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Uncapacitated Lot Sizing with Backlogging: The Convex Hull

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Abstract An explicit description of the convex hull of solutions to the uncapacitated lot-sizing problem with backlogging, in its natural space of production, setup, inventory and backlogging variables, has been an open question for many years. In this paper, we identify valid inequalities that subsume all previously known valid inequalities for this problem. We show that these inequalities are enough to describe the convex hull of solutions. We give polynomial separation algorithms for some special cases. Finally, we report a summary of computational experiments with our inequalities that illustrates their effectiveness.

Keywords Lot sizing · backlogging · convex hull · separation algorithms · computation

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1 Introduction

The *uncapacitated lot-sizing problem with backlogging* (ULSB) is to determine the production, inventory and backlog quantities in each period so that demand for a single product in each time period is met over a finite horizon and the sum of production, holding and backlogging costs over the horizon is minimized. It is assumed that production, inventory and backlog quantities have no upper bounds. There are polynomial-time algorithms for ULSB [4],[17],[18].

Pochet and Wolsey [9] provide the first polyhedral study of ULSB. The authors give extended formulations for ULSB. In addition, the authors give a class of inequalities for ULSB valid for the natural space of production, inventory, backlogging and setup variables. They give a separation heuristic for this class of inequalities. Later, Pochet and Wolsey [11] give another class of inequalities for ULSB and show that the proposed inequalities are enough to solve the problem as a linear program if there are no speculative motives for holding inventory or backlogging demand. In this paper, we give a class of inequalities for ULSB that subsumes previously known classes of inequalities. We show that adding the proposed inequalities to the natural formulation is enough to give the convex hull of solutions to ULSB. In addition, we give the first combinatorial exact separation algorithm for the special case of our inequalities that is equivalent to those proposed by Pochet and Wolsey [9].

For a finite planning horizon n, let the nonnegative demand d_t, variable production cost c_t, and fixed production (setup) cost f_t, variable inventory holding cost h_t, and variable backlogging cost g_t for time periods $t \in \{1, \ldots, n\}$ be given. Let variable y_t denote the production quantity in time period t, and variables s_t and r_t denote the inventory and backlog quantity at the end of period t, respectively. Also let x_t be the fixed-charge variable for production in period t. Throughout, we let $[i, j] := \{t \in \mathbb{Z} : i \leq t \leq j\}$, and let $\mathbb{R}_+$ and $\mathbb{Z}_+$ represent the nonnegative reals and integers, respectively. Finally, let $d_{t\ell} = \sum_{j=t}^{\ell} d_j$ for $t \in [1, \ell]$ and $d_{t\ell} = 0$ for $t > \ell$. (See Figure 1 for the fixed-charge network representation of ULSB with $n = 6$.) ULSB can be formulated as

$$Z^{BL} := \min \quad \sum_{t=1}^{n} (f_t x_t + c_t y_t + g_t r_t + h_t s_t)$$

$$s_{t-1} + y_t - r_{t-1} = d_t + s_t - r_t, \; t \in [1, n] \tag{1}$$

$$y_t \leq d_{1n} x_t, \qquad t \in [1, n] \tag{2}$$

$$r_0 = s_0 = r_n = s_n = 0, \tag{3}$$

$$y \in \mathbb{R}^n_+, \; s \in \mathbb{R}^{n+1}_+, \; r \in \mathbb{R}^{n+1}_+ \tag{4}$$

$$x \in \{0, 1\}^n. \tag{5}$$

We let $\mathcal{S}$ denote the convex hull of the feasible solutions to ULSB and $\mathcal{P}$ denote the set of feasible solutions to the linear programming relaxation of (1)−(5). Observe that, $\dim(\mathcal{S}) = 3n - 2$. In addition, if $g_t + h_t < 0$ for some $t \in [1, n - 1]$, then the problem is unbounded.

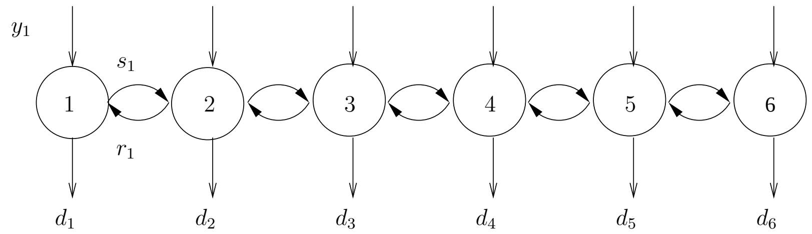

Fig. 1 Fixed-charge network for lot-sizing with backlogging.

Pochet and Wolsey [9] show that inequalities

$$\sum_{j\in S} y_j \le \sum_{j\in S} d_{(k(j,1)+1)k'(j,1)}x_j + \sum_{j\in L} r_j + \sum_{j\in R} s_j, \tag{6}$$

where $S \subseteq [1,n]$ and $L, R \subseteq [1, n-1]$ and $k(j,1) = \max\{t \in L : t < j\}$ (if $t \ge j$ for all $t \in L$, then let $k(j,1) = 0$) and $k'(j,1) = \min\{t \in R : t \ge j\}$ (if $t < j$ for all $t \in R$, then let $k'(j,1) = n$) are valid for (1)–(5). To see the validity of inequalities (6), let $\bar{y}_j$ be the portion of production in period j that is used to satisfy the demands in $[k(j,1)+1, k'(j,1)]$ and $\tilde{y}_j$ be the portion of production in period j that goes through $r_{k(j,1)}$ and $\hat{y}_j$ be the portion of production in period j that goes through $s_{k'(j,1)}$. Clearly, $y_j = \bar{y}_j + \tilde{y}_j + \hat{y}_j$. Furthermore, $\bar{y}_j \le d_{(k(j,1)+1)k'(j,1)}x_j$, $r_t \ge \sum_{j\in S:k(j,1)=t} \tilde{y}_j$ and $s_t \ge \sum_{j\in S:k'(j,1)=t} \hat{y}_j$. Thus,

$$\begin{aligned}\sum_{j\in S} y_j &= \sum_{j\in S}(\bar{y}_j + \tilde{y}_j + \hat{y}_j) \\ &\le \sum_{j\in S} d_{(k(j,1)+1)k'(j,1)}x_j + \sum_{t\in L}\sum_{j\in S:k(j,1)=t} \tilde{y}_j + \sum_{t\in R}\sum_{j\in S:k'(j,1)=t} \hat{y}_j,\end{aligned}$$

which implies inequality (6). The authors show that inequalities (6) are not enough to describe $\mathcal{S}$.

Example 1 Inequality (6) with $S = \{3,4,5\}, L = \{2\}, R = \{4,5\}$ given by

$$y_3 + y_4 + y_5 \le d_{34}x_3 + d_{34}x_4 + d_{35}x_5 + r_2 + s_4 + s_5, \tag{7}$$

is valid and facet-defining for $\mathcal{S}$. Note that $k(3,1) = k(4,1) = k(5,1) = 2$, $k'(3,1) = k'(4,1) = 4$ and $k'(5,1) = 5$. (See Figure 2.) However, the facet

$$y_3 + 2y_4 + y_5 \le d_{34}x_3 + (d_{34} + d_{25})x_4 + d_{35}x_5 + r_1 + r_2 + s_4 + s_5 \tag{8}$$

cannot be obtained from inequalities (6). □

Pochet and Wolsey [11] give another class of inequalities that is sufficient to solve ULSB as a linear program if the holding and backlogging costs satisfy the Wagner-Whitin property (i.e., when $h_t + p_t \ge p_{t+1}$ and $p_{t+1} + g_t \ge p_t$, for $t \in [1, n-1]$). However, these inequalities are not enough to describe $\mathcal{S}$ for general costs. We discuss the inequalities proposed in [11] in more

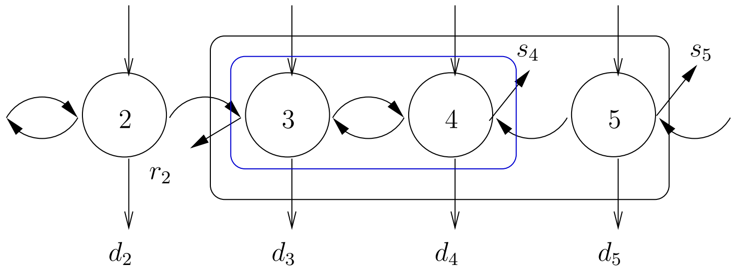

Fig. 2 Coefficients of x_j, $j \in S$ in inequality (7).

detail in Section 2. Agra and Constantino [1] extend these inequalities for ULSB with start-up costs in addition to the setup costs. Constantino [3] gives inequalities for constant capacity lot-sizing with backlogging and start-up costs in the natural space of production, setup, start-up, inventory and backlogging variables. Finally, van Vyve [13] gives extended formulations for the constant capacity lot-sizing problem with backlogging.

Pochet and Wolsey [10], Wolsey [16] and Guan et al. [5] demonstrate that a good understanding of the polyhedral structure of single item lot-sizing problems can be very useful in solving more complicated problems, involving multiple items and stages, and uncertain demand. Single item lot-sizing polyhedra have been of interest to researchers also because they are special cases of fixed-charge network flow problems. For uncapacitated fixed-charge network flows, van Roy and Wolsey [12] give network inequalities that are based on path substructures. Ortega and Wolsey [7] present a computational study on the performance of network inequalities in solving the uncapacitated fixed-charge network flow problem. The network inequalities have 0-1 coefficients for the continuous flow variables. In this paper, we give inequalities for ULSB that have general integer coefficients for the continuous variables. These valid inequalities for ULSB can be generalized to valid inequalities for path substructures in general fixed-charge network flow problems, thereby generalizing earlier work [7], [12].

Outline. In Section 2, we give valid inequalities for ULSB and show that they subsume all previously known inequalities. In Section 3 we explore the facility location reformulation given by Pochet and Wolsey [9] to derive a relationship between this extended formulation and the facets of ULSB in its natural space of production, setup, inventory and backlogging variables. We show that adding the proposed inequalities to the natural formulation is enough to give the convex of solutions to ULSB. In Section 4 we give a polynomial-time separation algorithm for a special case of the proposed inequalities. In Section 5 we summarize our computational experiments with the proposed inequalities. Finally, we conclude with Section 6.

2 Valid Inequalities for ULSB

To illustrate the inequalities proposed in this section, we first give an example.

Example 1 (cont.) Consider inequality (8). Let $L = [1,2], R = [4,5]$ and $S = [3,5]$. Recall the definitions of $\bar{y}_j, \tilde{y}_j, \hat{y}_j,\ j \in S$. Also let $\bar{y}_4^2$ be the portion of production in period 4 to satisfy demands in [2,5]; $\tilde{y}_4^2$ be the portion of production in period 4 that goes through r_1 (the backlog quantity in the second largest period in L before period 4); and $\hat{y}_4^2$ be the portion of production in period 4 that goes through s_5 (the inventory quantity in the second smallest period in R on or after period 4). Therefore, $y_4 = \bar{y}_4 + \tilde{y}_4 + \hat{y}_4 = \bar{y}_4^2 + \tilde{y}_4^2 + \hat{y}_4^2$. Observe that $\bar{y}_4 \leq d_{34}x_4$, $\bar{y}_4^2 \leq d_{25}x_4$, $\bar{y}_3 \leq d_{34}x_3$, $\bar{y}_5 \leq d_{35}x_5$, $r_2 \geq \tilde{y}_3 + \tilde{y}_4 + \tilde{y}_5$, $s_4 \geq \hat{y}_3 + \hat{y}_4$, $r_1 \geq \tilde{y}_4^2$ and $s_5 \geq \hat{y}_4^2 + \hat{y}_5$. (See Figure 3.) Therefore,

$$\begin{aligned} y_3 + 2y_4 + y_5 &= \sum_{j=3}^{5}(\bar{y}_j + \tilde{y}_j + \hat{y}_j) + \bar{y}_4^2 + \tilde{y}_4^2 + \hat{y}_4^2 \\ &\leq d_{34}x_3 + (d_{34} + d_{25})x_4 + d_{35}x_5 + r_1 + r_2 + s_4 + s_5, \end{aligned}$$

is valid for $\mathcal{S}$. Using similar arguments we can also show that the inequality

$$\begin{aligned} y_2 + 2y_3 + 3y_4 + y_5 + y_7 \leq & d_{25}x_2 + (d_{25} + d_{27})x_3 + d_{45}x_5 + d_{47}x_7 \\ & + (d_{45} + d_{27} + d_{28})x_4 \\ & + 2r_1 + r_3 + s_5 + s_7 + s_8, \end{aligned} \tag{9}$$

is valid for $\mathcal{S}$. Here, a coefficient 2 for r_1 (instead of 1) allows for a coefficient $(d_{25} + d_{27})$ for x_3 (instead of $(d_{25} + d_{17})$) and a coefficient $(d_{45} + d_{27} + d_{28})$ for x_4 (instead of $(d_{45} + d_{27} + d_{18})$).

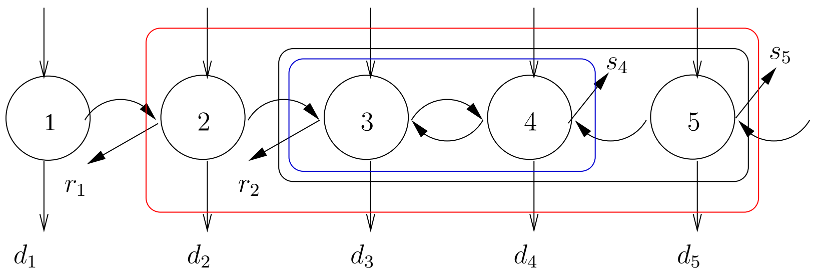

Fig. 3 Coefficients of x_j, $j \in S$ in inequality (8).

□

Theorem 1 *For $S \subseteq [1,n]$, $L, R \subseteq [0,n]$, the inequality*

$$\sum_{t\in S} u_t y_t \leq \sum_{t\in S}(\sum_{i=1}^{u_t} d_{(k(t,i)+1)k'(t,i)})x_t + \sum_{t\in L}\gamma_t r_t + \sum_{t\in R}\beta_t s_t, \tag{10}$$

is valid for $\mathcal{S}$, where

(i) $\gamma_t \in \mathbb{Z}_+$, $t \in L$, *and* $\beta_t \in \mathbb{Z}_+$, $t \in R$,
(ii) $u_t \in [1, q_t]$, $t \in S$ *with* $q_t = \min\{\sum_{i\in L: i<t}\gamma_i, \sum_{i\in R: i\geq t}\beta_i\}$,

(iii) $k(t,i) = \max\{k_i \in L \cap [0, t-1] : \sum_{j \in L \cap [k_i, t-1]} \gamma_j \geq i\}$, *$t \in S$ and $i \in [1, u_t]$,*

(iv) $k'(t,i) = \min\{k'_i \in R \cap [t, n] : \sum_{j \in R \cap [t, k'_i]} \beta_j \geq i\}$, *$t \in S$ and $i \in [1, u_t]$.*

Proof Let $\tilde{y}_{tp}$ be the production in period $t \in [1, n]$ to satisfy demand in period $p \in [0, n+1]$, where for ease of notation $d_0 = d_{n+1} = 0$. Then

$$\begin{aligned}
\sum_{t \in S} u_t y_t &= \sum_{t \in S} u_t \Big(\sum_{p \in [0, n+1]} \tilde{y}_{tp} \Big) \\
&= \sum_{t \in S} \sum_{i \in [1, u_t]} \Big(\sum_{p \in [0, k(t,i)]} \tilde{y}_{tp} + \sum_{p \in [k(t,i)+1, k'(t,i)]} \tilde{y}_{tp} + \sum_{p \in [k'(t,i)+1, n+1]} \tilde{y}_{tp} \Big) \\
&\leq \sum_{t \in S} \sum_{i \in [1, u_t]} d_{(k(t,i)+1)k'(t,i)} x_t + \sum_{t \in S} \sum_{i \in [1, u_t]} \sum_{p \in [0, k(t,i)]} \tilde{y}_{tp} \\
&\quad + \sum_{t \in S} \sum_{i \in [1, u_t]} \sum_{p \in [k'(t,i)+1, n+1]} \tilde{y}_{tp} \\
&\leq \sum_{t \in S} \sum_{i \in [1, u_t]} d_{(k(t,i)+1)k'(t,i)} x_t + \sum_{t \in L} \gamma_t r_t + \sum_{t \in R} \beta_t s_t,
\end{aligned}$$

where the second to last inequality follows because for $t \in S$ and $i \in [1, u_t]$, we have $\sum_{p \in [k(t,i)+1, k'(t,i)]} \tilde{y}_{tp} \leq d_{(k(t,i)+1)k'(t,i)} x_t$. The last inequality follows, because

$$\begin{aligned}
\gamma_t r_t \geq \gamma_t \sum_{j \in [t+1, n]} \sum_{p \in [0, t]} \tilde{y}_{jp} &\geq \sum_{j \in S \cap [t+1, n]} \gamma_t \left(\sum_{p \in [0, t]} \tilde{y}_{jp} \right) \\
&\geq \sum_{j \in S} \left(\sum_{i \in [1, u_j] : t = k(j,i)} 1 \right) \left(\sum_{p \in [0, t]} \tilde{y}_{jp} \right) \\
&= \sum_{j \in S} \left(\sum_{i \in [1, u_j] : t = k(j,i)} \left(\sum_{p \in [0, t]} \tilde{y}_{jp} \right) \right)
\end{aligned}$$

where the last inequality holds because $\gamma_t \geq |\{i \in [1, u_j] : t = k(j,i)\}|$ for $j > t$, and $|\{i \in [1, u_j] : t = k(j,i)\}| = 0$ for $j \leq t$. Similarly,

$$\begin{aligned}
\beta_t s_t \geq \beta_t \sum_{j \in [1, t]} \sum_{p \in [t+1, n+1]} \tilde{y}_{jp} &\geq \sum_{j \in S \cap [1, t]} \beta_t \left(\sum_{p \in [t+1, n+1]} \tilde{y}_{jp} \right) \\
&\geq \sum_{j \in S} \left(\sum_{i \in [1, u_j] : t = k'(j,i)} \left(\sum_{p \in [t+1, n+1]} \tilde{y}_{jp} \right) \right) .
\end{aligned}$$

where the last inequality holds because $\beta_t \geq |\{i \in [1, u_j] : t = k'(j,i)\}|$ for $j \leq t$, and $|\{i \in [1, u_j] : t = k'(j,i)\}| = 0$ for $j > t$.

Therefore,

$$\sum_{t\in L}\gamma_t r_t \geq \sum_{t\in L}\sum_{j\in S}\left(\sum_{i\in[1,u_j]:t=k(j,i)}\left(\sum_{p\in[0,t]}\tilde{y}_{jp}\right)\right)$$
$$=\sum_{j\in S}\sum_{i\in[1,u_j]}\sum_{p\in[0,k(j,i)]}\tilde{y}_{jp},\ \text{and,}$$
$$\sum_{t\in R}\beta_t s_t \geq \sum_{t\in R}\sum_{j\in S}\left(\sum_{i\in[1,u_j]:t=k'(j,i)}\left(\sum_{p\in[t+1,n+1]}\tilde{y}_{jp}\right)\right)$$
$$=\sum_{j\in S}\sum_{i\in[1,u_j]}\sum_{p\in[k'(j,i)+1,n+1]}\tilde{y}_{jp}.$$

where the above equalities hold because for each $j\in S$ and each $i\in[1,u_j]$, there exists exactly one $t\in L$ with $t=k(j,i)$, and one $t\in R$ with $t=k'(j,i)$. □

Remark 1 Note that inequalities (6) are special cases of inequalities (10) where $u_t=1$ for all $t\in S$, $\gamma_t=1$ for all $t\in L$ and $\beta_t=1$ for all $t\in R$. □

Pochet and Wolsey [11] propose a class of valid inequalities for ULSB, and prove that they suffice to solve ULSB as a linear program if there are no speculative motives for inventory holding or backlogging. We prove here that these inequalities are a special case of inequalities (10).

Proposition 1 *(Pochet and Wolsey [11]) The inequalities*

$$\sum_{\ell=\bar{k}_1+1}^{\bar{k}'_1}\sum_{i\in[1,u_\ell]}d_\ell(1-\sum_{t=\bar{k}(\ell,i)+1}^{\bar{k}'(\ell,i)}x_t)\leq\sum_{t\in L'}s_t+\sum_{t\in R'}r_t \qquad (11)$$

are valid for ULSB, where for an elementary directed cycle, C, on a complete digraph $D=(V,A)$ with $V=\{0,\ldots,n\}$:

(i) $\bar{k}_1<\bar{k}_2<\cdots<\bar{k}_p$ *are the tail nodes of the forward arcs (i,j) in C, $i<j$,*
(ii) $\bar{k}'_1>\bar{k}'_2>\cdots>\bar{k}'_b$ *are the tail nodes of backwards arcs (i,j) in C, $i>j$,*
(iii) $L'=\{\bar{k}_i : i\in[1,p]\}, R'=\{\bar{k}'_i : i\in[1,b]\}, L'\cap R'=\emptyset$,
(iv) for each node $\ell\in V$, u_ℓ is the cardinality of the cut across $(\ell-1,\ell)$, taking only the forward arcs into account ($u_{\bar{k}_1}=u_{\bar{k}'_1+1}=0$),
(v) $\bar{k}(\ell,i)$ *is the ith largest $\bar{k}_i$, $i\in[1,p]$ with $\bar{k}_i<\ell$ and $\bar{k}'(\ell,i)$ is the ith smallest $\bar{k}'_i$, $i\in[1,b]$ with $\bar{k}'_i\geq\ell$.*

Example 1 (cont.) See Figure 4 for an illustration of a subgraph of D with $\bar{k}_1=1$, $\bar{k}_2=2$, $\bar{k}_3=3$, $\bar{k}'_1=5$, $\bar{k}'_2=4$, and an elementary directed cycle given by the solid arcs for which $L' = [1, 3]$ and $R' = [4, 5]$. The corresponding inequality (11) is

$$\begin{aligned} s_1 + s_2 + s_3 + r_4 + r_5 \geq & d_2(1 - \sum_{t=2}^{4} x_t) + d_3(1 - \sum_{t=3}^{4} x_t) \\ & + d_3(1 - \sum_{t=2}^{5} x_t) + d_4(1 - x_4) \\ & + d_4(1 - \sum_{t=3}^{5} x_t) + d_5(1 - \sum_{t=4}^{5} x_t). \end{aligned} \tag{12}$$

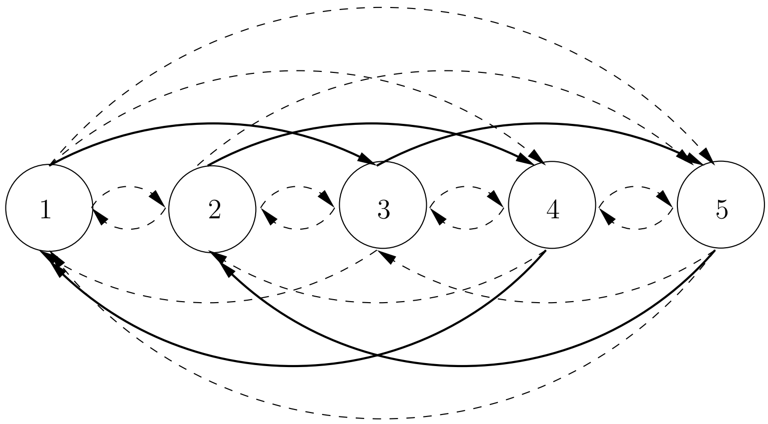

Fig. 4 Subgraph of D and the directed cycle that generates inequality (12).

Proposition 2 *Inequalities* (11) *are special cases of inequalities* (10) *with* $S = [\bar{k}_1 + 1, \bar{k}'_1]$, $u_t = q_t = \min\{|\{i \in L : i < t\}|, |\{i \in R : i \geq t\}|\}$ *for all* $t \in S$, $\gamma_t = 1$ *for all* $t \in L$ *and* $\beta_t = 1$ *for all* $t \in R$ *, and for some appropriate choice of* L *and* R *(given in the proof).*

Proof Let $U = \max_{\ell \in [\bar{k}_1+1, \bar{k}'_1]}\{u_\ell\}$ and $S_j = \{\ell \in [\bar{k}_1 + 1, \bar{k}'_1] : u_\ell \geq j\}$ for $j \in [1, U]$. Adding the aggregated flow balance equality

$$\sum_{j \in [1,U]} \sum_{\ell \in S_j} (s_{\ell-1} + y_\ell - r_{\ell-1}) = \sum_{j \in [1,U]} \sum_{\ell \in S_j} (s_\ell + d_\ell - r_\ell)$$

and inequality (11), we obtain

$$\sum_{j\in[1,U]}\sum_{\ell\in S_j} y_\ell \le \sum_{j\in[1,U]}\sum_{\ell\in S_j} d_\ell - \sum_{\ell=\bar{k}_1+1}^{\bar{k}'_1}\sum_{i\in[1,u_\ell]} d_\ell$$
$$+ \sum_{j\in L'} s_j + \sum_{j\in[1,U]}\sum_{\ell\in S_j}(s_\ell - s_{\ell-1})$$
$$+ \sum_{j\in R'} r_j + \sum_{j\in[1,U]}\sum_{\ell\in S_j}(r_{\ell-1} - r_\ell)$$
$$+ \sum_{\ell=\bar{k}_1+1}^{\bar{k}'_1} d_\ell \sum_{i\in[1,u_\ell]}\sum_{j\in[\bar{k}(\ell,i)+1,\bar{k}'(\ell,i)]} x_j.$$

Observe that for the elementary directed cycle, C, we must have $u_j - u_{j+1} \in \{-1,0,1\}$ for all $j \in [\bar{k}_1, \bar{k}'_1]$. Let $L^+ = R^+ = \{j \in [\bar{k}_1, \bar{k}'_1] : u_{j+1} - u_j = 1\}$ and $L^- = R^- = \{j \in [\bar{k}_1, \bar{k}'_1] : u_j - u_{j+1} = 1\}$. Note that $L^+ \subseteq L'$, $L^- \cap L' = \emptyset$, $R^- \subseteq R'$ and $R^+ \cap R' = \emptyset$. Cancelling common terms and rearranging, we get

$$\sum_{t\in[\bar{k}_1+1,\bar{k}'_1]} u_t y_t \le \sum_{t\in[\bar{k}_1+1,\bar{k}'_1]} (\sum_{i=1}^{u_t} d_{(k(t,i)+1)k'(t,i)}) x_t$$
$$+ \sum_{t\in(L'\setminus L^+)\cup L^-} s_t + \sum_{t\in(R'\setminus R^-)\cup R^+} r_t, \qquad (13)$$

where $k(t,i)$ and $k'(t,i)$ are as defined in Theorem 1, with $S = [\bar{k}_1+1, \bar{k}'_1]$, $L = (R' \setminus R^-) \cup R^+$ and $R = (L' \setminus L^+) \cup L^-$. We get an inequality of the form (10) in which $u_t = q_t$ for all $t \in S$, $\gamma_t = 1$ for all $t \in L$ and $\beta_t = 1$ for all $t \in R$. To see why $u_t = q_t$ for all $t \in S$, observe that the head nodes of the forward arcs in the directed cycle C give the set R and the head nodes of the backward arcs in C give the set L. Hence, the cardinality of the cut across $(t-1,t)$ is given by $u_t = q_t = \min\{|\{i \in L : i < t\}|, |\{i \in R : i \ge t\}|\}$. We use this observation in Section 4 to propose separation algorithms for inequalities (10) with $S \subseteq [k_1+1, k'_1]$ and $u_t = q_t$ for all $t \in S$, $\gamma_t = 1$ for all $t \in L$ and $\beta_t = 1$ for all $t \in R$. Finally, note that the proof of Theorem 1 provides a new proof of validity for inequalities (11).

□

Example 1 (cont.) Adding inventory balance equalities for periods in $[2,5]$ and for periods in $[3,4]$ to inequality (12), we get inequality (10) with $S = [2,5]$, $L = [1,2]$, $R = [3,5]$ and $u_t = q_t$ for $t \in S$:

$$y_2 + 2y_3 + 2y_4 + y_5 \le d_{23}x_2 + (d_3 + d_{24})x_3 + (d_{34} + d_{25})x_4$$
$$+ d_{35}x_5 + r_1 + r_2 + s_3 + s_4 + s_5. \qquad (14)$$

However, inequalities (8) and (9) cannot be obtained from inequalities (11). Similarly, inequality (10) with $S = [2,5]$, $L = [1,2]$, $R = [3,5]$ and $1 = u_3 < q_3 = 2$:

$$y_2+y_3+2y_4+y_5 \le d_{23}x_2+d_3x_3+(d_{34}+d_{25})x_4+d_{35}x_5+r_1+r_2+s_3+s_4+s_5,$$

cannot be obtained from inequalities (11).

□

We study the strength of inequalities (10) in Section 3.

3 Linear Description of the Convex Hull

Pochet and Wolsey [9] give shortest path and facility location linear programming reformulations of ULSB. In particular, the facility location reformulation is given by (FL):

$$Z^{FL} := \min \sum_{t=1}^{n} (f_t x_t + c_t y_t + g_t r_t + h_t s_t)$$

$$\sum_{k=1}^{n} \tilde{y}_{kt} = d_t \quad \text{for } t \in [1,n] \tag{15}$$

$$\sum_{k=1}^{n} \tilde{y}_{tk} = y_t \quad \text{for } t \in [1,n] \tag{16}$$

$$\tilde{y}_{kt} \le d_t x_k \quad \text{for } k,t \in [1,n] \tag{17}$$

$$x_t \le 1 \quad \text{for } t \in [1,n] \tag{18}$$

$$s_t - \sum_{k=1}^{t} \sum_{j=t+1}^{n} \tilde{y}_{kj} - \lambda_t = 0 \quad \text{for } t \in [1,n-1] \tag{19}$$

$$r_t - \sum_{k=t+1}^{n} \sum_{j=1}^{t} \tilde{y}_{kj} - \lambda_t = 0 \quad \text{for } t \in [1,n-1] \tag{20}$$

$$\tilde{y}, y, s, r, x, \lambda \ge 0, \tag{21}$$

where $\tilde{y}_{kt}$ for $k,t \in [1,n]$ represents the amount produced in period k to satisfy the demand in period t. Note that λ_t has to be added to the definition of s_t and r_t to represent an additional amount of flow between periods t and $t+1$. Such a flow λ_t does not satisfy any demand, but is required to obtain a correct reformulation of ULSB (i.e., ULSB is unbounded if $g_t + h_t < 0$). Let $\mathcal{Q}$ be the set of feasible solutions to (15)–(21).

Proposition 3 *(Pochet and Wolsey [9])* $\mathcal{S} = \text{proj}_{y,s,r,x}(\mathcal{Q}) = \{(y,s,r,x) \in \mathbb{R}^{4n-2} : (y,s,r,x) \in \mathcal{P}$ *and* $\mathcal{T}'(y,s,r,x) \neq \emptyset\}$*, with* $\mathcal{T}'(y,s,r,x) = \{(\tilde{y},y,s,r,x) \in \mathbb{R}^{n^2+4n-2} : (22)-(27)\}$*, where*

$$-\sum_{k=1}^{n} \tilde{y}_{kt} = -d_t \quad \textit{for } t \in [1,n] \tag{22}$$

$$-\sum_{k=1}^{n} \tilde{y}_{tk} = -y_t \quad \textit{for } t \in [1,n] \tag{23}$$

$$\sum_{k=1}^{t} \sum_{j=t+1}^{n} \tilde{y}_{kj} \le s_t \quad \textit{for } t \in [1,n-1] \tag{24}$$

$$\sum_{k=t+1}^{n} \sum_{j=1}^{t} \tilde{y}_{kj} \le r_t \quad \textit{for } t \in [1,n-1] \tag{25}$$

$$\tilde{y}_{kt} \le d_t x_k \quad \textit{for } k,t \in [1,n] \tag{26}$$

$$\tilde{y}_{kt} \ge 0 \quad \textit{for } k,t \in [1,n]. \tag{27}$$

By Proposition 3 and Farkas' Lemma, we obtain directly the following complete implicit linear description of $\mathcal{S}$.

Proposition 4 *(Pochet and Wolsey [9])* $\mathcal{S} = \{(y, s, r, x) \in \mathcal{P} : \sum_{t=1}^{n} \varepsilon_t^i d_t + \sum_{t=1}^{n} \alpha_t^i y_t \leq \sum_{t=1}^{n} \sigma_t^i s_t + \sum_{t=1}^{n} \rho_t^i r_t + \sum_{k=1}^{n} \sum_{t=1}^{n} \delta_{kt}^i d_t x_k, i \in I\}$*, where* $(\varepsilon^i, \alpha^i, \sigma^i, \rho^i, \delta^i)$*,* $i \in I$ *are the extreme rays of the dual cone of* (22)–(27) *given by*

$$-\varepsilon_t - \alpha_j + \sum_{k=j}^{t-1} \sigma_k + \delta_{jt} \geq 0 \text{ for } 1 \leq j \leq t \leq n \quad (28)$$

$$-\varepsilon_t - \alpha_j + \sum_{k=t}^{j-1} \rho_k + \delta_{jt} \geq 0 \text{ for } 1 \leq t < j \leq n \quad (29)$$

$$\sigma_j, \rho_j \geq 0 \qquad \text{for } j \in [1, n-1] \quad (30)$$

$$\delta_{jt} \geq 0 \qquad \text{for } j, t \in [1, n] \quad (31)$$

We use Proposition 4 to prove the following result, which is a strengthening of Proposition 12 in [9].

Proposition 5 *If inequality*

$$\sum_{t=1}^{n} \varepsilon_t d_t + \sum_{t=1}^{n} \alpha_t y_t \leq \sum_{t=1}^{n-1} \sigma_t s_t + \sum_{t=1}^{n-1} \rho_t r_t + \sum_{k=1}^{n} \sum_{t=1}^{n} \delta_{kt} d_t x_k \quad (32)$$

is a facet of $\mathcal{S}$ *such that* $(\varepsilon, \alpha, \sigma, \rho, \delta)$ *satisfy* (28)–(31) *with* $\varepsilon_t = 0$ *for all* $t \in [1, n]$*, then the facet is of the form* (10)*, with* $u = \lambda\alpha$*,* $\beta = \lambda\sigma$*,* $\gamma = \lambda\rho$ *for some* $\lambda \in \mathbb{R}_+$*.*

Proof If inequality (32) is a facet, then from Proposition 4, $(\varepsilon, \alpha, \sigma, \rho, \delta)$ with $\varepsilon_t = 0$ for all $t \in [1, n]$ is an extreme ray of (28)–(31). Note that for $\varepsilon_t = 0$ for all $t \in [1, n]$, we must have $\alpha_t \geq 0$ for all $t \in [1, n]$ for $(\varepsilon, \alpha, \sigma, \rho, \delta)$ to be an extreme ray of (28)–(31). For fixed $\alpha \in \mathbb{Z}_+^n$, $(\sigma, \delta_{kt}$ for $k \leq t)$ must be an extreme point of

$$\sum_{k=j}^{t-1} \sigma_k + \delta_{jt} \geq \alpha_j \text{ for } 1 \leq j \leq t \leq n \quad (33)$$

$$\delta_{jt} \geq 0 \qquad \text{for } 1 \leq j \leq t \leq n \quad (34)$$

$$\sigma_j \geq 0 \qquad \text{for } 1 \leq j \leq n-1. \quad (35)$$

The constraint matrix given by (33)–(35) is totally unimodular. Therefore, for integral α, $(\sigma, \delta_{kt}$ for $k \leq t)$ is integral. Similarly $(\rho, \delta_{kt}$ for $k > t)$ is integral. (Therefore, condition *(i)* of Theorem 1 is satisfied.) Let $a^+ = \max\{0, a\}$. Extreme points of (33)–(35) are of the form

$$\delta_{jt} = (\alpha_j - \sum_{k=j}^{t-1} \sigma_k)^+. \quad (36)$$

Similarly for $j > t$,

$$\delta_{jt} = (\alpha_j - \sum_{k=t}^{j-1} \rho_k)^+. \quad (37)$$

Let $\rho_0 = \max_{t\in[1,n]}\{(\alpha_t - \sum_{k=1}^{t-1}\rho_k)^+\}$ and $\sigma_n = \max_{t\in[1,n]}\{(\alpha_t - \sum_{k=t}^{n-1}\sigma_k)^+\}$. (Condition *(ii)* of Theorem 1 is satisfied with this choice of ρ_0 and σ_n.) Observe that for each $j \in [1,n]$ we have $\sum_{t=1}^{n}\delta_{jt}d_t = \sum_{i=1}^{\alpha_j} d_{(k(j,i)+1)k'(j,i)}$, where $k(j,i) = \max\{t \in [0, j-1] : \sum_{k\in[t,j-1]}\rho_k \geq i\}$, and $k'(j,i) = \min\{t \in [j,n] : \sum_{k\in[j,t]}\sigma_k \geq i\}$. (Therefore, conditions *(iii)* and *(iv)* of Theorem 1 are satisfied.) As a result, the facet (32) with integral α is of the form (10) where $\beta = \sigma$, $\gamma = \rho$ and $u = \alpha$.

Finally, we need to argue that considering integral α in inequality (32) with $\varepsilon_t = 0$ for all t is sufficient. Note that the constraint matrix (28)−(31) is not necessarily totally unimodular. Therefore, we could have fractional α_t for some t. For instance, the determinant of the following submatrix corresponding to the variables $(\alpha_2, \alpha_3, \alpha_4, \sigma_4, \sigma_5, \rho_1, \rho_2, \rho_3)$ is -2:

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

However, note that given a fractional extreme ray of (28)−(31) with $\varepsilon_t = 0$ and $\alpha_t \geq 0$ for all t, there exists a scaling such that the extreme ray $(\varepsilon, \alpha, \sigma, \rho, \delta)$ is integral, because the associated cone is pointed at the origin. In other words, inequalities (10) are positive multiples of inequalities (32) with $\varepsilon_t = 0$ for all $t \in [1,n]$.

□

The following theorem states that to generate $\mathcal{S}$ it suffices to consider inequalities (32) given by the rays of the dual cone (28)–(31), where $\varepsilon_t = 0$ for all t, which, from Proposition 5, are positive multiples of inequalities (10). Therefore, we have an explicit description of $\mathcal{S}$.

Theorem 2 $\mathcal{S} = \{(x,s,r,y) \in \mathcal{P} : (x,s,r,y)$ *satisfies* $(10)\} = \{(x,s,r,y) \in \mathbb{R}_+^{4n+2} : (x,s,r,y) \in X\}$ *where* X *is described by the linear constraints*

$$y_t + (s_{t-1} - r_{t-1}) = d_t + (s_t - r_t) \qquad \text{for } t \in [1,n] \qquad (38)$$

$$x_t \leq 1 \qquad \text{for } t \in [1,n] \qquad (39)$$

$$\sum_{k=1}^{n}((\sum_{t=1}^{n}\delta_{kt}d_t)x_k - \alpha_k y_k + \beta_k s_k + \gamma_k r_k) \geq 0 \quad \text{for } (\alpha,\beta,\gamma,\delta) \in \Gamma \qquad (40)$$

$$s_0 = r_0 = s_n = r_n = 0$$

$$x, s, r, y \geq 0,$$

where Γ *is described by the linear constraints*

$$\delta_{jt} = (\alpha_j - \textstyle\sum_{\ell=j}^{t-1}\beta_\ell)^+ \quad \text{for } 1 \leq j \leq t \leq n \qquad (41)$$

$$\delta_{jt} = (\alpha_j - \textstyle\sum_{\ell=t}^{j-1}\gamma_\ell)^+ \quad \text{for } 1 \leq t < j \leq n \qquad (42)$$

$$\alpha, \beta, \gamma, \delta \geq 0.$$

We give a primal-dual proof of this theorem. The primal formulation corresponding to the feasible set X, denoted by (P) is:

$$Z = \min\{\sum_{t=1}^{n}(c_t y_t + h_t s_t + g_t r_t + f_t x_t) : (x, s, r, y) \in X\} \tag{43}$$

Letting v_t, $-z_t$ and $u(\alpha\beta\gamma)$ be the dual variables associated with each constraint (38), (39) and (40), respectively, we obtain the corresponding dual formulation, (D):

$$W = \max\{\sum_{i=1}^{n} d_i v_i - \sum_{i=1}^{n} z_i : (u, v, z) \text{ satisfies } (44) - (48)\},$$

where

$$-z_i + \textstyle\sum_{\alpha,\beta,\gamma} \left(\sum_{j=1}^{n} \delta_{ij}^{\alpha\beta\gamma} d_j\right) u(\alpha\beta\gamma) \le f_i \quad \text{for } i \in [1, n] \tag{44}$$

$$v_i - \textstyle\sum_{\alpha,\beta,\gamma} \alpha_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) \le c_i \quad \text{for } i \in [1, n] \tag{45}$$

$$v_{i+1} - v_i + \textstyle\sum_{\alpha,\beta,\gamma} \beta_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) \le h_i \quad \text{for } i \in [1, n-1] \tag{46}$$

$$v_i - v_{i+1} + \textstyle\sum_{\alpha,\beta,\gamma} \gamma_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) \le g_i \quad \text{for } i \in [1, n-1] \tag{47}$$

$$z, u \ge 0, \tag{48}$$

where $\alpha_i^{\alpha\beta\gamma}$ represents the ith element of the α vector for given (α, β, γ) ($\beta_i^{\alpha\beta\gamma}$ and $\gamma_i^{\alpha\beta\gamma}$ are defined similarly, and $\delta_{ij}^{\alpha\beta\gamma}$ is given by (41)–(42)). We have to prove that for any primal objective coefficients (c, h, g, f)

$$Z = W = Z^{BL}.$$

If $h_t + g_t < 0$ for some t we know that $\mathcal{P}$ is unbounded, and it is easy to check that (P) is unbounded as well. Hence it remains to show that $Z = W = Z^{BL}$ for any coefficients (c, h, g, f) with $h_t + g_t \ge 0$ for all t.

The following proposition is needed in the proof of Theorem 2. Let (P*) be the formulation

$$Z^* = \min \sum_{k=1}^{n}\sum_{t=1}^{n} q_{kt}\tilde{y}_{kt} + \sum_{t=1}^{n} f_t x_t + \sum_{t=1}^{n-1} h_t \eta_t + \sum_{t=1}^{n-1} g_t \nu_t$$

$$\sum_{k=1}^{n} \tilde{y}_{kt} - (\eta_t - \eta_{t-1}) + (\nu_t - \nu_{t-1}) = d_t \quad \text{for } t \in [1, n] \tag{49}$$

$$\tilde{y}_{kt} \le d_t x_k \quad \text{for } k, t \in [1, n] \tag{50}$$

$$x_t \le 1 \quad \text{for } t \in [1, n] \tag{51}$$

$$x, \tilde{y}, \eta, \nu \ge 0,$$

where $\eta_0 = \nu_0 = \eta_n = \nu_n = 0$, $q_{kk} = c_k$, $q_{kt} = (c_k + h_k + \cdots + h_{t-1})$ if $k < t$ and $q_{kt} = (c_k + g_{k-1} + \cdots + g_t)$ if $k > t$. Letting v_t, $-w_{kt}$ and $-z_t$ be the dual variables associated with each constraint (49), (50) and (51), respectively, we obtain the corresponding dual formulation, (D*):

$$\begin{aligned} W^* = \max \textstyle\sum_{i=1}^n d_i v_i - \sum_{i=1}^n z_i & \\ -z_i + \textstyle\sum_{j=1}^n d_j w_{ij} \le f_i \quad & \text{for } i \in [1,n] \\ v_j - w_{ij} \le q_{ij} \quad & \text{for } i,j \in [1,n] \\ v_{i+1} - v_i \le h_i \quad & \text{for } i \in [1,n-1] \\ v_i - v_{i+1} \le g_i \quad & \text{for } i \in [1,n-1] \\ w, z \ge 0. & \end{aligned}$$

Proposition 6 *If $h_t + g_t \ge 0$ for all t, then (P^*) has an optimal solution with $\eta_t = \nu_t = 0$ for all $t \in [1, n-1]$.*

The consequence of this proposition that will be used in the proof of Theorem 2 is given in the following corollary.

Corollary 1 *If $h_t + g_t \ge 0$ for all t there exist numbers $v_1, \ldots, v_n$ and $z_1, \ldots, z_n \ge 0$ such that*

$$\begin{aligned} Z^{BL} = \textstyle\sum_{i=1}^n d_i v_i - \sum_{i=1}^n z_i & \\ -g_i \le v_{i+1} - v_i \le h_i \quad & \text{for } i \in [1,n-1] \\ \textstyle\sum_{j=1}^n d_j (v_j - q_{ij})^+ - z_i \le f_i \quad & \text{for } i \in [1,n]. \end{aligned}$$

Proof If $h_t + g_t \ge 0$ for all t, then we know that there exists an optimal solution to (FL) with $\lambda_t = 0$ for all t and that $Z^{FL} = Z^{BL}$ [9]. Proposition 6 shows that $Z^* = Z^{FL}$ under the assumption that $h_t + g_t \ge 0$ for all t, because there always exists an optimal solution to (P*) that is optimal in (FL). Hence, $W^* = Z^* = Z^{FL} = Z^{BL}$. Finally, note that there exists an optimal solution to (D*) with $w_{ij} = (v_j - q_{ij})^+$ for all $i, j \in [1, n]$. □

Proof **[Proof of Proposition 6.]** Consider an optimal solution $(x^*, \tilde{y}^*, \eta^*, \nu^*)$ to problem (P*) with $\sum_{t=1}^{n-1}(\eta_t^* + \nu_t^*)$ being minimal. In this solution we must have $\eta_t^* \cdot \nu_t^* = 0$ for all t (otherwise it is possible to decrease strictly $\sum_{t=1}^{n-1}(\eta_t^* + \nu_t^*)$). We build a graph $G' = (V', A')$ with vertices $V' = \{1, \ldots, n\}$ and oriented arc set A' such that $(i, i+1) \in A'$ if $\eta_i^* > 0$ and $(i+1, i) \in A'$ if $\nu_i^* > 0$. Define $K(i) = \{k \in V' \mid \tilde{y}_{ki}^* > 0\}$. (In particular $k \in K(i)$ implies $x_k > 0$.) We must have $\sum_{k \in K(i)} \tilde{y}_{ki}^* = d_i + (\eta_i^* - \eta_{i-1}^*) - (\nu_i^* - \nu_{i-1}^*)$. Hence without changing the values η^*, ν^* there always exists an optimal solution $(x^*, \tilde{y}^*, \eta^*, \nu^*)$ to (P^*) with $\tilde{y}_{ki}^* = d_i x_k^*$ for all $k \in K(i)$ except at most one.

Now consider one arc $(i, i+1) \in A'$ (so $\eta_i^* > 0$) and $k \in K(i)$ (so $x_k^* > 0$ and $\tilde{y}_{ki}^* > 0$). We claim that $k \in K(i+1)$ and $\tilde{y}_{k(i+1)}^* = d_{i+1} x_k^*$.

Case 1. ($k \le i$.) Suppose that $\tilde{y}_{k(i+1)}^* < d_{i+1} x_k^*$. Then a new solution is $\tilde{y}_{ki} = \tilde{y}_{ki}^* - \varepsilon$, $\tilde{y}_{k(i+1)} = \tilde{y}_{k(i+1)}^* + \varepsilon$ and $\eta_i = \eta_i^* - \varepsilon$, for some $\varepsilon > 0$. This new solution is feasible and also optimal because the change of the objective value is $-\varepsilon q_{ki} + \varepsilon q_{k(i+1)} - \varepsilon h_i = 0$. Furthermore, $\sum_{t=1}^{n-1}(\eta_t + \nu_t)$ strictly decreases and this is a contradiction.

Case 2. ($k \ge i+1$.) Suppose that $\tilde{y}_{k(i+1)}^* < d_{i+1} x_k^*$. Then a new solution is $\tilde{y}_{ki} = \tilde{y}_{ki}^* - \varepsilon$, $\tilde{y}_{k(i+1)} = \tilde{y}_{k(i+1)}^* + \varepsilon$ and $\eta_i = \eta_i^* - \varepsilon$. This new solution is feasible and also optimal because the change of the objective value is $-\varepsilon q_{ki} + \varepsilon q_{k(i+1)} - \varepsilon h_i \leq -\varepsilon q_{ki} + \varepsilon q_{k(i+1)} + \varepsilon g_i = 0$ (where the last inequality holds because $h_i + g_i \geq 0$). Again, the contradiction follows from a strict decrease in $\sum_{t=1}^{n-1}(\eta_t + \nu_t)$.

By the same argument, if $(i+1, i) \in A'$ and $k \in K(i+1)$, then we must have $k \in K(i)$ with $\tilde{y}^*_{ki} = d_i x^*_k$.

Now suppose that a path exists in G' (i.e., $A' \neq \emptyset$). Consider a longest directed path $i_1, \ldots, i_r$ in G' and define

$$Y(i_s) = \sum_{\substack{k \in K(i_s): \\ \tilde{y}^*_{ki_s} = d_{i_s} x^*_k > 0}} x^*_k + \frac{\tilde{y}^*_{\tilde{k}i_s}}{d_{i_s}} \quad \text{for } s \in [1, r],$$

where $\tilde{k} \in K(i_s)$ and $0 < \tilde{y}^*_{\tilde{k}i_s} < d_{i_s} x^*_{\tilde{k}}$. (This term disappears if no such $\tilde{k}$ exists.) Note that a longest path always exists since $\eta^*_i \cdot \nu^*_i = 0$ for all i. We claim that $0 \leq Y(i_1) \leq Y(i_2) \leq \cdots \leq Y(i_r) < 1$.

As $(i_s, i_{s+1}) \in A'$, we know that $k \in K(i_s)$ implies that $k \in K(i_{s+1})$ and $\tilde{y}^*_{ki_{s+1}} = d_{i_{s+1}} x^*_k$. Hence,

$$Y(i_s) \leq \sum_{k \in K(i_s)} x^*_k \leq Y(i_{s+1})$$

where the first inequality holds because $\frac{\tilde{y}^*_{\tilde{k},i_s}}{d_{i_s}} < x^*_{\tilde{k}}$ if $\tilde{k}$ exists. In addition,

$$\begin{aligned}\sum_{k \in V'} \tilde{y}^*_{ki_r} &= d_{i_r} + (\eta^*_{i_r} - \eta^*_{i_r-1}) - (\nu^*_{i_r} - \nu^*_{i_r-1}) \\ &= d_{i_r} - \eta^*_{i_r-1} - \nu^*_{i_r} \\ &< d_{i_r},\end{aligned}$$

where the second equality is because $\eta^*_{i_r} = \nu^*_{i_r-1} = 0$ as i_r is the last node of the longest path and the last inequality is because if a path exists in G' we must have $\eta^*_{i_r-1} + \nu^*_{i_r} > 0$.

On the other hand, $\sum_{k \in V'} \tilde{y}^*_{ki_r} = \sum_{k \in K(i_r)} \tilde{y}^*_{ki_r} = d_{i_r} Y(i_r)$, where the last equality holds by definition of $Y(i_r)$. We have then $Y(i_r) < 1$, which implies that $Y(i_1) < 1$. But we also have

$$d_{i_1} Y(i_1) = \sum_{k \in K(i_1)} \tilde{y}^*_{ki_1} = \sum_{k \in V'} \tilde{y}^*_{ki_1} = d_{i_1} + (\eta^*_{i_1} - \eta^*_{i_1-1}) - (\nu^*_{i_1} - \nu^*_{i_1-1}) > d_{i_1},$$

where the last inequality holds because $\eta^*_{i_1-1} = \nu^*_{i_1} = 0$ as node i_1 is the first node of a longest directed path and $\eta^*_{i_1} + \nu^*_{i_1-1} > 0$ because a path starting from node i_1 exists.

The contradiction we have obtained ($1 > Y(i_1) > 1$) implies that no path exists in G' (i.e., $A' = \emptyset$). Therefore, $\eta^*_i = \nu^*_i = 0$ for all i. □

Proof [**Proof of Theorem 2.**] Given that $h_t + g_t \geq 0$ for all t, we must find a solution of (D) with $W = Z^{BL}$. We know by Corollary 1 that there are numbers $v_1, \ldots, v_n$ and $z_1, \ldots, z_n \geq 0$ such that

$$\begin{aligned}
Z^{BL} = & \sum_{i=1}^{n} d_i v_i - \sum_{i=1}^{n} z_i & \\
& v_{i+1} - v_i \leq h_i & i \in [1, n-1] \\
& v_i - v_{i+1} \leq g_i & i \in [1, n-1] \\
& w_{ij} = (v_j - q_{ij})^+ & i, j \in [1, n] \\
& \sum_{j=1}^{n} d_j w_{ij} - z_i \leq f_i & i \in [1, n].
\end{aligned}$$

We construct a feasible solution to (D) with one variable $u(\alpha\beta\gamma) = 1$ corresponding to the following values of α, β, γ :

$$\begin{aligned}
\alpha_i &= w_{ii} \geq 0 \text{ for } i \in [1, n] \\
\beta_i &= h_i + v_i - v_{i+1} \geq 0 \text{ for } i \in [1, n-1] \\
\gamma_i &= g_i + v_{i+1} - v_i \geq 0 \text{ for } i \in [1, n-1].
\end{aligned}$$

All other $u(\alpha\beta\gamma)$ variables are equal to zero. The values of the variables v, z are the values used in Corollary 1. This implies that the objective value corresponding to this solution is Z^{BL}. It remains to show that this solution is feasible in (D).

By definition

$$\sum_{\alpha,\beta,\gamma} \beta_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) = h_i + v_i - v_{i+1}, \text{ (46) is satisfied;}$$

$$\sum_{\alpha,\beta,\gamma} \gamma_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) = g_i + v_{i+1} - v_i, \text{ (47) is satisfied;}$$

$$\sum_{\alpha,\beta,\gamma} \alpha_i^{\alpha\beta\gamma} u(\alpha\beta\gamma) = w_{ii} = (v_i - q_{ii})^+ = (v_i - c_i)^+ \geq v_i - c_i, \text{ (45) is satisfied.}$$

The δ_{ij} values are defined in (41)–(42). It remains to show that $-z_i + \sum_{\alpha,\beta,\gamma} (\sum_{j=1}^{n} \delta_{ij}^{\alpha\beta\gamma} d_j) u(\alpha\beta\gamma) = -z_i + \sum_{j=1}^{n} \delta_{ij} d_j \leq f_i$. We prove it by showing that $\delta_{ij} = w_{ij}$ for all i, j.

For each $i \in [1, n]$ we have $\delta_{ii} = \alpha_i = w_{ii}$. For $j > i$ we have

$$\begin{aligned}
\delta_{ij} &= (\alpha_i - \sum_{\ell=i}^{j-1} \beta_\ell)^+ \\
&= ((v_i - c_i) - (h_i + v_i - v_{i+1}) - \cdots - (h_{j-1} + v_{j-1} - v_j))^+ \\
&= (v_j - c_i - h_i - h_{i+1} - \cdots - h_{j-1})^+ \\
&= (v_j - q_{ij})^+ = w_{ij}.
\end{aligned}$$

Finally, for $j < i$ we have

$$\begin{aligned}\delta_{ij} &= (\alpha_i - \sum_{\ell=j}^{i-1} \gamma_\ell)^+ \\ &= ((v_i - c_i) - (g_{i-1} + v_i - v_{i-1}) - \cdots - (g_j + v_{j+1} - v_j))^+ \\ &= (v_j - c_i - g_{i-1} - \cdots - g_j)^+ \\ &= (v_j - q_{ij})^+ = w_{ij}.\end{aligned}$$

From Corollary 1 we know that $\sum_{j=1}^{n} d_j w_{ij} \leq f_i + z_i$. This completes the proof. □

Inequalities (10) are enough to provide a complete linear description of $\mathcal{S}$. Although we do not give general conditions under which these inequalities define facets of $\mathcal{S}$, we conclude this section by showing that the coefficients of the variables can grow very large in facet-defining inequalities. In particular, we give an example showing that the coefficients of a facet-defining inequality for $\mathcal{S}$ with n time periods can be as large as the $(n-2)^{th}$ number in the Fibonacci series.

Example 2 Consider an instance of ULSB with $n = 10$ time periods, and the inequality (10) defined by $S = [2, 9]$, $L = [1, 5]$, $R = [6, 9]$, and:

$$\begin{array}{lllll} \gamma_1 = 21, & \gamma_2 = 8, & \gamma_3 = 3, & \gamma_4 = 1, & \gamma_5 = 1, \\ \beta_6 = 1, & \beta_7 = 2, & \beta_8 = 5, & \beta_9 = 13, & \\ u_2 = 21, & u_3 = 8, & u_4 = 3, & u_5 = 1, & \\ u_6 = 1, & u_7 = 2, & u_8 = 5, & u_9 = 13, & \end{array}$$

The corresponding facet-defining inequality (10) is:

$$\begin{aligned} & 21y_2 + 8y_3 + 3y_4 + 1y_5 + 1y_6 + 2y_7 + 5y_8 + 13y_9 \\ \leq\ & 21r_1 + 8r_2 + 3r_3 + 1r_4 + 1r_5 \\ & + 1s_6 + 2s_7 + 5s_8 + 13s_9 \\ & + (d_{26} + 2d_{27} + 5d_{28} + 13d_{29})\ x_2 \\ & + (d_{36} + 2d_{37} + 5d_{38})\ x_3 \\ & + (d_{46} + 2d_{47})\ x_4 \\ & + (d_{56})\ x_5 \\ & + (d_6)\ x_6 \\ & + (d_{67} + d_{57})\ x_7 \\ & + (d_{68} + d_{58} + 3d_{48})\ x_8 \\ & + (d_{69} + d_{59} + 3d_{49} + 8d_{39})\ x_9. \end{aligned}$$

□

In our computational experiments, summarized in Section 5, we observe that the coefficients of the variables are not very large in the facets that are generated for the test instances.

4 Separation

From Proposition 4, there is a linear programming based separation algorithm for ULSB, which according to Proposition 5 and Theorem 2 will generate inequalities of type (10).

Proposition 7 *The separation problem for (a positive multiple of) inequalities* (10) *can be solved as a linear program (LP) with the objective*

$$\max \sum_{t=1}^{n} \alpha_t y_t - (\sum_{t=1}^{n-1} \sigma_t s_t + \sum_{t=1}^{n-1} \rho_t r_t + \sum_{k=1}^{n} \sum_{t=1}^{n} \delta_{kt} d_t x_k), \tag{52}$$

subject to (28)−(31) *and* $\varepsilon_t = 0$ *for all* $t \in [1, n]$. *If for a given point* (y, x, s, r) *the objective function value of the separation LP is unbounded, then the direction of unboundedness given by an extreme ray of the LP identifies a violated inequality* (10).

In Section 5, we summarize our computational experiments on using the linear program in Proposition 7 to solve the separation problem for inequalities (10). Although Proposition 7 gives a polynomial-time separation algorithm for inequalities (10), it is preferable to have a combinatorial algorithm to solve the separation problem in practice. Pochet and Wolsey [9] give a separation heuristic for the special case of inequalities (10) where $u_t = 1$ for $t \in S$. Here, we give an exact algorithm for this special case.

Theorem 3 *Separation problem for inequalities* (10) *for* $u_t = 1$ *for all* $t \in S$ *can be solved in* $O(n^4)$.

Proof The inequalities (10) with $u_t = 1$ for all $t \in S$ can be rewritten as

$$\sum_{t \in S} y_t - \sum_{t \in S} d_{(k(t,1)+1)k'(t,1)} x_t - \sum_{t \in L} r_t - \sum_{t \in R} s_t \leq 0, \tag{53}$$

For a given point (y, x, s, r), we find sets $S \subseteq [1, n]$ and $L, R \subseteq [0, n]$ such that the left-hand side of (53) is maximized. We formulate this problem as a longest-path problem on a directed acyclic (layered) network.

Consider a directed graph $G = (V, A)$ with a source vertex $0 \in V$ and a sink vertex $(n+1) \in V$. Let $(i, j, t^S, t^L), (i, j, t^{\bar{S}}, t^L), (i, j, t^S, t^{\bar{L}}), (i, j, t^{\bar{S}}, t^{\bar{L}}) \in V$ for $0 \leq i < t \leq j \leq n$, where we let i be the largest period smaller than t that is included in L and j be the smallest period greater than or equal to t that is included in R. We let $0 \in L$ and $n \in R$. Let $\bar{S} = [1, n] \setminus S$ and $\bar{L} = [1, n] \setminus L$.

There is an arc $(0, (0, j, 1^W, 1^Z)) \in A$ for each $j \in [1, n], W \in \{S, \bar{S}\}, Z \in \{L, \bar{L}\}$ so that if the path includes this arc, then $j \in R, 1 \in W \cup Z$. Also, let $((i, n, n^W, n^{\bar{L}}), (n+1)) \in A$ for $i \in [0, n-1]$ and $W \in \{S, \bar{S}\}$ so that if the longest path includes this arc, then $n \in W$. For $0 \leq i < t < p \leq n$, $j \in \{t, p\}$, $U, W \in \{S, \bar{S}\}$ and $Z \in \{L, \bar{L}\}$, the arc $((i, j, t^U, t^L), (t, p, (t+1)^W, (t+1)^Z))$ is in A and if the longest path includes this arc, then $t \in L, p \in R$ and $(t+1) \in W \cup Z$. Also, for $0 \leq i < t < p \leq n$, $j \in \{t, p\}$, $U, W \in \{S, \bar{S}\}$

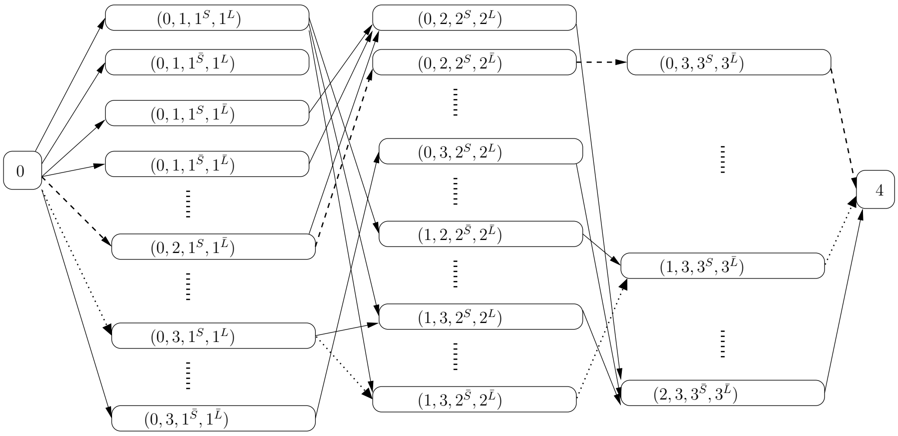

Fig. 5 Graph G for separation for inequalities (53).

and $Z \in \{L, \bar{L}\}$, the arc $((i, j, t^U, t^{\bar{L}}), (i, p, (t+1)^W, (t+1)^Z))$ is in A; if the longest path includes this arc, then $t \in \bar{L}, p \in R$ and $(t+1) \in W \cup Z$. Figure 5 depicts G for $n = 3$.

Next, we assign length to the arcs in A. For each $j \in [1, n]$, let the length of the arc $(0, (0, j, 1^W, 1^Z)) \in A$ be

$$c_{0,(0,j,1^W,1^Z))} = \begin{cases} y_1 - d_{1j}x_1 - s_j & \text{if } j = 1, W = S, Z \in \{L, \bar{L}\} \\ y_1 - d_{1j}x_1 & \text{if } j > 1, W = S, Z \in \{L, \bar{L}\} \\ -s_j & \text{if } j = 1, W = \bar{S}, Z \in \{L, \bar{L}\} \\ 0 & \text{if } j > 1, W = \bar{S}, Z \in \{L, \bar{L}\}. \end{cases}$$

Also let the length of the arcs $((i, n, n^W, n^Z), (n+1)) \in A$ for $i \in [0, n-1]$ and $W \in \{S, \bar{S}\}$ and $Z \in \{L, \bar{L}\}$ be zero.

For $0 \leq i < t < p \leq n$, $j \in \{t, p\}$, and $U, W \in \{S, \bar{S}\}$, let the length of the arc $a = ((i, j, t^U, t^L), (t, p, (t+1)^W, (t+1)^Z))$ for $Z \in \{L, \bar{L}\}$ be

$$c_a = \begin{cases} -r_t + y_{t+1} - d_{(t+1)p}x_{t+1} - s_p & \text{if } p = t+1, U \in \{S, \bar{S}\}, W = S \\ -r_t + y_{t+1} - d_{(t+1)p}x_{t+1} & \text{if } p > t+1, U \in \{S, \bar{S}\}, W = S \\ -r_t - s_p & \text{if } p = t+1, U \in \{S, \bar{S}\}, W = \bar{S} \\ -r_t & \text{if } p > t+1, U \in \{S, \bar{S}\}, W = \bar{S}. \end{cases}$$

Finally, for $0 \leq i < t < p \leq n$, $j \in \{t, p\}$ and $U, W \in \{S, \bar{S}\}$, the arc $a = ((i, j, t^U, t^{\bar{L}}), (i, p, (t+1)^W, (t+1)^Z))$ for $Z \in \{L, \bar{L}\}$ has length

$$c_a = \begin{cases} y_{t+1} - d_{(i+1)p}x_{t+1} - s_p & \text{if } p = t+1, U \in \{S, \bar{S}\}, W = S \\ y_{t+1} - d_{(i+1)p}x_{t+1} & \text{if } p > t+1, U \in \{S, \bar{S}\}, W = S \\ -s_p & \text{if } p = t+1, U \in \{S, \bar{S}\}, W = \bar{S} \\ 0 & \text{if } p > t+1, U \in \{S, \bar{S}\}, W = \bar{S}. \end{cases}$$

We solve the longest path problem on this directed acyclic graph using Dijkstra's algorithm. There exists a violated inequality (10) if and only if the longest path is strictly positive. Observe that G has $O(n^3)$ vertices and

$O(n^4)$ arcs. Because we solve a longest path problem on a directed acyclic graph, the overall running time of the separation algorithm for inequality (53) is $O(n^4)$.

□

For example, in Figure 5, the dashed path corresponds to the inequality

$$y_1 + y_2 + y_3 \leq d_{12}x_1 + d_{12}x_2 + d_{13}x_3 + r_0 + s_2 + s_3,$$

and the dotted path corresponds to the inequality

$$y_1 + y_3 \leq d_{13}x_1 + d_{23}x_3 + r_0 + r_1 + s_3.$$

Furthermore, separation for inequalities (10) with $\gamma_t, \beta_t \in \{0,1\}, t \in [0,n]$ is easy when (L,R) is known. For given (L,R) where $L = \{k_1, k_2, \ldots, k_p\} \subseteq [0,n]$ and $R = \{k'_1, k'_2, \ldots, k'_b\} \subseteq [0,n]$, the separation for inequalities (10) can be done in $O(n^2)$. To see this, observe that if (L,R) is known, then for each $t \in [1,n]$ we know the values of $q_t = \min\{|\{i \in L : i < t\}|, |\{i \in R : i \geq t\}|\}$, $k(t,i)$ and $k'(t,i)$, $i \in [1,q_t]$. We let $u_t = \text{argmax}\{jy_t - (\sum_{i=1}^{j} d_{(k(t,i)+1)k'(t,i)})x_t, j \in [1,q_t]\}$ and $S = \{t \in [k_1+1, k'_1] : u_t y_t > \sum_{i=1}^{u_t} d_{(k(t,i)+1)k'(t,i)} x_t\}$.

Finally, for given S the separation problem for inequalities (10) with $\gamma_t, \beta_t \in \{0,1\}, t \in [0,n]$ in which $u_t = q_t$ for all $t \in S$ can be solved by finding a minimum cost negative cycle in a digraph, which is polynomial [2]. Let $H = (V,A)$ be a complete directed graph with $V = \{0,\ldots,n\}$. The arcs $(k,t) \in A$ with $k < t$ have cost $s_t - \sum_{j \in S \cap [k+1,t]} (y_j - d_{jt}x_t)$ and the arcs $(k,t) \in A$ with $k > t$ have cost $r_t + \sum_{j \in S \cap [t+2,k]} d_{(t+1)(j-1)} x_j$. If the minimum cost negative elementary directed cycle, C, contains the arc (k,t) for $k < t$, then let $t \in R$ and if C contains the arc (k,t) for $k > t$, then let $t \in L$. Finally, for each $t \in S$, u_t is the cardinality of the cut across $(t-1,t)$. This is a generalization of the separation algorithm in [11] given for inequalities (13).

5 Computations

To test the effectiveness of the inequalities described in Section 2 in solving ULSB in practice, we implement a branch-and-cut algorithm that incorporates inequalities (10). All computations are done on a 1.86 GHz Linux workstation with 3600 CPU seconds time and 512 MB memory limits.

The data used in the experiments has the following properties: Demands are generated from discrete uniform distribution between 0 and 30. Production costs are generated from discrete uniform distribution between 1 and 10. Let f be the ratio of production fixed cost to variable inventory cost and c be the upper bound on the holding costs. To test the performance of our branch-and-cut algorithm for varying cost parameters, we let $c \in \{5, 10, 20\}$ and $f \in \{500, 1000, 2000, 5000\}$ and generate five random instances for each combination.

A summary of these experiments is reported in Tables 1 and 2. In the third column of the tables we report the average integrality gap, which is

$100 \times (\texttt{zub} - \texttt{zinit})/\texttt{zub}$, where `zinit` is the objective value of the initial LP relaxation and `zub` is the objective value of the best integer solution. In the fourth column we compare the average percentage improvement of the integrality gap at the root node (`% gapimp`), which is $100 \times (\texttt{zroot} - \texttt{zinit})/(\texttt{zub} - \texttt{zinit})$, where `zroot` is the objective value of the LP at the root node after the cuts are added. Columns `cuts` and `nodes` compare the average number of cuts added, and the average number of branch-and-cut tree nodes explored, respectively.

The first set of experiments summarized in Table 1 is on solving ULSB with linear programming based exact separation for inequalities (10) given in Proposition 7. Our goal in these experiments is to test the maximum coefficients of the production, inventory and backlogging variables in inequalities (10). For these instances, we let holding costs be discrete uniform random variables between $-c$ and c and the backorder costs to be discrete uniform random variables between $-2c$ and $2c$ with the restriction that $g_t + h_t \geq 0$. Note that without loss of generality, we can assume that the production costs are nonnegative. The problem instances are solved with the MIP solver of CPLEX[1] Version 10.0. CPLEX cuts are disabled in the experiments with the branch-and-cut algorithm using inequalities (10) (denoted by `LSB`) to underline the impact of the inequalities discussed in this paper. However, in order to see how CPLEX cuts would perform we also solve the same instances with the default settings of CPLEX (`Def`) without adding any user cuts. We solve problem instances with $n = 50$. As the separation LP's are large, the exact separation is slow in practice. Therefore, for these runs we do not report the solution times.

We note that our inequalities are enough to solve ULSB as a linear program, so we do not report the percentage gap improvement of 100% and the number of branch-and-cut tree nodes which is zero for all instances. Also, in the last column of Table 1, denoted by $u_{\max}, \beta_{\max}$ and $\gamma_{\max}$, we report the maximum coefficients of the production, inventory and backlogging variables in inequalities (10), respectively. We observe that in all problem instances, there exist violated facets where one or more of the continuous variables have a coefficient that is greater than one.

In the second set of experiments, we test the effectiveness of our inequalities in solving multi-item lot sizing instances with a single setup per period. These problems are called *small bucket* problems [8] and are formulated as:

$$\begin{aligned}
\min \sum_{i=1}^{m} \sum_{t=1}^{n} (f_t^i x_t^i + c_t^i y_t^i + g_t^i r_t^i + h_t^i s_t^i) & \\
s_{t-1}^i + y_t^i - r_{t-1}^i = d_t^i + s_t^i - r_t^i, \quad & t \in [1,n],\ i \in [1,m] \\
y_t^i \leq d_{1n}^i x_t^i, \quad & t \in [1,n],\ i \in [1,m] \\
r_0^i = s_0^i = r_n^i = s_n^i = 0, \quad & i \in [1,m] \\
\textstyle\sum_{i=1}^{m} x_t^i \leq 1 \quad & t \in [1,n] \\
y^i \in \mathbb{R}_+^n,\ s^i \in \mathbb{R}_+^{n+1},\ r^i \in \mathbb{R}_+^{n+1} \quad & i \in [1,m] \\
x^i \in \{0,1\}^n. \quad & i \in [1,m],
\end{aligned} \tag{54}$$

[1] CPLEX is a trademark of ILOG, Inc.

Table 1 ULSB with inequalities (10) – exact separation.

f	c	gap	% gapimp	cuts		nodes	Coeff. in (10)		
			Def	Def	LSB	Def	$u_{\max}$	$\beta_{\max}$	$\gamma_{\max}$
500	5	50.7	67.1	68.4	309.4	82.4	3	3	2
	10	60.3	69.6	71.6	312.2	73.8	2	2	2
	20	88.0	78.5	71.0	216.6	53.4	3	2	2
1000	5	48.1	59.7	53.2	301.6	77.0	3	2	2
	10	57.9	61.7	64.4	423.4	152.0	5	2	2
	20	77.8	64.4	68.6	344.4	111.8	5	2	2
2000	5	46.4	54.7	50.4	253.0	62.0	3	3	2
	10	52.9	57.9	53.6	358.0	58.2	3	2	2
	20	70.0	55.2	60.0	383.6	136.8	2	2	2
5000	5	44.1	40.4	39.4	195.6	40.0	4	2	1
	10	47.8	59.1	40.4	235.4	33.6	3	2	1
	20	60.2	53.3	45.6	340.0	75.2	4	2	2

where the superscript i refers to item i, $i = 1, \ldots, m$. The small bucket problem can be seen as m single-item problems linked by the mode constraint (54).

We use similar data for each item as before, except, we let all holding costs be discrete uniform random variables between 1 and c and backlogging costs be discrete uniform random variables between 1 and $2c$. (The data files are available at `http://www.sie.arizona.edu/faculty/simge/data`.) To solve multi-item problem instances effectively, we propose a heuristic based on the algorithm given in Theorem 3 for inequalities (10) with $u_t, \gamma_t, \beta_t \in \{0, 1\}$. This heuristic relies on the observation that, in most cases, the production in a period is not used to satisfy demands in much earlier or much later periods. This observation is related to approximate extended formulations [14], however, the inequalities proposed in our study are in the original space of the variables. Instead of solving the separation problem exactly — an $O(n^4)$ running time — we solve a truncated version of the separation problem over intervals of length 10. In other words, for all $k \in [0, n-10]$ we let k be the smallest period included in L, $k+10$ be the largest period included in R and we let $S \subseteq [k+1, k+10]$. Therefore, the network depicted in Figure 5 has only 10 layers. We run this separation heuristic for each item at the root node and every 10,000 nodes of the branch-and-cut tree. We report our results for $n = 50$ and $m = 5$ in Table 2. In the last column of Table 2 we report the average CPU time elapsed (in seconds). If the problem is not solved within the time and memory limits, then we also report, in parenthesis, the average percentage gap between the best lower bound and the best integer solution found in the search tree (`endgap`).

Table 2 Multi-item ULSB with inequalities (10) – heuristic separation.

f	c	gap	% gapimp		cuts		nodes		time (endgap)	
			Def	LSB	Def	LSB	Def	LSB	Def	LSB
500	5	56.1	56.3	89.7	831.6	4027.8	1852533.0	2082.0	2439.7 (3.1)	373.7
	10	64.4	72.7	89.5	878.0	4217.4	1180885.4	3889.6	1596.7 (5.7)	962.1
	20	73.0	83.3	88.1	904.4	4043.2	868675.0	10671.0	1682.0 (1.5)	2783.0 (1.0)
1000	5	54.7	56.1	87.4	748.8	4176.4	1283602.0	3266.4	1172.7 (8.2)	342.3
	10	62.2	64.5	89.6	818.2	4264.4	1107742.4	1263.4	1189.3 (7.1)	341.8
	20	71.4	74.0	88.7	862.0	4197.4	1058535.6	4824.8	1358.4 (5.5)	1223.0
2000	5	52.8	45.9	83.5	595.4	4116.2	1398232.6	3173.6	1041.9 (8.8)	141.4
	10	59.3	53.2	87.2	746.8	4228.0	1616655.8	3474.8	1503.7 (10.0)	258.9
	20	67.7	63.0	89.4	812.2	4502.0	1300764.6	1844.4	1412.5 (9.1)	384.8
5000	5	49.0	38.1	74.9	403.6	3891.4	1584370.4	7224.6	1205.1 (2.0)	55.2
	10	55.1	42.0	81.1	503.2	4142.4	1662693.2	4054.4	1159.0 (7.7)	77.7
	20	62.2	49.4	85.5	656.2	4444.8	1234228.2	4878.8	892.3 (12.5)	230.3

All set of instances, but one, can be solved to optimality well within an hour time limit with our inequalities, whereas many problem instances cannot be solved within the time and memory limits with default CPLEX. For this special set of instances ($f = 500$ and $c = 20$) the average optimality gap at termination is one per cent with our branch-and-cut algorithm and 1.5% with default CPLEX. For most of the other problem instances, the average optimality gap at termination is over 5% with default CPLEX and the main reason for termination is the memory limit. As default CPLEX exhaustively searches over a million of nodes for almost every instance, it reaches the memory limit in less than half an hour. We note that the percentage gap improvement using our inequalities and an efficient separation heuristic is on average over 85% at the root node. This reduces the number of branch-and-cut tree nodes explored drastically, from over a million nodes to a few thousand nodes.

In summary,

(a) Inequalities with general integer coefficients on some of production, inventory and backlogging variables are necessary. Earlier work considers general integer coefficients only on a restricted choice of the production variables.
(b) The incorporation of the single-item inequalities (10) with the proposed separation heuristic improves the performance of the branch-and-cut algorithm significantly for the small-bucket multi-item problems.

6 Concluding Remarks

In this paper, we give a class of facets for ULSB that subsumes previously known classes of inequalities. We show that adding the proposed facets to the formulation gives an explicit description of the convex hull of solutions to ULSB in its natural space. In addition, we give the first polynomial-time combinatorial separation algorithm for the special case of our inequalities that are equivalent to those in [9].

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