{A}$ is monotone increasing, $L^\infty_+ \subseteq \mathcal{A}$, and $L^\infty_- \cap \mathcal{A} = \{\mathbf{0}\}$.

The axiom $L^\infty_+ \subseteq \mathcal{A}$ means that any portfolio with no shortfall is acceptable, whereas $L^\infty_- \cap \mathcal{A} = \{\mathbf{0}\}$ means that any portfolio that has no excess but does have some shortfall is unacceptable. Acceptance sets that say which portfolios are desirable to their owners should satisfy Condition 4.1. Condition 4.1 implies that the risk measure $\rho_{\mathcal{A}}$ is cash-additive, monotone non-increasing, and normalized.

As an alternative to Condition 4.1, consider a property of excess invariance for acceptance sets. The following proposition shows that the definitions of excess invariance for risk measures and acceptance sets are compatible.

Definition 4.2. *Excess invariance* of an acceptance set $\mathcal{A}$ means $\mathbf{X} \in \mathcal{A} \Leftrightarrow \mathbf{X}' \in \mathcal{A}$ for all $\mathbf{X}, \mathbf{X}' \in L^\infty$ such that $\mathbf{X}^- = (\mathbf{X}')^-$.

Proposition 4.1. *The functional $\rho : L^\infty \to \mathbb{R}$ is excess-invariant as a risk measure if and only if the level set $\{\mathbf{X} : \rho(\mathbf{X}) \leq \beta\}$ is excess-invariant as an acceptance set for all $\beta \in \mathbb{R}$.*

Proof. If ρ is excess-invariant and $\mathbf{X}^- = (\mathbf{X}')^-$, then $\rho(\mathbf{X}) = \rho(\mathbf{X}')$. Therefore $\{\mathbf{X} : \rho(\mathbf{X}) \leq \beta\}$ contains $\mathbf{X}$ and $\mathbf{X}'$ for $\beta \geq \rho(\mathbf{X})$ and contains neither of them for $\beta < \rho(\mathbf{X})$.

If $\mathcal{A}(\beta) = \{\mathbf{X} : \rho(\mathbf{X}) \leq \beta\}$ is excess-invariant as an acceptance set for all $\beta \in \mathbb{R}$ and $\mathbf{X}^- = (\mathbf{X}')^-$, then $\rho(\mathbf{X}) = \inf\{\beta \in \mathbb{R} : \mathbf{X} \in \mathcal{A}(\beta)\} = \inf\{\beta \in \mathbb{R} : \mathbf{X}' \in \mathcal{A}(\beta)\} = \rho(\mathbf{X}')$. □

Condition 4.1 does not fit well with excess invariance, as the following proposition shows. These two conditions on acceptance sets are appropriate to different interpretations of "acceptable," namely "desirable" or "tolerable." Excess invariance is a suitable property for a risk measure or acceptance set in applications in which one wishes to describe which portfolios have a tolerable amount of risk. Of course, shortfall is undesirable in the absence of excess; however, shortfall need not be intolerable in the absence of excess. An excess-invariant acceptance set is appropriate when the amount of shortfall that is tolerable, from a regulatory or supervisory standpoint, does not depend on how much excess there is.

Proposition 4.2. *The only acceptance set satisfying Condition 4.1 and excess invariance is L^∞_+.*

Proof. Let $\mathcal{A}$ be an excess-invariant acceptance set that satisfies Condition 4.1, so $L^\infty_+ \subseteq \mathcal{A}$. For any $\mathbf{X} \notin L^\infty_+$, $-\mathbf{X}^-$ is in L^∞_- and is not equal to $\mathbf{0}$. Therefore $\mathbf{X} \notin \mathcal{A}$. □

It is typical to construct an acceptance set from a cash-additive risk measure ρ via $\mathcal{A} = \{\mathbf{X} \in L^\infty : \rho(X) \leq 0\}$ (Föllmer and Schied, 2002a), where zero risk serves as the boundary between desirable and undesirable portfolios. In constructing an acceptance set $\mathcal{A}(\beta) = \{\mathbf{X} \in L^\infty : \rho(X) \leq \beta\}$ from a shortfall risk measure ρ, it may well be necessary to choose the threshold β to be positive, because it would be too conservative to say that it is unacceptable to have any risk.

Example 4.1. Suppose that a bank regulator wishes to limit a bank's default probability to $\beta > 0$. Let $\mathbf{X}$ represent the net value of the bank. The regulator's acceptance set is $\mathcal{A}(\beta) = \{\mathbf{X} \in L^\infty : \boldsymbol{P}(\mathbf{X} < 0) \leq \beta\}$. It is excess invariant, but it is not convex. Suppose that a deposit insurer wishes to limit a bank's mean shortfall (Example 3.2) to $\beta' > 0$. The deposit insurer's acceptance set $\mathcal{A}'(\beta') = \{\mathbf{X} \in L^\infty : \mathrm{E}_{\boldsymbol{P}}[\mathbf{X}^-] \leq \beta'\}$ is excess invariant and convex.

4.2 Cash-Additivity Subject to Positivity

Non-negativity is compatible with cash subadditivity (Example 3.4), although not cash additivity (Proposition 2.1). It is possible to construct non-negative risk measures, and indeed shortfall risk measures, that have an essential property of the Artzner et al. (1999) construction of a cash-additive risk measure in Equation (4.1). Suppose the infimum in Equation (4.1) is attained. If it is non-negative, it is interpreted as the minimum amount of cash that must be added to the portfolio to make it acceptable, as specified by the acceptance set $\mathcal{A}$. The risk measure $\rho^+_{\mathcal{A}}$ given by

$$\rho^+_{\mathcal{A}}(\mathbf{X}) = 0 \vee \rho_{\mathcal{A}}(\mathbf{X}) = \inf\{\alpha > 0 : \mathbf{X} + \alpha\mathbf{1} \in \mathcal{A}\} \tag{4.2}$$

is non-negative and has the essential property for capital adequacy applications: $\rho^+_{\mathcal{A}}(\mathbf{X})$ is the answer to the question "How much cash must we add to $\mathbf{X}$ to get an acceptable portfolio?" The answer is non-negative because we do not contemplate the withdrawal of cash. What is lost by using $\rho^+_{\mathcal{A}}$ is the less essential property that $-\rho_{\mathcal{A}}(\mathbf{X})$ is the maximum amount of cash that can be withdrawn from the portfolio while keeping it acceptable, if $\rho_{\mathcal{A}}(\mathbf{X})$ is negative.

Suppose that the acceptance set $\mathcal{A}$ is monotone. Then $\rho_{\mathcal{A}}(\mathbf{X})$ measures how far $\mathbf{X}$ is from the boundary of $\mathcal{A}$ while traveling along the line $\{\mathbf{X} + \alpha\mathbf{1} : \alpha \in \mathbb{R}\}$, whereas $\rho^+_{\mathcal{A}}(\mathbf{X})$ measures how far $\mathbf{X}$ is from $\mathcal{A}$ along that line. The distance from $\mathbf{X}$ to $\mathcal{A}$ is zero if $\mathbf{X}$ is acceptable, i.e., $\mathbf{X} \in \mathcal{A}$.

The preceding discussion leads the way to a weakening of the axiom of cash additivity. The point of the condition $\rho(\mathbf{X}) > (0 \vee \alpha)$ in Definition 4.3 is to ensure that both $\rho(\mathbf{X})$ and $\rho(\mathbf{X}) - \alpha$ are positive.

Definition 4.3. A risk measure ρ is *cash-additive subject to positivity* if $\rho(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X}) - \alpha$ for all $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}$ such that $\rho(\mathbf{X}) > (0 \vee \alpha)$.

Proposition 4.3. *If a risk measure ρ is cash-additive, then $\rho^+ = 0 \vee \rho$ is cash-additive subject to positivity and it is cash-subadditive.*

Proof. Consider any $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}$ such that $\rho^+(\mathbf{X}) > (0 \vee \alpha)$. Then $\rho^+(\mathbf{X}) = \rho(\mathbf{X})$. By cash additivity, $\rho(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X}) - \alpha$. This equals $\rho^+(\mathbf{X}) - \alpha$, which is positive. Therefore $\rho^+(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X} + \alpha\mathbf{1}) = \rho(\mathbf{X}) - \alpha = \rho^+(\mathbf{X}) - \alpha$. This establishes cash additivity subject to positivity.

For any $\mathbf{X} \in L^\infty$ and $\alpha \in \mathbb{R}_+$,

$$\rho^+(\mathbf{X} + \alpha\mathbf{1}) = 0 \vee \rho(\mathbf{X} + \alpha\mathbf{1}) = 0 \vee (\rho(\mathbf{X}) - \alpha) \geq (0 \vee \rho(\mathbf{X})) - \alpha = \rho^+(\mathbf{X}) - \alpha,$$

which establishes cash subadditivity. □

The following proposition provides a procedure for constructing a risk measure with desired properties.

Proposition 4.4. *If $\mathcal{A}$ is a monotone increasing acceptance set that contains $\mathbf{0}$, then $\rho^+_{\mathcal{A}}$ defined in Equation (4.2) is non-negative, monotone non-increasing, cash-additive subject to positivity, and cash-subadditive. If $\mathcal{A}$ is also excess-invariant, convex, or positively homogeneous, then $\rho^+_{\mathcal{A}}$ has the same property.*

Proof. If $\mathcal{A}$ is a monotone increasing acceptance set that contains $\mathbf{0}$, then it contains L^∞_+. Therefore $\rho_{\mathcal{A}}$ defined in Equation (4.1) is real-valued, monotone non-increasing, and cash-additive. It follows immediately that $\rho^+_{\mathcal{A}}$ is real-valued, non-negative, and monotone non-increasing. By Proposition 4.3, it is cash-additive subject to positivity and cash-subadditive.

Suppose $\mathcal{A}$ is also excess-invariant. Consider any $\mathbf{X}, \mathbf{X}' \in L^\infty$ such that $\mathbf{X}^- = (\mathbf{X}')^-$. For any $\alpha \in \mathbb{R}_+$, $(\mathbf{X} + \alpha\mathbf{1})^- = (\mathbf{X}' + \alpha\mathbf{1})^-$, so $\mathcal{A}$ contains $(\mathbf{X} + \alpha\mathbf{1})^-$ if and only if it contains $(\mathbf{X}' + \alpha\mathbf{1})^-$. Therefore $\rho^+_{\mathcal{A}}(\mathbf{X}) = \rho^+_{\mathcal{A}}(\mathbf{X}')$, so $\rho^+_{\mathcal{A}}$ is excess-invariant.

Suppose instead that $\mathcal{A}$ is also convex. Then $\rho_{\mathcal{A}}$ is convex, so $\rho^+_{\mathcal{A}}$ is convex.

The same reasoning shows that $\rho^+_{\mathcal{A}}$ is positively homogeneous if $\mathcal{A}$ is. □

Example 4.2 (worst conditional mean shortfall and capital adequacy). Worst conditional mean shortfall (Example 3.1) is not cash-additive subject to positivity. In the particular example with $p = 5\%$ and three outcomes, -30 with probability 1%, 10 with probability 7%, and 20 with probability 92%, the worst conditional mean shortfall is $\tilde{\tau}_p(\mathbf{X}) = 3.75$, and $\tilde{\tau}_p(\mathbf{X} - \mathbf{1}) = 3.875$. To get a measure of capital adequacy based on worst conditional mean shortfall, choose $\beta \geq 0$ and let $\mathcal{A}(\beta) = \{\mathbf{X} \subseteq L^\infty : \tilde{\tau}_p(\mathbf{X}) \leq \beta\}$ be the set of portfolios with tolerable worst conditional mean shortfall. Because worst conditional mean shortfall is a shortfall risk measure and convex, $\mathcal{A}(\beta)$ contains $\mathbf{0}$ and is monotone, excess-invariant, and convex. Then $\rho^+_{\mathcal{A}(\beta)}$ defined via Equation (4.2) is a shortfall risk measure, convex, and cash-additive subject to positivity. It measures the amount of capital that must be added to a portfolio to reduce its worst conditional mean shortfall to a tolerable level. Continuing the particular example and taking $\beta = 1$, $\rho^+_{\mathcal{A}(1)}(\mathbf{X}) = 22$. Whereas $\tilde{\tau}_p(\mathbf{X}) = 3.75$ is a measurement of how bad the shortfall is, $\rho^+_{\mathcal{A}(1)}(\mathbf{X}) = 22$ is a measurement of how expensive it would be to reduce the shortfall to the tolerable level of 1.

5 Characterization

The purpose of this section is to establish theorems that characterize shortfall risk measures that are convex and positively homogeneous. These include worst conditional mean shortfall (Example 3.1) and robust mean shortfall (Example 3.2). The theorems characterize such risk measures as support functions (Theorem 5.2) and as Minkowski functionals (Theorem 5.3). The representation as a support function is a subdifferential representation (Pflug, 2006). Applications of subdifferential representations include optimization (e.g., Ruszczyński and Shapiro, 2006) and risk attribution (e.g., Cherny and Orlov, 2011). The representation as a Minkowski functional can be considered as an alternative to the representation (4.1) of a convex risk measure as the amount of cash to add to a portfolio to make it barely acceptable (i.e., desirable). It represents the risk measure as the factor by which a portfolio must be shrunk for it to become barely acceptable (i.e., tolerable). It can be used to represent a convex, positively homogeneous shortfall risk measure in terms of a set of tolerable portfolios whose risk is not too large.

As a preliminary, Section 5.1 reviews the structure of functionals on L^∞ that are real-valued, monotone non-decreasing, convex, and positively homogeneous, and those that are also non-negative.

Section 5.2 establishes propositions about the relationship between a risk measure and its excess-invariant counterpart in terms of their representations. They are ingredients for the representation theorems in Section 5.3.

5.1 Preliminaries without Excess Invariance

The following theorem summarizes what is already known about subdifferential representations of real-valued, monotone non-decreasing, convex, positively homogeneous functionals on L^∞. The mathematical framework, and indeed the first half of the proof of Theorem 5.1, are drawn from Ruszczyński and Shapiro (2006, §§2–3.1). However, cash-additivity is omitted, and only the special case of the space $L^\infty = L^\infty(\Omega, \mathcal{F}, \boldsymbol{P})$ is treated. It is paired with its dual space $\mathrm{ba} = \mathrm{ba}(\Omega, \mathcal{F}, \boldsymbol{P})$ of bounded, finitely additive, signed measures defined on $\mathcal{F}$ and absolutely continuous with respect to $\boldsymbol{P}$. The pairing is made by the bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle \mathbf{X}, \mu \rangle = \int \mathbf{X} \, \mathrm{d}\mu$ for any $\mathbf{X} \in L^\infty$ and $\mu \in \mathrm{ba}$. The Banach space L^∞ has the norm $\| \cdot \|_\infty$, and this induces a norm on ba defined by $\|\mu\| = \sup\{|\langle \mathbf{X}, \mu \rangle| : \|\mathbf{X}\|_\infty \leq 1\}$.

Definition 5.1. The *directional derivative* of a functional ψ on L^∞ at $\mathbf{X}$ in the direction $\mathbf{X}'$ is $\psi'(\mathbf{X}; \mathbf{X}') = \lim_{\lambda \downarrow 0} (\psi(\mathbf{X} + \lambda \mathbf{X}') - \psi(\mathbf{X}))/\lambda$.

Lemma 5.1. *If a functional $\psi : L^\infty \to \mathbb{R}$ is positively homogeneous, then for any $\mathbf{X} \in L^\infty$, $\psi(\mathbf{X}) = \psi'(\mathbf{0}; \mathbf{X})$.*

Definition 5.2. The set of *subgradients* of a functional ψ on L^∞ at $\mathbf{X}$ is $\partial\psi(\mathbf{X}) = \{\mu \in \mathrm{ba} : \psi(\mathbf{X}') \geq \psi(\mathbf{X}) + \langle \mathbf{X}' - \mathbf{X}, \mu \rangle, \ \forall \mathbf{X}' \in L^\infty\}$.

Lemma 5.2. *If a functional ψ on L^∞ is convex and $\psi(\mathbf{X}) \in \mathbb{R}$, then $\partial\psi(\mathbf{X}) = \{\mu \in \mathrm{ba} : \langle \mathbf{X}', \mu \rangle \leq \psi'(\mathbf{X}; \mathbf{X}') \ \forall \mathbf{X}' \in L^\infty\}$.*

Proof. This is shown by Rockafellar (1974, p. 33). □

Definition 5.3. The *support function* $\delta^*_{\mathcal{M}}$ of a set $\mathcal{M} \subset \mathrm{ba}$ is given by $\delta^*_{\mathcal{M}}(\mathbf{X}) = \sup\{\langle \mathbf{X}, \mu \rangle : \mu \in \mathcal{M}\}$ for any $\mathbf{X} \in L^\infty$.

Theorem 5.1. *Any monotone non-decreasing, convex, positively homogeneous functional $\psi : L^\infty \to \mathbb{R}$ is the support function of the set $\partial\psi(\mathbf{0})$ of subgradients of ψ at $\mathbf{0}$, and $\partial\psi(\mathbf{0})$ is a non-empty, weakly compact, convex subset of* ba_+. *The support function of any non-empty, weakly compact, convex set $\mathcal{M} \subset \mathrm{ba}_+$ is a monotone non-decreasing, convex, positively homogeneous, real-valued functional on L^∞.*

Proof. Suppose $\psi : L^\infty \to \mathbb{R}$ is monotone non-decreasing, convex, and positively homogeneous. Then ψ is strongly continuous (Ruszczyński and Shapiro, 2006, Prop. 3.1). Theorem 11 of Rockafellar (1974) implies that $\partial\psi(\mathbf{0})$ is non-empty and weakly compact (as well as convex) and that for all $\mathbf{X} \in L^\infty$, the directional derivative $\psi'(\mathbf{0}; \mathbf{X})$ equals $\delta^*_{\partial\psi(\mathbf{0})}(\mathbf{X})$. By Lemma 5.1, $\psi(\mathbf{X}) = \psi'(\mathbf{0}; \mathbf{X})$. Monotonicity implies that $\mathcal{M} \subseteq \mathrm{ba}_+$ (Ruszczyński and Shapiro, 2006, Thm. 2.2).

Suppose that $\mathcal{M} \subseteq \mathrm{ba}_+$ is non-empty, weakly compact, and convex. By Corollary 7 and Remark 8 of Frittelli and Rosazza Gianin (2002), its support function is monotone non-decreasing, convex, and positively homogeneous. Weak compactness implies strong boundedness (Dunford and Schwartz, 1958, V.4.3). For any $\mathbf{X} \in L^\infty \setminus \{\mathbf{0}\}$,

$$
\begin{aligned}
\delta^*_{\mathcal{M}}(\mathbf{X}) = \|\mathbf{X}\|_\infty \delta^*_{\mathcal{M}}(\mathbf{X}/\|\mathbf{X}\|_\infty) &= \|\mathbf{X}\|_\infty \sup\{\langle \mathbf{X}/\|\mathbf{X}\|_\infty, \mu \rangle : \mu \in \mathcal{M}\} \\
&\leq \|\mathbf{X}\|_\infty \sup\{\langle \mathbf{X}', \mu \rangle : \|\mathbf{X}'\|_\infty \leq 1, \mu \in \mathcal{M}\} \\
&= \|\mathbf{X}\|_\infty \sup\{\|\mu\| : \mu \in \mathcal{M}\} < \infty
\end{aligned}
$$

by strong boundedness of $\mathcal{M}$. This shows that the support function takes on only real values, as is mentioned by Rockafellar (1974, p. 31). $\square$

The concept of weak compactness of a set $\mathcal{M} \subset \mathrm{ba}$ can be elucidated by its connection to the property that there exists $\mu^0 \in \mathrm{ba}_+$ such that for any sequence of events $\{E_n\}_{n\in\mathbb{N}}$, if $\mu^0(E_n) \to 0$, then $\mu(E_n) \to 0$ uniformly over all $\mu \in \mathcal{M}$ (Dunford and Schwartz, 1958, IV.9.12, V.6.1).

Next we summarize what is known about non-negative functionals that have the properties above, with regard to their representation as Minkowski functionals of subsets of L^∞.

Definition 5.4. The *Minkowski functional* $\gamma_{\mathcal{X}}$ of a subset $\mathcal{X} \subseteq L^\infty$ is given by $\gamma_{\mathcal{X}}(\mathbf{X}) = \inf\{\alpha > 0 : \mathbf{X} \in \alpha\mathcal{X}\}$ for any $\mathbf{X} \in L^\infty$.

Definition 5.5. A subset $\mathcal{X} \subseteq L^\infty$ is *absorbing* if, for all $\mathbf{X} \in L^\infty$, there exists $\alpha > 0$ such that $\mathbf{X} \in \alpha\mathcal{X}$.

Proposition 5.1. *If $\psi : L^\infty \to \mathbb{R}$ is monotone non-decreasing, convex, positively homogeneous, and non-negative, then it is the Minkowski functional of the level set $\mathcal{X} = \{\mathbf{X} \in L^\infty : \psi(\mathbf{X}) \leq 1\}$, and $\mathcal{X}$ is absorbing, convex, and monotone decreasing. If $\mathcal{X} \subseteq L^\infty$ is absorbing, convex, and monotone decreasing, then its Minkowski functional $\gamma_{\mathcal{X}}$ is real-valued, monotone non-decreasing, convex, positively homogeneous, and non-negative.*

Proof. Suppose that ψ has the specified properties. For any $\mathbf{X} \in L^\infty$, $\gamma_{\mathcal{X}}(\mathbf{X}) = \inf\{\alpha > 0 : \mathbf{X}/\alpha \in \mathcal{X}\} = \inf\{\alpha > 0 : \psi(\mathbf{X}/\alpha) \leq 1\} = \inf\{\alpha > 0 : \psi(\mathbf{X}) \leq \alpha\} = \psi(\mathbf{X})$. Because ψ is convex, its level set $\mathcal{X}$ is convex. Because $\gamma_{\mathcal{X}} = \psi$ is real-valued, $\mathcal{X}$ is absorbing. Consider $\mathbf{X} \in \mathcal{X}$ and $\mathbf{X}' \leq \mathbf{X}$. By monotonicity of ψ, $\psi(\mathbf{X}') \leq \psi(\mathbf{X}) \leq 1$, so $\mathbf{X}' \in \mathcal{X}$.

Suppose instead that $\mathcal{X}$ has the specified properties. Then its Minkowski functional $\gamma_{\mathcal{X}}$ is real-valued, convex, and positively homogeneous (Wouk, 1979, Lem. 13.1.1). Clearly it is non-negative. Consider $\alpha > 0$, $\mathbf{X} \in \alpha\mathcal{X}$, and $\mathbf{X}' \leq \mathbf{X}$. Because $\alpha\mathcal{X}$ is monotone decreasing, $\mathbf{X}' \in \alpha\mathcal{X}$. This implies $\gamma_{\mathcal{X}}(\mathbf{X}') \leq \gamma_{\mathcal{X}}(\mathbf{X})$. $\square$

5.2 Excess-Invariant Counterpart

This section establishes the relationship between a risk measure ρ and its excess-invariant counterpart in terms of the subdifferential representation and the representation as a Minkowski functional. Example 5.1 illustrates the relationship. Define $\ell : L^\infty \to L^\infty_+$ and $n : L^\infty \to L^\infty$ by $\ell(\mathbf{X}) = \mathbf{X}^-$ and $n(\mathbf{X}) = -\mathbf{X}$ for all $\mathbf{X} \in L^\infty$. Define $\psi : L^\infty \to \mathbb{R}$ by $\psi = \rho \circ n$. Then the excess-invariant counterpart $\tilde{\rho}$ of ρ is given by $\tilde{\rho} = \rho \circ n \circ \ell = \psi \circ \ell$.[2]

Proposition 5.2. *If $\psi : L^\infty \to \mathbb{R}$ is monotone non-decreasing, convex, and positively homogeneous, then $\tilde{\rho} = \psi \circ \ell$ is the support function of $-\{\mu \in \mathrm{ba}_+ : \exists\mu' \in \partial\psi(\mathbf{0}) \ni \mu' \geq \mu\}$.*

Proof. Let $\mathcal{M} = \{\mu \in \mathrm{ba}_+ : \exists\mu' \in \partial\psi(\mathbf{0}) \ni \mu \leq \mu'\}$. For any $\mathbf{X} \in L^\infty$ and $\mu \in \mathrm{ba}$, let $\mu_{\mathbf{X}} \in \mathrm{ba}$ be given by $\mu_{\mathbf{X}}(E) = \mu(E \cap \{\mathbf{X} > 0\})$ for all $E \in \mathcal{F}$. If $\mu \in \mathrm{ba}_+$, then $\mu_{\mathbf{X}} \leq \mu$. For all $\mathbf{X} \in L^\infty$, the support function $\delta^*_{\mathcal{M}}$ satisfies

$$\begin{aligned}\delta^*_{\mathcal{M}}(\mathbf{X}) &= \sup\{\langle \mathbf{X}, \mu\rangle : \mu \in \mathcal{M}\} = \sup\{\langle \mathbf{X}, \mu'_X\rangle : \mu' \in \partial\psi(\mathbf{0})\} = \sup\{\langle \mathbf{X}^+, \mu'\rangle : \mu' \in \partial\psi(\mathbf{0})\} \\ &= \psi(\mathbf{X}^+),\end{aligned}$$

because Theorem 5.1 implies that ψ is the support function of $\partial\psi(\mathbf{0})$. This shows that the support function of $\mathcal{M}$ equals $\psi \circ \ell \circ n$, which is given by $(\psi \circ \ell \circ n)(\mathbf{X}) = \psi(\mathbf{X}^+)$ for all $\mathbf{X} \in L^\infty$. Therefore $\tilde{\rho} = \psi \circ \ell$ is the support function of $-\mathcal{M}$. $\square$

[2] An anonymous referee suggested relying on this construction to simplify the proofs.

Proposition 5.3. *If $\psi : L^\infty \to \mathbb{R}$ is monotone non-decreasing, convex, and positively homogeneous, and $\mathcal{X} = \{\mathbf{X} \in L^\infty : \psi(\mathbf{X}) \le 1\}$, then $\tilde{\rho} = \psi \circ \ell$ is the Minkowski functional of $-\{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \ge \mathbf{X}\}$.*

Proof. It is equivalent to show that $\psi \circ \ell \circ n$ is the Minkowski functional of $\tilde{\mathcal{X}} = \{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \ge \mathbf{X}\}$. The functional $\psi \circ \ell \circ n$ is real-valued, monotone non-decreasing, convex, and positively homogeneous. It is non-negative because $(\psi \circ \ell \circ n)(\mathbf{X}) = \psi(\mathbf{X}^+)$ for all $\mathbf{X} \in \mathcal{X}$, ψ is monotone non-decreasing, and $\psi(\mathbf{0}) = 0$. By Proposition 5.1, $\psi \circ \ell \circ n$ is the Minkowski functional of the level set $\mathcal{X}_1 = \{\mathbf{X} \in L^\infty : (\psi \circ \ell \circ n)(\mathbf{X}) \le 1\} = \{\mathbf{X} \in L^\infty : \psi(\mathbf{X}^+) \le 1\}$. Clearly $\mathcal{X}_1 \subseteq \tilde{\mathcal{X}}$. For any $\mathbf{X} \in \tilde{\mathcal{X}}$, there exists $\mathbf{X}' \in \mathcal{X} \cap L^\infty_+$ such that $\mathbf{X}' \ge \mathbf{X}$. Because $\mathbf{X}' \ge \mathbf{0}$, $\mathbf{X}^+ \le \mathbf{X}'$. By monotonicity of ψ, $\psi(\mathbf{X}^+) \le \psi(\mathbf{X}') \le 1$, so $\mathbf{X} \in \mathcal{X}_1$. This establishes $\mathcal{X}_1 = \tilde{\mathcal{X}}$. □

Example 5.1. In Example 3.2, the robust mean shortfall $\tilde{\rho}_{\mathcal{Q},0}$, given by $\tilde{\rho}_{\mathcal{Q},0}(\mathbf{X}) = \sup\{\mathrm{E}_{\boldsymbol{Q}}[\mathbf{X}^-] : \boldsymbol{Q} \in \mathcal{Q}\}$, is a convex, positively homogeneous shortfall risk measure. It is the excess-invariant counterpart of the coherent risk measure $\rho_{\mathcal{Q},0}$, given by $\rho_{\mathcal{Q},0}(\mathbf{X}) = \sup\{\mathrm{E}_{\boldsymbol{Q}}[-\mathbf{X}] : \boldsymbol{Q} \in \mathcal{Q}\}$. In the right panel of Figure 1, $\mathcal{Q} = \{(1/3, 2/3), (2/3, 1/3)\}$, and $\rho = \rho_{\mathcal{Q},0}$ is the support function of $-\mathcal{M}$, where $\mathcal{M}$ is the convex hull of $\mathcal{Q}$, the blue line segment connecting $(1/3, 2/3)$ and $(2/3, 1/3)$. It is a subset of the dotted line segment $\{\mu \in \mathbb{R}^2_+ : Y_1 + Y_2 = 1\}$ of probability mass vectors. The set $\mathcal{X} = \{\mathbf{X} : \rho_{\mathcal{Q},0}(-\mathbf{X}) \le 1\}$ appears in the left panel of Figure 1 as the union of the red and blue sets. Turning to the shortfall risk measure, $\tilde{\rho}_{\mathcal{Q},0}(\mathbf{X}) = \sup\{\int \mathbf{X}\, \mathrm{d}\mu : \mu \in -\tilde{\mathcal{M}}\}$ where $\tilde{\mathcal{M}} = \{\mu \in \mathrm{ba}_+ : \exists \mu' \in \partial\psi(\mathbf{0}) \ni \mu' \ge \mu\}$ is the red set in the right panel of Figure 1, including the blue line segment $\mathcal{M}$ as part of its boundary. The red set in the left panel of Figure 1 is $\tilde{\mathcal{X}} = \{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \ge \mathbf{X}\} = \{\mathbf{X} : \tilde{\rho}_{\mathcal{Q},0}(-\mathbf{X}) \le 1\}$, and $\tilde{\rho}_{\mathcal{Q},0}$ is the Minkowski functional of $-\tilde{\mathcal{X}}$.

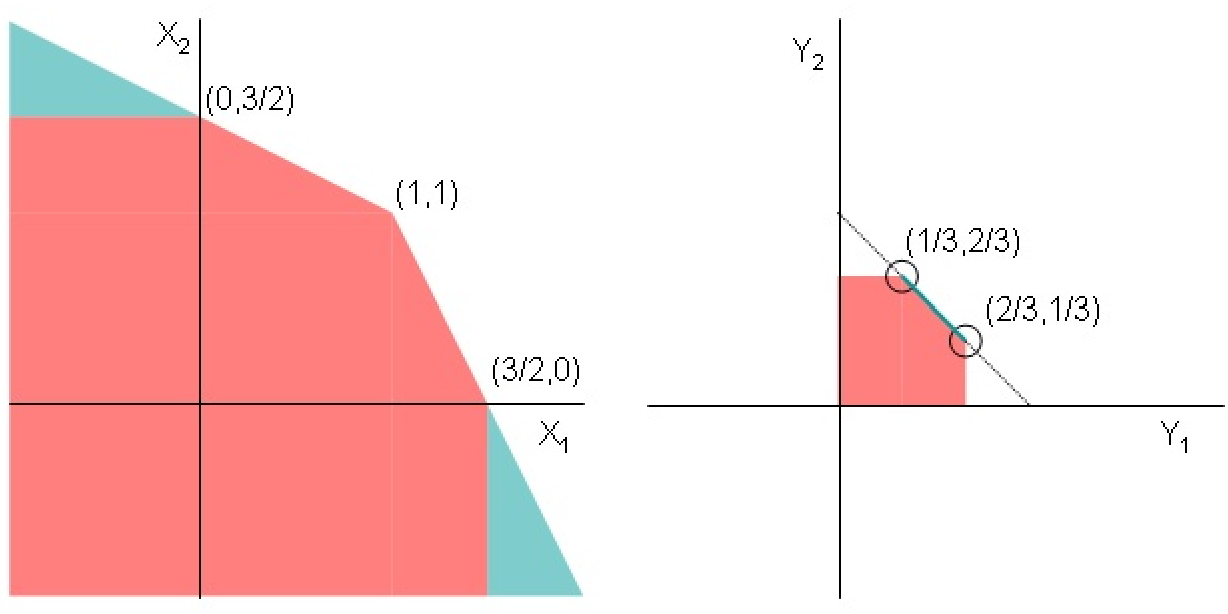

Figure 1: Robust mean shortfall arises from the Minkowski functional of the red set in the left panel and from the support function of the red set in the right panel. See Example 5.1.

5.3 Representation Theorems

An excess-invariant risk measure is its own excess-invariant counterpart. Therefore, in the case of convex, positively homogeneous risk measures, excess-invariance is equivalent to invariance of the set $\mathcal{M}$ in the support function representation (Definition 5.3) and of the set $\mathcal{X}$ in the Minkowski functional representation (Definition 5.4) under the operations applied in Section 5.2.

Theorem 5.2. *A functional $\rho : L^\infty \to \mathbb{R}$ is a convex, positively homogeneous shortfall risk measure if and only if $\psi = \rho \circ n$ is the support function of a non-empty, weakly compact, convex set $\mathcal{M} \subseteq \mathrm{ba}_+$ such that $\mathcal{M} = \{\mu \in \mathrm{ba}_+ : \exists \mu' \in \mathcal{M} \ni \mu \leq \mu'\}$.*

Proof. Suppose ψ is the support function of a non-empty, weakly compact, convex set $\mathcal{M} \subseteq \mathrm{ba}_+$ such that $\mathcal{M} = \{\mu \in \mathrm{ba}_+ : \exists \mu' \in \mathcal{M} \ni \mu \leq \mu'\}$. By Theorem 5.1, ψ is monotone non-decreasing, convex, positively homogeneous, and real-valued, so $\rho = \psi \circ n$ is monotone non-increasing, convex, positively homogeneous, and real-valued. It follows from Proposition 5.2 that ρ equals its own excess-invariant counterpart, and therefore is excess-invariant. By Proposition 3.1, ρ is a shortfall risk measure.

Suppose instead that a functional $\rho : L^\infty \to \mathbb{R}$ is a convex, positively homogeneous shortfall risk measure. Then $\rho = \psi \circ n$ equals its excess-invariant counterpart $\tilde{\rho}$. Because $\psi = \rho \circ n$ is monotone non-decreasing, convex, positively homogeneous, and real-valued, by Theorem 5.1, ψ is the support function of the non-empty, weakly compact, convex set $\partial\psi(\mathbf{0}) \subseteq \mathrm{ba}_+$. By Proposition 5.2, $\tilde{\rho}$ is the support function of $\{\mu \in \mathrm{ba}_- : \exists \mu' \in \partial\psi(\mathbf{0}) \ni \mu' \geq -\mu\}$. Therefore $\psi = \rho \circ n = \tilde{\rho} \circ n$ is the support function of $\{\mu \in \mathrm{ba}_+ : \exists \mu' \in \partial\psi(\mathbf{0}) \ni \mu \leq \mu'\}$. □

Theorem 5.3. *A functional $\rho : L^\infty \to \mathbb{R}$ is a convex, positively homogeneous shortfall risk measure if and only if it is the Minkowski functional of $-\mathcal{X}$, where $\mathcal{X} \subseteq L^\infty$ is absorbing and convex, and satisfies $\mathcal{X} = \{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \geq \mathbf{X}\}$.*

Proof. Suppose that $\rho : L^\infty \to \mathbb{R}$ is a convex, positively homogeneous shortfall risk measure. Then $\psi = \rho \circ n$ is real-valued, monotone non-decreasing, convex, positively homogeneous, and non-negative. By Proposition 5.1, ψ is the Minkowski functional of its level set $\mathcal{X} = \{\mathbf{X} \in L^\infty : \psi(\mathbf{X}) \leq 1\} = \{\mathbf{X} \in L^\infty : \rho(-\mathbf{X}) \leq 1\}$, which is absorbing, convex, and monotone decreasing. Then ρ is the Minkowski functional of $-\mathcal{X}$. For any $\mathbf{X} \in \mathcal{X}$, using excess-invariance, $\rho(-\mathbf{X}^+) = \rho(-\mathbf{X}) \leq 1$, which implies $\mathbf{X}^+ \in \mathcal{X}$. Therefore, for any $\mathbf{X} \in \mathcal{X}$, we have $\mathbf{X} \leq \mathbf{X}^+$ and $\mathbf{X}^+ \in \mathcal{X} \cap L^\infty_+$. This shows that $\mathcal{X} \subseteq \{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \geq \mathbf{X}\}$. The monotone decreasing property of $\mathcal{X}$ shows that $\{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \geq \mathbf{X}\} \subseteq \mathcal{X}$.

Suppose instead that ρ is the Minkowski functional of $-\mathcal{X}$, where $\mathcal{X} \subseteq L^\infty$ is absorbing and convex, and satisfies $\mathcal{X} = \{\mathbf{X} \in L^\infty : \exists \mathbf{X}' \in \mathcal{X} \cap L^\infty_+ \ni \mathbf{X}' \geq \mathbf{X}\}$. Then $\mathcal{X}$ is monotone decreasing. By Proposition 5.1, $\psi = \rho \circ n$ is real-valued, monotone non-decreasing, convex, positively homogeneous, and non-negative; $\rho = \psi \circ n$ has the same properties except that it is monotone non-increasing. By Proposition 5.3, the excess-invariant counterpart $\tilde{\rho} = \psi \circ \ell$ of ρ is the Minkowski functional of $-\mathcal{X}$. Therefore $\rho = \tilde{\rho}$, so ρ is excess-invariant. □

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