

LOT SIZING WITH INVENTORY BOUNDS AND FIXED COSTS: POLYHEDRAL STUDY AND COMPUTATION

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ABSTRACT. We investigate the polyhedral structure of the lot-sizing problem with inventory bounds. We consider two models, one with linear cost on inventory, the other with linear and fixed costs on inventory. For both models, we identify facet-defining inequalities that make use of the inventory bounds explicitly and give exact separation algorithms. We also describe a linear programming formulation of the problem when the order and inventory costs satisfy the Wagner-Whitin nonspeculative property. We present computational experiments that show the effectiveness of the results in tightening the linear programming relaxations of the lot-sizing problem with inventory bounds and fixed costs.

Key words: Lot sizing, facets, separation algorithms, computation.

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1. INTRODUCTION

Given the demand for an item for each time period over a finite discrete horizon, the lot-sizing problem is to determine the order and inventory quantity in each time period so that the sum of order and inventory holding costs is minimized. In this paper we study the problem with linear and fixed costs on order as well as on inventory. Furthermore, we impose upper bounds on the inventory carried in each period. We refer to this problem as the *lot-sizing problem with bounded inventory*. Throughout we assume that there is no upper bound on the order quantity.

Motivation. Even though in almost every study on the lot-sizing problem inventory is assumed to be unbounded, for many practical applications the amount of inventory that is carried from one period to the next is bounded due to either physical constraints such as warehouse capacity or managerial policies.

Another aspect of the problem studied here, which has not received much attention in the literature, is the inventory fixed costs. Inventory fixed costs and capacities play a significant role in situations where warehousing is outsourced. The lease cost of storage space in an off-site warehouse is a fixed charge that cannot be treated as part of the standard variable holding cost, which is commonly taken as the cost of capital investment in inventory. Inventory fixed costs and capacities also arise in situations where manufacturers rent shelf space at the retailers. These fixed rental costs and limited storage capacities are considerations that cannot be overlooked.

It has been demonstrated in earlier studies (Pochet and Wolsey, 1991, Wolsey, 2002) that a good understanding of the polyhedral structure of single item lot-sizing problems can be very useful in solving more complicated problems, involving multiple products and stages. Since the

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single item lot-sizing polyhedron is contained as a fundamental substructure in those problems, investigating inventory bounds and fixed costs within the lot-sizing context is meaningful from a practical point of view.

Although the polyhedron of lot-sizing with inventory bounds and fixed costs is of interest in its own right, a secondary goal in this study is to gain a better understanding of the polyhedral structure of paths in capacitated fixed-charge networks. In this context, due to its simple path network representation, the lot-sizing problem with inventory bounds and fixed costs is the first natural problem to investigate.

Relevant literature. The lot-sizing problem lies at the core of production/order and inventory planning applications and has been studied extensively. Wagner and Whitin (1958) give an $O(n^2)$ algorithm for the lot-sizing problem with no bounds on order quantity or inventory (*uncapacitated lot-sizing problem*), where n is the number of time periods. Federgruen and Tzur (1991), Wagelmans et al. (1992), Aggarwal and Park (1993) give $O(n \log n)$ algorithms for this problem.

The first polyhedral study of the uncapacitated lot-sizing problem is due to Barany et al. (1984). They give a complete linear description of the convex hull of the solutions. Variants of the uncapacitated problem that include sales and safety stocks (Loparic et al., 2001), order/production lower bounds (Constantino, 1998), backlogging and start-ups (Pochet and Wolsey, 1988, Constantino, 1996, Agra and Constantino, 1999), piecewise linear concave costs (Aghezzaf and Wolsey, 1994) have been studied.

The lot-sizing problem with upper bounds on production/order quantities is \mathcal{NP} -hard (Florian et al., 1971). Pochet (1988), Miller et al. (2000), Loparic et al. (2003), Atamtürk and Munoz (2004) give inequalities for lot-sizing with order capacities and uncapacitated inventory. If order capacities are constant, the problem can be solved in polynomial time (Florian and Klein, 1980, Van Hoesel and Wagelmans, 1996). Leung et al. (1989), Pochet and Wolsey (1993) describe inequalities for this case. However, a complete linear description of the convex hull of the solutions for the constant-capacity lot-sizing is unknown. Atamtürk and Hochbaum (2001) study constant-capacity lot-sizing with capacity acquisition and subcontracting.

Belvaux and Wolsey (2000, 2001) discuss strong formulations and a specialized branch-and-cut system for practical lot-sizing problems. Recently, van Vyve and Ortega (2003) give a linear description of the convex hull of the solutions to uncapacitated lot-sizing with inventory fixed costs. This is the only polyhedral study we are aware of that considers inventory fixed costs.

In all of the studies mentioned above, inventory is assumed to be uncapacitated. Love (1973) proposes a polynomial algorithm for the lot-sizing problem with unbounded order quantity and bounded inventory. To the best of our knowledge, the only polyhedral study on lot-sizing that considers inventory capacities is Pochet and Wolsey (1994), in which the authors study lot-sizing problems under Wagner-Whitin nonspeculative cost structure. They give a linear programming formulation of lot-sizing problems with uncapacitated and constant-capacity production and inventory upper bounds.

Lot-sizing problems are special fixed-charge network flow problems. For uncapacitated fixed-charge network flows, Van Roy and Wolsey (1985) give inequalities that are based on path substructures. Ortega and Wolsey (2000) present a computational study on the performance of path and dicut inequalities in solving the uncapacitated fixed-charge network flow problem. Padberg et al. (1985), Stallaert (1997), Gu et al. (1999), Atamtürk (2001) describe valid inequalities for capacitated fixed-charge networks based on single-node relaxations. These inequalities do

not capture the path substructure of capacitated networks. Submodular inequalities of Wolsey (1989) take into account capacities, but not the fixed charges on arcs along a path.

Outline. In Section 2 we formally present the lot-sizing problem with inventory bounds and fixed costs and introduce the notation used in the paper. Section 3 is devoted to the special case with linear inventory costs. In Section 4 we study the polyhedron for linear and fixed inventory costs and extend the inequalities defined in Section 3 to this more general case. We give polynomial exact separation algorithms for these inequalities. In addition, we give a linear programming formulation for the lot-sizing problem with inventory bounds and fixed costs if order and inventory costs satisfy the Wagner-Whitin nonspeculative property. Section 5 summarizes our computational experiments on testing the effectiveness of the inequalities when used as cuts for solving the problem. We conclude with Section 6.

2. LOT-SIZING PROBLEM WITH BOUNDED INVENTORY

For a finite planning horizon n , given the demand d_t , variable order cost p_t , and fixed order cost f_t for time periods $t \in \{1, 2, \dots, n\}$; and inventory capacity u_t , variable inventory holding cost h_t , and fixed inventory holding cost g_t for $t \in \{0, 1, \dots, n\}$, the *lot-sizing problem with bounded inventory* (LSBI) is to determine the order quantity and inventory in each period so that the sum of order and inventory holding costs over the horizon is minimized. We assume that order quantity is unbounded, although given the demand, there is an implicit upper bound on the order quantity due to the inventory capacities. Throughout we let $[i, j] := \{t \in \mathbb{Z} : i \leq t \leq j\}$.

Let y_t denote the order quantity in time period t , i_t denote the inventory at the end of period t . Also let x_t and z_t be the fixed-charge variables for order and inventory in period t , respectively. Then LSBI can be formulated as

$$\begin{aligned}
 \text{(LSBI)} \quad & \min \quad \sum_{t=1}^n (f_t x_t + p_t y_t + g_t z_t + h_t i_t) + g_0 z_0 + h_0 i_0 \\
 & \text{s.t.} \quad \begin{array}{ll} i_{t-1} + y_t - i_t = d_t, & t \in [1, n] \\ 0 \leq i_t \leq u_t z_t, & t \in [0, n] \\ 0 \leq y_t \leq (d_t + u_t) x_t, & t \in [1, n] \end{array} \\
 & y \in \mathbb{R}^n, \quad x \in \{0, 1\}^n, \quad i \in \mathbb{R}^{n+1}, \quad z \in \{0, 1\}^{n+1}.
 \end{aligned}$$

Let $d_{t\ell} = \sum_{j=t}^{\ell} d_j$ for $t \in [1, \ell]$ and $d_{t\ell} = 0$ for $t > \ell$. We let \mathcal{Q} denote the convex hull of the feasible solutions to LSBI. Observe that for the optimization problem LSBI, if desired, by assigning sufficiently high values to h_0 and h_n , one may ensure that $i_0 = i_n = 0$ in an optimal solution. Therefore, for generality, we keep i_0 and i_n in the formulation for the polyhedral analysis. Throughout the paper we assume that the data of the model consists of rational numbers and satisfy the following:

Assumptions.

- (A1) $u_t > 0$ for $t \in [0, n]$,
- (A2) $u_{t-1} \geq d_t$ for $t \in [1, n]$,
- (A3) $u_{t-1} \leq d_t + u_t$ for $t \in [1, n]$,
- (A4) $d_t \geq 0$ for $t \in [1, n]$.

Assumptions (A1) and (A3) are made without loss of generality. None of the validity proofs use assumption (A2). Therefore, all inequalities in this paper are valid without (A2), which is used only in facet proofs for convenience. If $u_t = 0$ for $t \in [0, n]$, then $i_t = 0$ and the problem decomposes into two subproblems for $t \in [1, n-1]$. If $u_{t-1} < d_t$ for $t \in [1, n]$, then $x_t = 1$ in

every feasible solution. If $u_{t-1} > d_t + u_t$ for $t \in [1, n]$, then u_{t-1} can be reduced to $d_t + u_t$ without changing the feasible set of solutions. Finally, $d_t < 0$ is not sensible for the lot-sizing problem.

Definition 1. For a given point (y, x, i, z) in \mathcal{Q} , a consecutive sequence of time periods $[k, \ell]$ is called a *block* if $i_{k-1} \in \{0, u_{k-1}\}$, $i_\ell \in \{0, u_\ell\}$, and $0 < i_t < u_t$ for all $t \in [k, \ell - 1]$.

The block definition leads to a characterization of the extreme points of \mathcal{Q} and a polynomial dynamic programming algorithm (Love, 1973) for LSBI, which can be implemented in $O(n^2)$. We represent the four types of blocks by $[k, \ell]_{\beta}^{\alpha}$, where $\alpha \in \{0, u_{k-1}\}$ and $\beta \in \{0, u_\ell\}$. It follows from the network structure of the problem that (y, x, i, z) is an extreme point of \mathcal{Q} if and only if there is at most one period with positive order quantity in every block $[k, \ell]$. Consequently, the order quantity in such a period equals $d_{k\ell} + \beta - \alpha$.

Example 1. Suppose LSBI is given as $(d_1, \dots, d_5) = (11, 12, 13, 14, 15)$ and $(u_0, \dots, u_5) = (\infty, 30, \infty, \infty, \infty, 5)$. Then we can strengthen the inventory bounds as $(u_0, \dots, u_5) = (41, 30, 47, 34, 20, 5)$ to satisfy assumption **(A3)**. Figure 1 illustrates the four types of blocks in different extreme point solutions.

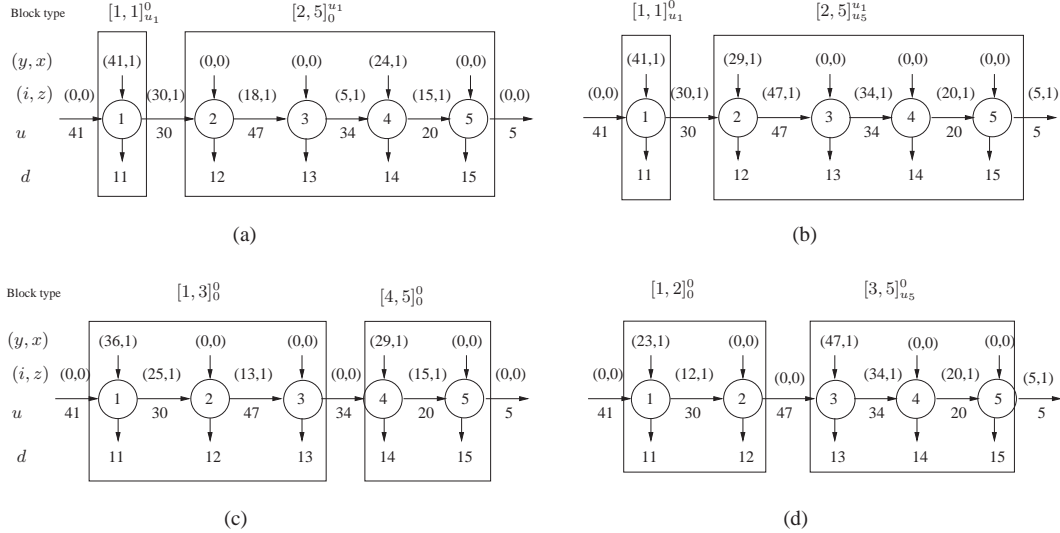


FIGURE 1. Four types of blocks in extreme point solutions of \mathcal{Q} .

The linear programming (LP) relaxation of LSBI has the same block structure; however, the fixed-charge variables take the values $x_t \in \{1, y_t/(d_t + u_t)\}$ and $z_t \in \{1, i_t/u_t\}$ and are typically highly fractional. The LP extreme point solution in Figure 1 (a) has $(x_1, \dots, x_5) = (1, 0, 0, 24/34, 0)$ and $(z_1, \dots, z_5) = (0, 1, 18/47, 5/34, 3/4, 0)$. \square

In Section 3 we consider the case with linear costs on inventory, i.e., the restriction of \mathcal{Q} , where inventory fixed-charge variables are one. We describe valid inequalities that cut off all extreme points of the LP relaxation with fractional order fixed-charge variables. In Section 4 we generalize these inequalities to incorporate the inventory fixed-charge variables as well.

Notation. We introduce the following notation, which will be used throughout the paper: For $1 \leq k \leq \ell \leq n$ let

$$p = \min\{t \in [k, \ell] : d_{kt} > u_{k-1}\} \quad (p = \ell + 1 \text{ if } u_{k-1} \geq d_{k\ell}),$$

$$q = \min\{t \in [k, \ell] : d_{kt} \geq u_{k-1}\} \quad (q = \ell + 1 \text{ if } u_{k-1} > d_{k\ell}),$$

$$r = \max\{t \in [k, p] : u_t < d_{(t+1)\ell}\} \quad (r = k - 1 \text{ if } u_t \geq d_{(t+1)\ell} \text{ for all } t \in [k, p]).$$

3. LINEAR INVENTORY COSTS

In this section we address the special case of LSBI with linear holding costs. When $g = 0$, the convex hull of the feasible set of interest is $\mathcal{P} = \{(y, x, i, z) \in \mathcal{Q} : z = 1\}$ or

$$\mathcal{P} = \text{conv} \left\{ \begin{array}{ll} i_{t-1} + y_t - i_t = d_t, & t \in [1, n] \\ 0 \leq i_t \leq u_t, & t \in [0, n] \\ 0 \leq y_t \leq (d_t + u_t)x_t, & t \in [1, n] \\ y \in \mathbb{R}^n, x \in \{0, 1\}^n, i \in \mathbb{R}^{n+1} \end{array} \right\}.$$

3.1. Uncapacitated inequalities. For the special case of \mathcal{P} with no inventory capacities Barany et al. (1984) give the so-called (ℓ, S) inequalities

$$(1) \quad \sum_{t \in S} y_t \leq \sum_{t \in S} d_{t\ell} x_t + i_\ell, \quad \text{where } S \subseteq [1, \ell] \text{ and } \ell \in [1, n].$$

They show that adding inequalities (1) to the LP relaxation suffices to describe the convex hull of feasible solutions. Example 1 illustrates that in the presence of inventory upper bounds, (ℓ, S) inequalities may be weak and are not sufficient to describe \mathcal{P} .

Example 1 (cont.) It is easy to check that extreme point solution of the LP relaxation with $(x_1, \dots, x_5) = (1, 29/59, 0, 0, 0)$ (in Figure 1(b)) cannot be cut off by any (ℓ, S) inequality (1). Observe that the (ℓ, S) inequality

$$y_1 + \dots + y_5 \leq 65x_1 + 54x_2 + 42x_3 + 29x_4 + 15x_5 + i_5$$

with $\ell = 5$ and $S = [1, 5]$ can be strengthened as

$$y_1 + \dots + y_5 \leq 41x_1 + 54x_2 + 42x_3 + 29x_4 + 15x_5 + i_5$$

since $y_1 \leq 11 + 30$. However, this inequality is dominated by the variable upper bound constraint $y_1 \leq 41x_1$ and inequality (1) with $\ell = 5$ and $S = [2, 5]$. Thus, if $d_t + u_t < d_{t\ell}$ for some $t \in S$, then inequality (1) and even the strengthening of (1)

$$\sum_{t \in S} y_t \leq \sum_{t \in S} \min\{d_{t\ell}, d_t + u_t\} x_t + i_\ell$$

are weak. □

3.2. Capacitated inequalities. In Section 3.1 we have illustrated that (ℓ, S) inequalities (1) may not cut off fractional LP extreme solutions if for a block incoming or outgoing inventory is at capacity. Motivated by this observation, we obtain new inequalities by saturating incoming and outgoing inventory variables for a block.

Example 1 (cont.) In order to derive a strong inequality that uses inventory upper bounds, we observe that due to u_1 , it is not possible to meet the total demand $d_2 + d_3 + d_4 = 39$ from inventory i_1 . Therefore, an order must be placed in periods 2, 3, or 4, which is stated by the inequality

$$x_2 + x_3 + x_4 \geq 1,$$

cutting off the fractional solution in Figure 1(b) that cannot be cut by uncapacitated inequalities. □

In general for any $1 \leq k \leq \ell \leq n$ such that $u_{k-1} < d_{k\ell}$, the ‘‘cut-set type’’ inequality

$$(2) \quad \sum_{t \in [k, \ell]} x_t \geq 1$$

is valid for \mathcal{P} . Notice that if $u_{k-1} < d_{k(\ell-1)}$, then (2) is dominated by $\sum_{t \in [k, \ell-1]} x_t \geq 1$; therefore, strong inequalities among (2) must satisfy $d_{k(\ell-1)} \leq u_{k-1} < d_{k\ell}$, implying that there are only at most n strong inequalities among (2). Next, we introduce inequalities that generalize (2).

The first class of inequalities is obtained by setting the incoming inventory variable i_{k-1} for $[k, \ell]$ to u_{k-1} for $u_{k-1} \leq d_{k\ell}$ (Figure 2). Then, due to the exogenous supply u_{k-1} in time period k , the effective total demand in periods $t, t+1, \dots, \ell$ is $\min\{d_{k\ell} - u_{k-1}, d_{t\ell}\}$ for $t \in [k, \ell]$. Also observing, in this case, that the order quantity in period t cannot exceed $d_{kt} - u_{k-1} + u_t$ due to the inventory capacity u_t , we obtain the following inequalities.

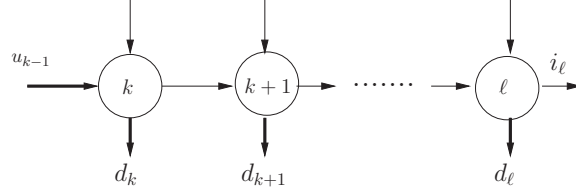


FIGURE 2. Saturating incoming inventory.

For $1 \leq k \leq \ell \leq n$ such that $u_{k-1} \leq d_{k\ell}$ and $S \subseteq [k, \ell]$, consider the inequality

$$(3) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\} x_t + i_\ell.$$

Proposition 1. *Inequality (3) is valid for \mathcal{P} .*

Proof. If $u_{k-1} = d_{k\ell}$, inequality follows from $i_{k-1} + \sum_{t \in S} y_t \leq d_{k\ell} + i_\ell = u_{k-1} + i_\ell$. Otherwise, let $p = \min\{t \in [k, \ell] : u_{k-1} < d_{kt}\}$. Observe that $d_{k\ell} - u_{k-1} \leq d_{t\ell}$ for $t \in [k, p]$ and $d_{k\ell} - u_{k-1} > d_{t\ell}$ for $t \in [p+1, \ell]$. For $(y, x, i) \in \mathcal{P}$, let $b = \max\{t \in S : d_{kt} + u_t - u_{k-1} < \min\{d_{k\ell} - u_{k-1}, d_{t\ell}\}\}$ and $x_t = 1$ (if no such t exists, let $b = k-1$) and let $h = \min\{t \in [b+1, \ell] \cap S : x_t = 1\}$ (if no such t exists, let $h = \ell+1$). If $h \leq p$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S} y_t &\leq d_{k\ell} + i_\ell = d_{kb} + d_{(b+1)\ell} + u_b - u_b + u_{k-1} - u_{k-1} + i_\ell \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\} x_t + i_\ell. \end{aligned}$$

The last inequality follows since $x_h = 1$, $d_{k\ell} - u_{k-1} \leq d_{h\ell}$ for $h \leq p$, $d_{(b+1)\ell} - u_b \leq d_{k\ell} - u_{k-1}$ (by assumption **(A3)**), $d_{k\ell} - u_{k-1} \geq 0$, and $d_{kt} + u_t - u_{k-1} \geq 0$ (by assumption **(A3)**).

On the other hand, if $h > p$, then

$$\begin{aligned} i_{k-1} + \sum_{t \in S} y_t &\leq d_{kb} + u_b + d_{h\ell} + i_\ell + u_{k-1} - u_{k-1} \\ &\leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\} x_t + i_\ell. \quad \square \end{aligned}$$

Remark 1. Note that assumption **(A2)** is not used to prove validity of inequality (3). If $u_{k-1} < d_k$ for some $k \in [1, n]$, then we write the inequality (3) for $k = \ell$ and $S = \{k\}$ as

$$i_{k-1} + y_k \leq u_{k-1} + (d_k - u_{k-1})x_k + i_k.$$

Adding the flow balance equality $d_k + i_k = y_k + i_{k-1}$, we get $d_k - u_{k-1} \leq (d_k - u_{k-1})x_k$ or $x_k \geq 1$.

Remark 2. If there exists $j \in S \cap [p + 1, \ell]$ with $u_j < d_{(j+1)\ell}$, then inequality (3) can be strengthened as

$$(4) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}, d_t + u_t\}x_t + i_\ell.$$

However, inequality (4) is weak, because it is dominated by

$$i_{k-1} + \sum_{t \in S \setminus \{j\}} y_t \leq u_{k-1} + \sum_{t \in S \setminus \{j\}} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell,$$

and the constraint $y_j \leq (d_j + u_j)x_j$. It follows that $u_j \geq d_{(j+1)\ell}$ for $j \in [p + 1, \ell]$ is a necessary facet condition for inequalities (3); see Proposition 13.

Remark 3. We note that inequalities (3) dominate inequalities (2). To see this, suppose $u_{k-1} < d_{k\ell}$, hence inequality (2) is valid for \mathcal{P} . Rewriting inequality (3) for $S = [k, \ell]$ by subtracting the aggregate flow balance equality

$$i_{k-1} + \sum_{t \in [k, \ell]} y_t = d_{k\ell} + i_\ell$$

we obtain

$$\sum_{t \in [k, \ell]} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t \geq d_{k\ell} - u_{k-1},$$

which is at least as strong as inequality (2). \square

The next class of inequalities is obtained by saturating incoming as well as outgoing inventory variables i_{k-1} and i_ℓ for a block $[k, \ell]$ (Figure 3). Then due the exogenous supply in time period k , the effective total demand in periods $t, t + 1, \dots, \ell$ is $\min\{d_{k\ell} - u_{k-1} + u_\ell, d_{t\ell} + u_\ell\}$ for $t \in [k, \ell]$. However, observing, in this case, that the order quantity in period t cannot exceed $d_{kt} - u_{k-1} + u_t$ due to the inventory capacity u_t , and that $d_{kt} - u_{k-1} + u_t \leq d_{k\ell} - u_{k-1} + u_\ell$ by assumption **(A3)**, we obtain the following inequalities.

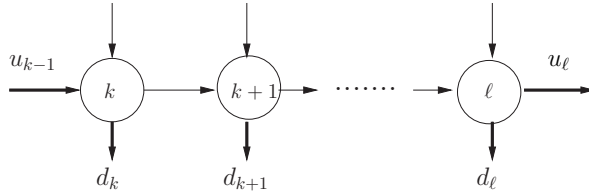


FIGURE 3. Saturating incoming and outgoing inventory.

For $k \geq 1$ and $S \subseteq [k, n]$ consider the inequality

$$(5) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} (d_{kt} - u_{k-1} + u_t)x_t.$$

Proposition 2. *Inequality (5) is valid for \mathcal{P} .*

Proof. Let $(y, x, i) \in \mathcal{P}$. If $x_t = 0$ for all $t \in S$, then inequality is trivially valid. Otherwise, let $\omega = \max\{t \in S : x_t = 1\}$. Then

$$i_{k-1} + \sum_{t \in S} y_t \leq d_{k\omega} + u_\omega + u_{k-1} - u_{k-1} \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t. \quad \square$$

Remark 4. Suppose $S \subseteq [k, \ell]$ for $\ell \geq k$. Then inequality

$$(6) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{t\ell} + u_\ell\} x_t$$

is dominated by inequality (5) with $S' = S \cap [k, q]$, where $q = \min\{t \in [k, \ell] : d_{kt} \geq u_{k-1}\}$ and inequalities $y_t \leq (d_t + u_t)x_t$ for $t \in S \setminus S'$ since $d_t + u_t \leq d_{t\ell} + u_\ell$ by assumption **(A3)**. It follows that $S \subseteq [k, q]$ is a necessary facet condition for inequalities (5); see Proposition 14. \square

In the Appendix we study the strength of the inequalities (1), (3), and (5) with respect to \mathcal{P} .

Example 1 (cont.) For $[k, \ell] = [2, 5]$ we have $p = q = 4$. Then all facet-defining inequalities (3) of \mathcal{P} for $[k, \ell] = [2, 5]$ with $|S| \leq 2$ are

$$\begin{aligned} i_1 + y_2 &\leq 30 + 24x_2 + i_5 \\ i_1 + y_3 &\leq 30 + 24x_3 + i_5 \\ i_1 + y_4 &\leq 30 + 24x_4 + i_5 \\ i_1 + y_2 + y_3 &\leq 30 + 24x_2 + 24x_3 + i_5 \\ i_1 + y_2 + y_4 &\leq 30 + 24x_2 + 24x_4 + i_5 \\ i_1 + y_3 + y_4 &\leq 30 + 24x_3 + 24x_4 + i_5 \\ i_1 + y_2 + y_5 &\leq 30 + 24x_2 + 15x_5 + i_5 \\ i_1 + y_3 + y_5 &\leq 30 + 24x_3 + 15x_5 + i_5 \\ i_1 + y_4 + y_5 &\leq 30 + 24x_4 + 15x_5 + i_5. \end{aligned}$$

On the other hand, all facet-defining inequalities (5) of \mathcal{P} for $[k, \ell] = [2, 5]$ are

$$\begin{aligned} i_1 &\leq 30 \\ i_1 + y_2 &\leq 30 + 29x_2 \\ i_1 + y_3 &\leq 30 + 29x_3 \\ i_1 + y_4 &\leq 30 + 29x_4 \\ i_1 + y_2 + y_3 &\leq 30 + 29x_2 + 29x_3 \\ i_1 + y_2 + y_4 &\leq 30 + 29x_2 + 29x_4 \\ i_1 + y_3 + y_4 &\leq 30 + 29x_3 + 29x_4. \end{aligned}$$

\square

3.3. Fractional extreme points of the LP relaxation. Now we show that the inequalities described in Sections 3.1 and 3.2 cut all fractional extreme point solutions of the linear programming relaxation of LSBI. Recall that extreme point solutions are characterized by blocks $[k, \ell]_\beta^\alpha$ for $1 \leq k \leq \ell \leq n$, where $\alpha \in \{0, u_{k-1}\}$ and $\beta \in \{0, u_\ell\}$. Note that in an extreme point solution there can be at most one period with positive order quantity in a block.

Proposition 3. *Inequalities (1), (3), and (5) cut all fractional extreme point solutions of the LP relaxation of LSBI.*

Proof. Let (y, x, i) be an extreme point solution of the LP relaxation of LSBI. For a block type $[k, \ell]_\beta^0$, there exists one period $t \in [k, \ell]$ with $y_t = d_{t\ell} = d_{k\ell}$. (Note that $t = k$ if $d_k > 0$.) Therefore, the order fixed-charge variable $x_t = d_{k\ell}/(d_t + u_t)$. If $0 < x_t < 1$, then the inequality (1)

$$y_t \leq d_{t\ell}x_t + i_\ell$$

cuts off this point.

For a block type $[k, \ell]_0^{u_{k-1}}$, there exists one period $t \in [k, \ell]$ with $y_t = d_{k\ell} - u_{k-1}$. Therefore, the order fixed-charge variable $x_t = (d_{k\ell} - u_{k-1})/(d_t + u_t)$. If $0 < x_t < 1$, then inequality (3)

$$i_{k-1} + y_t \leq u_{k-1} + (d_{k\ell} - u_{k-1})x_t + i_\ell$$

cuts off (y, x, i) .

For a block type $[k, \ell]_{u_\ell}^{u_{k-1}}$, there exists one period $t \in [k, \ell]$ with $y_t = d_{k\ell} + u_\ell - u_{k-1}$. Therefore, the order fixed-charge variable $x_t = (d_{k\ell} + u_\ell - u_{k-1})/(d_t + u_t)$. If $0 < x_t < 1$, then inequality (5)

$$i_{k-1} + y_t \leq u_{k-1} + (d_{k\ell} + u_\ell - u_{k-1})x_t$$

cuts off (y, x, i) .

Finally, for a block type $[k, \ell]_{u_\ell}^0$, there exists one period $t \in [k, \ell]$ with $y_t = d_{k\ell} + u_\ell$. In this case we must have $d_{k\ell} + u_\ell = d_t + u_t$. Therefore, the order fixed-charge variable is integral. \square

3.4. Separation. Now we discuss how to find inequalities (3) violated by a given point $(y, x, i) \in \mathbb{R}_+^{3n+1}$. Let $\Delta = \sum_{t \in S} (y_t - \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t)$. For fixed k and ℓ such that $u_{k-1} < d_{k\ell}$, Δ is maximized by placing $t \in [k, \ell]$ in S if and only if $y_t > \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t$. Since this can be done in linear time for each k and ℓ , the observation leads to an $O(n^3)$ separation algorithm for inequalities (3). Next we improve the computational complexity of separation.

Theorem 4. *There is an $O(n^2 \log n)$ algorithm to solve the separation problem for inequalities (3). There are $O(n \log n)$ algorithms to solve the separation problems for inequalities (1) and (5).*

Proof. For $k \in [1, n-1]$ let $p(k) = \min\{t \in [k, \ell] : u_{k-1} < d_{kt}\}$ ($p(k) = \ell + 1$ if $u_{k-1} \geq d_{k\ell}$). Also let $r' = \max\{t \in [1, \ell] : u_t < d_{(t+1)\ell}\}$. So in inequality (3) x_t for $t \in S$ has coefficient $d_{kt} + u_t - u_{k-1}$ if $t \in [k, \min\{r', p(k)\}]$, $d_{k\ell} - u_{k-1}$ if $t \in [r' + 1, p(k)]$, and $d_{t\ell}$ if $t \in [\max\{r', p(k)\} + 1, \ell]$. (Observe that in order to obtain a strong inequality we must have $S \cap [p(k) + 1, r'] = \emptyset$ for $r' > p(k)$.)

In the rest of the discussion we fix ℓ and decrement k from ℓ to 1. A set S that maximizes Δ for all $k \in [1, \ell]$ will be computed in $O(n \log n)$. The following observations are due to assumption **(A3)**: (a) $p(k-1) \leq p(k)$, (b) $d_{k\ell} - u_{k-1} \leq d_{(k-1)\ell} - u_{k-2}$.

Let $S(k)$ be a subset of $[k, \ell]$ maximizing Δ for k (and fixed ℓ). Observation (b) implies that if $t \in [k, r']$ and $r' \leq p(k)$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$; also if $t \in [r' + 1, p(k)]$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$ such that $t \leq p(j)$; however, t may be in $S(j)$ if $p(j) < t$. It is clear that if $t \in [\max\{r', p(k)\} + 1, \ell]$ and $t \notin S(k)$, then $t \notin S(j)$ for any $j \leq k$. Also note that $S(j) \cap [p(j) + 1, r'] = \emptyset$ for $r' > p(k)$.

Let T be the list of $t \in [k, r']$ for $r' \leq p(k)$ such that $\psi_t := y_t/x_t - (d_{kt} + u_t - u_{k-1}) > 0$, sorted by ψ_t , T' be the list of $t \in [r' + 1, p(k)]$ such that $\varepsilon_t := y_t/x_t - (d_{k\ell} - u_{k-1}) > 0$, sorted by ε_t , and finally T'' be the (unsorted) list of $t \in [\max\{r', p(k)\} + 1, \ell]$ such that $y_t - d_{t\ell}x_t > 0$. For $k' = k - 1$, if $k' \leq r'$, then we place k' into T in $O(\log n)$ time by binary search on ψ_t and delete permanently from T all $t \leq r'$ with $\psi'_t = \psi_t - (d_{k-1} - u_{k-2} + u_{k-1}) \leq 0$. On the other hand if $r' < k'$, then we place k' into T' in $O(\log n)$ time by binary search on ε_t . We delete from T' all $t > p(k')$ and all $t \in T'$ such that $\varepsilon'_t = \varepsilon_t - (d_{k-1} - u_{k-2} + u_{k-1}) \leq 0$ permanently. Finally $t \in [p(k') + 1, p(k)]$ is placed in T'' permanently if and only if $y_t - d_{t\ell}x_t > 0$. Thus $S(k') = T \cup T' \cup T''$.

Keeping separate sums $y(T) = \sum_{t \in T} y_t$, $x(T) = \sum_{t \in T} x_t$ and $y(T') = \sum_{t \in T'} y_t$, $x(T') = \sum_{t \in T'} x_t$ allows us to compute Δ for $S(k)$ in constant average time. Since variables are inserted into and deleted from each list at most once, all updates in Δ for the changes in the lists can be

done in $O(n)$. For the elements remaining in the lists, updates can be done in constant time per iteration. For instance, let $\bar{T} \subseteq T$ be the set of elements that remain from iteration to the next. Since the coefficients of x_i for all $i \in \bar{T}$ reduce by $d_{k-1} - u_{k-2} + u_{k-1}$, the required change for them in Δ is $-(d_{k-1} - u_{k-2} + u_{k-1})x(\bar{T})$.

There are at most n insertions to T and T' , all of which take total $O(n \log n)$. Also there are at most n deletions from T' and insertions to T'' , all of which can be done in total $O(n)$. Since Δ can be updated in a total of $O(n)$, the complexity of the algorithm is $O(n \log n)$ for each ℓ , giving an overall $O(n^2 \log n)$ separation algorithm for inequalities (3).

From the above algorithm it also follows that the separation for the simpler inequalities (5) can be done in $O(n \log n)$ by fixing $\ell = n$ only. Separation for inequalities (1) can be done similarly in $O(n \log n)$ by keeping the elements of T'' sorted by $y_t/x_t - d_{t\ell}$ and incrementing ℓ from 1 to n . \square

4. FIXED COSTS ON INVENTORY

In this section we expand the study to incorporate the inventory fixed-charge variables. Thus the polyhedron studied in this section is

$$\mathcal{Q} = \text{conv} \left\{ \begin{array}{ll} i_{t-1} + y_t - i_t = d_t & t \in [1, n] \\ 0 \leq i_t \leq u_t z_t & t \in [0, n] \\ 0 \leq y_t \leq (d_t + u_t)x_t & t \in [1, n] \\ y \in \mathbb{R}^n, x \in \{0, 1\}^n, i \in \mathbb{R}^{n+1}, z \in \{0, 1\}^{n+1} \end{array} \right\}.$$

4.1. Uncapacitated inequalities. As before, we first describe inequalities that do not consider the inventory upper bounds. For $t \in [0, n-1]$ let b_t be the first time period after t with positive demand, i.e., $b_t = \min\{k \in [t+1, n] : d_k > 0\}$. Then, since d_{b_t} is either satisfied from inventory at t or from an order in a later period up to b_t , inequality

$$(7) \quad z_t + x_{t+1} + \cdots + x_{b_t} \geq 1$$

is valid for \mathcal{Q} .

For $0 \leq k \leq \ell \leq n$ and $S \subseteq [k, \ell]$ inequality

$$(8) \quad i_{k-1} + \sum_{t \in S} y_t \leq d_{k\ell} z_{k-1} + \sum_{t \in S} d_{t\ell} x_t + i_\ell,$$

where i_{-1} and z_{-1} are taken as 0, generalizes (7) as well as the (ℓ, S) inequality (1). When there are no inventory upper bounds, van Vyve and Ortega (2003) show that it suffices to add inequalities (8) to the LP relaxation of the uncapacitated lot-sizing problem with inventory fixed costs to obtain the convex hull of the feasible solutions. Inequalities (8) are not sufficient to describe \mathcal{Q} ; however, they define facets of \mathcal{Q} as shown in the Appendix.

4.2. Capacitated inequalities. In order to obtain strong inequalities that use the inventory fixed costs as well as upper bounds, we introduce the inventory fixed-charge variables into inequalities (3) and (5).

For $1 \leq k \leq \ell \leq n$ such that $u_{k-1} \leq d_{k\ell}$, let $S \subseteq [k, \ell]$ and $T := \{t_1, t_2, \dots, t_\tau\} \subseteq [k-1, \ell-1]$. For $j \in T$ let $s(j) = \min\{t \in S \cup \{\ell+1\} : t > j\}$. Consider the inequality

$$(9) \quad i_{k-1} + \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t (1 - z_t) \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\} x_t + i_\ell,$$

where

$$(10) \quad \gamma_{t_j} = \begin{cases} u_{k-1} - d_{kt_\tau}, & \text{if } j = \tau \text{ and } s(t_j) > p, \\ d_{(t_j+1)(t_{j+1})}, & \text{if } j < \tau \text{ and } s(t_j) = s(t_{j+1}), \\ d_{(t_j+1)(s(t_j)-1)}, & \text{if } (j < \tau \text{ and } s(t_j) < s(t_{j+1})) \text{ or } (j = \tau \text{ and } s(t_j) \leq p). \end{cases}$$

Example 1 (cont.) Consider inequality (3) for $[k, \ell] = [2, 5]$ (where $p = 4$) and $S = \{2\}$:

$$(11) \quad i_1 + y_2 \leq 30 + 24x_2 + i_5,$$

which is facet-defining for the restriction $z_t = 1$ for $t \in [0, 5]$. Now if $z_2 = 0$, then we observe that $i_1 + y_2 \leq 12 = d_2$. So,

$$i_1 + y_2 + 18(1 - z_2) \leq 30 + 24x_2 + i_5,$$

is valid for \mathcal{Q} , where $T = \{2\}$, $t_\tau = 2$, $s(2) = \ell + 1 = 6 > p$ so $\gamma_2 = u_1 - d_2 = 18$.

Alternatively, for inequality (11), if $z_3 = 0$, then $i_1 + y_2 \leq 25 = d_{23}$. Therefore,

$$(12) \quad i_1 + y_2 + 5(1 - z_3) \leq 30 + 24x_2 + i_5,$$

where $\gamma_3 = u_1 - d_{23} = 5$. Now for (12) if $z_2 = 0$, then $i_1 + y_2 + 5(1 - z_3) \leq 12 + 5 = d_2 + \gamma_3$ and inequality

$$i_1 + y_2 + 13(1 - z_2) + 5(1 - z_3) \leq 12 + 5 = d_2 + \gamma_3,$$

which is equivalent to inequality (9) with $T = \{2, 3\}$, $s(2) = s(3) = 6$, and $\gamma_2 = d_3 = 13$. \square

This example illustrates that γ in (10) are sequence dependent lifting coefficients for z_t , $t \in T$. Next, we give a direct proof of validity.

Proposition 5. *Inequality (9) is valid for \mathcal{Q} .*

Proof. Let $(y, x, i, z) \in \mathcal{Q}$. If $z_t = 1$ for all $t \in T$, validity follows from Proposition 1. Therefore, we assume that $z_t = 0$ for some $t \in T$. For a given point $(y, x, i, z) \in \mathcal{Q}$, for $1 \leq k \leq \ell \leq n$, $S \subseteq [k, \ell]$ and $T \subseteq [k - 1, p - 1]$, if $z_t = 0$ for some $t \in T$, then let $V = \{v_1, v_2, \dots, v_\kappa\} = \cup_{j \in S \cup \{\ell+1\}} (\min\{t \in T : z_t = 0, s(t) = j\})$. Without loss of generality assume that $v_1 < v_2 < \dots < v_\kappa$; $v_0 = 0$ and $v_{\kappa+1} = n$. Thus $S \cap [s, p - 1]$ is partitioned as $(S_1, S_2, \dots, S_\kappa, S_{\kappa+1})$, where $S_j = \{t \in S : v_{j-1} < t \leq v_j\}$, and T is partitioned as $(T_1, T_2, \dots, T_\kappa, T_{\kappa+1})$, where $T_j = \{t \in T : s(v_{j-1}) \leq t < s(v_j)\}$ with $s(v_0) = k - 1$ and $s(v_{\kappa+1}) = n$. Observe that for $t_j \in T$ if $s(t_j) \leq p$,

$$(13) \quad \sum_{t \in T \cap [t_j, s(t_j)-1]} \gamma_t \leq d_{(t_j+1)(s(t_j)-1)}$$

and if $s(t_j) > p$,

$$(14) \quad \sum_{t \in T \cap [t_j, s(t_j)-1]} \gamma_t \leq u_{k-1} - d_{kt_j}.$$

First, we express the left-hand side of (9) as the sum of $\kappa + 1$ terms: $i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t(1 - z_t)$, $\sum_{t \in S: v_{j-1} < t \leq v_j} y_t + \sum_{t \in T: s(v_{j-1}) \leq t < s(v_j)} \gamma_t(1 - z_t)$ for $j \in [2, \kappa]$ and $\sum_{t \in S: t > v_\kappa} y_t$. Next, we show the following relations for the first κ of these terms.

Observe from (13) that if $s(v_1) \leq p$,

$$(15) \quad i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t(1 - z_t) \leq d_{kv_1} + d_{(v_1+1)(s(v_1)-1)} = d_{k(s(v_1)-1)},$$

since $i_{k-1} + \sum_{t \in S: t \leq v_1} y_t \leq d_{kv_1}$ when $z_{v_1} = i_{v_1} = 0$. Similarly, if $s(v_1) > p$ (in which case $\kappa = 1$), then

$$(16) \quad i_{k-1} + \sum_{t \in S: t \leq v_1} y_t + \sum_{t \in T: t < s(v_1)} \gamma_t(1 - z_t) \leq d_{kv_1} + d_{v_1 t_\tau} + u_{k-1} - d_{kt_\tau} = u_{k-1}.$$

For $v_j, j \in [2, \kappa]$ with $s(v_j) \leq p$, we have

$$(17) \quad \sum_{t \in S: v_{j-1} < t \leq v_j} y_t + \sum_{t \in T: s(v_{j-1}) \leq t < s(v_j)} \gamma_t(1 - z_t) \leq d_{(s(v_{j-1}))v_j} + d_{(v_j+1)(s(v_j)-1)} \\ = d_{(s(v_{j-1}))s(v_j)-1}.$$

On the other hand, for v_κ if $\kappa \geq 2$ and $s(v_\kappa) > p$, then

$$(18) \quad \sum_{t \in S: v_{\kappa-1} \leq t \leq v_\kappa} y_t + \sum_{t \in T: s(v_{\kappa-1}) \leq t < s(v_\kappa)} \gamma_t(1 - z_t) \leq d_{s(v_{\kappa-1})v_\kappa} + d_{(v_\kappa+1)t_\tau} + u_{k-1} - d_{kt_\tau} \\ = u_{k-1} - d_{k(s(v_{\kappa-1})-1)}.$$

To see the validity of inequalities (9), we consider six cases. Let b and h be defined as in the proof of Proposition 1. First suppose that $h \leq p$. If $s(v_\kappa) > p$, then from (15)–(18), we have

$$i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t \leq u_{k-1} + \sum_{t \in S: t > v_\kappa} y_t \\ \leq u_{k-1} + d_{(p+1)\ell} + i_\ell \\ \leq u_{k-1} + d_{k\ell} - u_{k-1} + i_\ell \quad (\text{since } u_{k-1} < d_{kp}) \\ \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell,$$

where the second inequality follows, since $\sum_{t \in S: t > v_\kappa} y_t \leq d_{(p+1)\ell} + i_\ell$ if $s(v_\kappa) > p$ and the last inequality follows, since $x_t \geq 0$ for $t \in S$, $x_h = 1$ and for $h \leq p$, $d_{k\ell} - u_{k-1} \leq \min\{d_{kt} + u_t - u_{k-1}, d_{t\ell}\}$.

Similarly, if $s(v_\kappa) \leq p$, then from (15) and (17)

$$i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t \leq d_{k(s(v_\kappa)-1)} + d_{s(v_\kappa)\ell} + i_\ell \\ = d_{k\ell} + u_{k-1} - u_{k-1} + i_\ell \\ \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell.$$

Now suppose that $h > p$. If $s(v_\kappa) > p$ and $b > v_\kappa$, then

$$i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq b} y_t + \sum_{t \in S: t \geq h} y_t \\ \leq u_{k-1} + d_{s(v_\kappa)b} + u_b + d_{h\ell} + i_\ell \\ = u_{k-1} + d_{kb} + u_b - u_{k-1} + d_{h\ell} + i_\ell + (u_{k-1} - d_{k(s(v_\kappa)-1)}) \\ \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell,$$

where the last inequality follows since $u_{k-1} < d_{k(s(v_\kappa)-1)}$ for $s(v_\kappa) > p$. If $s(v_\kappa) \leq p$ and $b > v_\kappa$, then

$$\begin{aligned}
 i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq b} y_t + \sum_{t \in S: t \geq h} y_t \\
 \leq d_{k(s(v_\kappa)-1)} + d_{s(v_\kappa)b} + u_b + d_{h\ell} + i_\ell + u_{k-1} - u_{k-1} \\
 \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell.
 \end{aligned}$$

If $s(v_\kappa) > p$ and $b \leq v_\kappa$, then

$$\begin{aligned}
 i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t \geq h} y_t \leq u_{k-1} + d_{h\ell} + i_\ell \\
 \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell.
 \end{aligned}$$

Finally if $s(v_\kappa) \leq p$ and $b \leq v_\kappa$, then

$$\begin{aligned}
 i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: t > v_\kappa} y_t \leq d_{k(s(v_\kappa)-1)} + d_{h\ell} + i_\ell \\
 \leq u_{k-1} + \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t + i_\ell.
 \end{aligned}$$

□

In the next class of inequalities we introduce inventory fixed-charge variables into inequalities (5). For $1 \leq k \leq \ell \leq n$ let $S \subseteq [k, \ell]$ and $T := \{t_1, t_2, \dots, t_\tau\} \subseteq [k-1, p-1]$, where $p = \min\{t \in [k, \ell] : u_{k-1} < d_{kt}\}$. For $t \in T$ let $s(j) = \min\{t \in S \cup \{\ell+1\} : t > j\}$. Then consider the inequality

$$(19) \quad i_{k-1} + \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t(1 - z_t) \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t,$$

where γ_t is defined as in (10).

Proposition 6. *Inequality (19) is valid for \mathcal{Q} .*

Proof. For $(y, x, i, z) \in Q$ if $x_t = 0$ for all $t \in S$, inequality follows from $i_{k-1} + \sum_{t \in T} \gamma_t(1 - z_t) \leq u_{k-1}$ due to (15)–(18). Otherwise, let $\omega = \max\{t \in S : x_t = 1\}$. Using the definitions in the proofs of Propositions 1 and 5, if $s(v_\kappa) > p$ and $\omega > v_\kappa$, then

$$\begin{aligned}
 i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq \omega} y_t \\
 \leq u_{k-1} + d_{s(v_\kappa)\omega} + u_\omega + (d_{kp} - u_{k-1}) \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t.
 \end{aligned}$$

If $s(v_\kappa) \leq p$ and $\omega > v_\kappa$, then

$$\begin{aligned}
 i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) + \sum_{t \in S: v_\kappa < t \leq \omega} y_t \\
 \leq d_{k(s(v_\kappa)-1)} + d_{s(v_\kappa)\omega} + u_\omega + u_{k-1} - u_{k-1} \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t.
 \end{aligned}$$

Finally if $\omega \leq v_\kappa$, then

$$i_{k-1} + \sum_{t \in S: t \leq v_\kappa} y_t + \sum_{t \in T} \gamma_t(1 - z_t) \leq u_{k-1} \leq u_{k-1} + \sum_{t \in S} (d_{kt} + u_t - u_{k-1})x_t.$$

□

In the Appendix we study the strength of the inequalities (9) and (19) with respect to \mathcal{Q} .

Example 1 (cont.) All facet-defining inequalities (19) of \mathcal{Q} for $[k, \ell] = [2, 5]$ with $|S| \leq 1$ are

$$\begin{aligned} i_1 + 12(1 - z_1) + 13(1 - z_2) + 5(1 - z_3) &\leq 30 \\ i_1 + 12(1 - z_1) + 18(1 - z_2) &\leq 30 \\ i_1 + 25(1 - z_1) + 5(1 - z_3) &\leq 30 \\ i_1 + 30(1 - z_1) &\leq 30 \\ i_1 + 18(1 - z_2) + y_2 &\leq 30 + 29x_2 \\ i_1 + 13(1 - z_2) + 5(1 - z_3) + y_2 &\leq 30 + 29x_2 \\ i_1 + 12(1 - z_1) + 5(1 - z_3) + y_3 &\leq 30 + 29x_3 \\ i_1 + 25(1 - z_1) + y_4 &\leq 30 + 29x_4 \\ i_1 + 12(1 - z_1) + 13(1 - z_2) + y_4 &\leq 30 + 29x_4. \end{aligned}$$

Observe that these inequalities are extensions of inequalities (5) with inventory fixed-charge variables. The first four inequalities are extensions of inequality $i_1 \leq 30$, the next two are extensions of $i_1 + y_2 \leq 30 + 29x_2$ and so on.

On the other hand, all facet-defining inequalities (9) of \mathcal{Q} for $[k, \ell] = [2, 5]$ with $|S| = 1$ are

$$\begin{aligned} i_1 + 18(1 - z_2) + y_2 &\leq 30 + 24x_2 + i_5 \\ i_1 + 13(1 - z_2) + 5(1 - z_3) + y_2 &\leq 30 + 24x_2 + i_5 \\ i_1 + 12(1 - z_1) + 5(1 - z_3) + y_3 &\leq 30 + 24x_3 + i_5 \\ i_1 + 25(1 - z_1) + y_4 &\leq 30 + 24x_4 + i_5 \\ i_1 + 12(1 - z_1) + 13(1 - z_2) + y_4 &\leq 30 + 24x_4 + i_5. \end{aligned}$$

Since the extreme point solutions of the LP relaxation have the same block structure with or without inventory fixed costs, it follows from Proposition 3 and assumption **(A3)** that inequalities (8), (9), and (19), which are generalizations of inequalities (1), (3), and (5), cut off all fractional extreme points of the LP relaxation of LSBI.

4.3. Separation. In this section we study separation problems for inequalities (9) and (19) and show that they can be solved in polynomial time.

Theorem 7. *There is an $O(n^4)$ algorithm to solve the separation problem for inequalities (9) and an $O(n^3)$ algorithm to solve the separation problem for inequalities (19).*

Proof. Rewriting inequality (9) as

$$(20) \quad \sum_{t \in S} y_t + \sum_{t \in T} \gamma_t(1 - z_t) - \sum_{t \in S} \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}, d_{t\ell}\}x_t \leq u_{k-1} - i_{k-1} + i_\ell,$$

for each k and ℓ such that $1 \leq k \leq \ell \leq n$ and $u_{k-1} < d_{k\ell}$, we wish to determine sets $S \subseteq [k, \ell]$ and $T \subseteq [k-1, p-1]$ such that the left-hand-side of (20) is maximized for a given point (y, x, i, z) .

We formulate this problem as a longest path problem on a directed acyclic network. The arcs on a longest path determine the sets S and T .

Consider a directed graph $G = (V, A)$ with a single source $(k-1) \in V$ and a single sink $(\ell+1) \in V$. Let $t \in V$ for $t \in [k, \ell] \setminus [p+1, r']$, where $r' = \max\{t \in [k, \ell] : u_t < d_{(t+1)\ell}\}$ ($r' = k-1$ if $u_t \geq d_{(t+1)\ell}$ for all $t \in [k, p]$).

There is an arc $(t, j)^a \in A$, $a \in \{T, S\}$, for each $t \in V$ and $j \in V$ such that $k-1 \leq t < j \leq p-1$ so that if the path includes arc $(t, j)^a$, then j is included in both S and T if $a = S$, j is included in T if $a = T$ ($k-1$ is assumed to be in T , since $\{k-1\} \cup (S \cap [k, p-1]) \subseteq T \subseteq [k-1, p-1]$ for strong inequalities). Also for $k \leq t \leq p-1 < j \leq \ell+1$ with $t, j \in V$, arc $(t, j)^S$ is in A and if the path includes this arc, then $j \in S$. For $p \leq t < j \leq \ell+1$ with $t, j \in V$, arc $(t, j)^S$ is in A and if the path includes this arc, then $j \in S$. Finally let $((k-1), p)^S$ be an arc in A if the path includes this arc, then $j \in S$. (Note that there is no arc $(k, j)^S$ for $j > p$, since we must have $S \cap [k, p] \neq \emptyset$ from condition 6 of Proposition 13.)

Now we assign lengths to the arcs. Let the length of arc $(t, j)^T$ for $k-1 \leq t < j \leq p-1$ be

$$c_{(t,j)^T} = d_{(t+1)j}(1 - z_t).$$

Also let the length of arc $(t, j)^S$ be

$$c_{(t,j)^S} = \begin{cases} d_{(t+1)(j-1)}(1 - z_t) + y_j - \min\{d_{kt} + u_t - u_{k-1}, d_{k\ell} - u_{k-1}\}x_j, & \text{if } k-1 \leq t < j \leq p, \\ (u_{k-1} - d_{kt})(1 - z_t) + y_j - d_{j\ell}x_j, & \text{if } k \leq t < p < j \leq \ell+1, \\ y_j - d_{j\ell}x_j, & \text{if } p \leq t < j \leq \ell+1, \end{cases}$$

where we use $y_{\ell+1} - d_{(\ell+1)\ell}x_{\ell+1} = 0$, for simplicity of notation. It is easy to see that the total length of a path from vertex $(k-1)$ to vertex $(\ell+1)$ equals the left-hand-side of the corresponding inequality (20).

Different arc types and costs for $k \leq j < t < p < v < w \leq \ell$ are illustrated in Figure 4. For simplicity not all the arcs in the subgraph induced by j, t, v, w is shown in the figure.

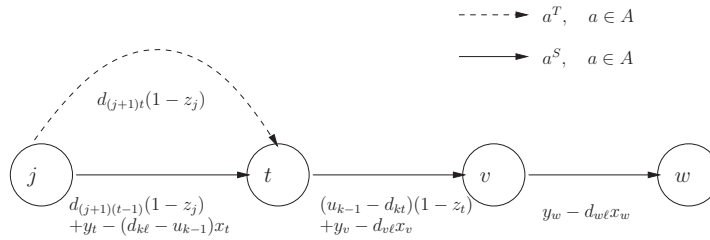


FIGURE 4. Network for separation for inequalities (9) ($k \leq j < t < p < v < w \leq \ell$).

Note that G is a directed acyclic graph with $O(n)$ vertices and $O(n^2)$ arcs. Therefore, the longest path in G can be found in $O(n^2)$ time (see Ahuja et al. (1993)). Since we solve a longest path problem for each k and ℓ such that $u_{k-1} < d_{k\ell}$, the overall complexity of the separation algorithm for inequalities (9) is $O(n^4)$. The graph G can be preprocessed to remove low length parallel arcs between two nodes, but this does not change the complexity of the algorithm.

It follows that separation for inequalities (19) is done $O(n^3)$ by letting $\ell = n$. \square

Separation of inequalities (8) is done in $O(n \log n)$ similar to separation for inequalities (1).

4.4. A special case: Wagner-Whitin nonspeculative costs. In this section we consider the special case of LSBI, in which the objective function satisfies the *Wagner-Whitin* nonspeculative property, i.e., $p_t + h_t \geq p_{t+1}$ for all $t \in [0, n]$, where $p_0 = p_{n+1} = 0$. For this case Pochet and Wolsey (1994) give an LP formulation for the lot-sizing problem with inventory bounds, but no inventory fixed costs. On the other hand, van Vyve and Ortega (2003) give an LP formulation for the lot-sizing with inventory fixed costs, but no bounds. In this section, we generalize these results to the lot-sizing problem with inventory bounds and fixed costs by observing that the intersection of the previously given LP formulations is an integral polyhedron.

Unlike in the general cost case, if Wagner-Whitin nonspeculative property holds, then there is an optimal *inventory-minimal* solution; that is, the demand in each period $t \in [1, n]$ is satisfied either from inventory or by ordering in that period, but not both, because otherwise, since the order quantity is unbounded, we can increase y_t until $i_{t-1} = 0$ without increasing the objective value.

Proposition 8. *If $p_t + h_t \geq p_{t+1}$ for all $t \in [0, n]$, where $p_0 = p_{n+1} = 0$, then $\min\{py + fx + hi + gz : (y, x, i, z) \in \mathcal{Q}\}$ has an optimal solution that satisfies $i_{t-1}y_t = 0$ for all $t \in [1, n]$.*

Thus if the costs satisfy the Wagner-Whitin nonspeculative property, then there exist an optimal solution that consists of blocks of type $[k, \ell]_0^0$ only, which implies that, in this case, LSBI can be solved as the uncapacitated lot-sizing problem by simply ignoring solutions with blocks $[k, \ell]_0^0$ that do not satisfy inventory upper bounds.

Next we describe a formulation of LSBI over the inventory-minimal solutions. Note that (y, x, i, z) is an inventory-minimal solution if and only if for all $t \in [1, n]$ we have $i_{t-1} = d_{tj}$ for some $j \in [t, n]$. Let ρ_{tk} be a binary variable such that $\rho_{tk} = 1$ if $i_t \geq d_{(t+1)k} > 0$ and $\rho_{tk} = 0$ otherwise for $k \in [t+1, n]$. Then inventory-minimal solutions can be modeled as $i_t = \sum_{k \in [t+1, n]} d_k \rho_{tk}$ and $\rho_{tk} = (1 - \sum_{j \in [t+1, k]} x_j)^+$ for $k \in [b_t, n]$, where b_t is the first period after t with positive demand, i.e., $b_t = \min\{k \in [t+1, n] : d_k > 0\}$. Observe that the second set of equalities imply $\rho_{tk} \geq \rho_{t(k+1)} \geq \dots \geq \rho_{tn}$; hence, $i_t = d_{(t+1)j}$ for some $j \in [t+1, n]$ (see Pochet and Wolsey (1994)).

Furthermore, for an inventory-minimal solution, $i_t \leq u_t z_t$ if and only if (24) $z_t \geq \rho_{tb_t}$ and (23) $\sum_{j \in [t+1, \ell_t]} x_j \geq 1$, where ℓ_t is the first time period after t in which demand cannot be completely satisfied by i_t , i.e., $\ell_t = \min\{k \in [t+1, n] : u_t < d_{(t+1)k}\}$. In order to see this, suppose $i_t \leq u_t z_t$ holds. Then, $d_{(t+1)(\ell_t-1)} \leq u_t < d_{(t+1)\ell_t}$, and we have (23). Also $z_t = 0$ implies $i_t = 0$ and from $i_t = \sum_{k \in [t+1, n]} d_k \rho_{tk}$ (inventory minimality), it follows that $\rho_{tk} = 0$ for all $k \in [t+1, n]$. Thus we have (24) as well. For the other direction, suppose (23)–(24) hold. From (23), $x_k = 1$ for some $k \in [t+1, \ell_t]$. Then for an inventory-minimal solution, from $\rho_{tk} = (1 - \sum_{j \in [t+1, k]} x_j)^+$, we have $\rho_{tk} = \dots = \rho_{tn} = 0$ and $i_t \leq d_{(t+1)(k-1)} \leq u_t$. From (24), if $z_t = 0$, then $\rho_{tb_t} = 0$, which, by $\rho_{tk} = (1 - \sum_{j=t+1}^k x_j)^+$ implies that $x_j = 1$ for some $j \in [t+1, b_t]$. Then, again by inventory minimality, $\rho_{tb_t} = \dots = \rho_{tn} = 0$ and $i_t = 0$ as $d_{(t+1)b_{t-1}} = 0$.

After eliminating the order variables y_t by substituting $d_t + i_t - i_{t-1}$, the inventory costs become $h'_t = p_t + h_t - p_{t+1}$ and the objective contains the constant term $K = \sum_{t \in [1, n]} p_t d_t$.

Since $h' \geq 0$ due to the Wagner-Whitin nonspeculative costs, the problem can be stated as

$$\begin{aligned}
 \min \quad & \sum_{t \in [1, n]} f_t x_t + \sum_{t \in [0, n]} (g_t z_t + h'_t i_t) + K \\
 (21) \quad & i_t \geq \sum_{k \in [t+1, n]} d_k \rho_{tk} \quad t \in [0, n-1], \\
 (22) \quad & \rho_{tk} + \sum_{j \in [t+1, k]} x_j \geq 1 \quad t \in [0, n-1], k \in [b_t, n], \\
 (23) \quad & \sum_{j \in [t+1, \ell_t]} x_j \geq 1 \quad t \in [0, n-1], \\
 (24) \quad & z_t - \rho_{tb_t} \geq 0 \quad t \in [0, n-1], \\
 (25) \quad & \rho_{tk} \geq 0 \quad t \in [0, n-1], k \in [t+1, n] \\
 (26) \quad & 0 \leq x_t \leq 1 \quad t \in [1, n], \\
 (27) \quad & 0 \leq z_t \leq 1 \quad t \in [0, n], \\
 (28) \quad & i \in \mathbb{R}_+^{n+1}, x \in \{0, 1\}^n, z \in \{0, 1\}^{n+1}, \rho \in \{0, 1\}^{\binom{n}{2}}.
 \end{aligned}$$

Pochet and Wolsey (1994) show that the constraint matrix for (22)–(23) and (25)–(27) is totally unimodular. Since each z_t appears once in (24), the following statement holds.

Proposition 9. *Let $w = (\rho, x, z)$ and let $Aw \geq b$ represent the constraints (22)–(27). The constraint matrix A is totally unimodular.*

Since the right-hand-side of the constraints (22)–(27) is integral and each inventory variable i_t appears only once in constraints (21), the extreme points of

$$\mathcal{Q}_{WW} = \{(x, i, z, \rho) : (x, i, z, \rho) \text{ satisfies (21) – (27)}\},$$

are the inventory-minimal solutions of LSBI.

Corollary 10. *The linear program $\min\{fx + h'i + gz + K : (x, i, z, \rho) \in \mathcal{Q}_{WW}\}$ is an extended formulation of LSBI with Wagner-Whitin nonspeculative costs.*

Projecting out the auxiliary variables ρ in \mathcal{Q}_{WW} and reintroducing order variables y , we obtain a linear programming formulation of the problem with the original variables.

Theorem 11. *LSBI with Wagner-Whitin nonspeculative costs can be formulated as the following linear program*

$$\begin{aligned}
 \min\{ & py + fx + hi + gz : (y, x, i, z) \text{ satisfies (29) – (35)}\}, \text{ where} \\
 (29) \quad & \sum_{j \in [t, k]} (y_j - d_{jk} x_j) \leq i_k \quad 1 \leq t \leq k \leq n, \\
 (30) \quad & \sum_{j \in [t, \ell_{t-1}]} x_j \geq 1 \quad t \in [1, n], \\
 (31) \quad & z_{t-1} + \sum_{j \in [t, b_{t-1}]} x_j \geq 1 \quad t \in [1, n], \\
 (32) \quad & i_{t-1} + y_t = d_t + i_t \quad t \in [1, n], \\
 (33) \quad & 0 \leq y_t \quad t \in [1, n], \\
 (34) \quad & 0 \leq x_t \leq 1 \quad t \in [1, n], \\
 (35) \quad & 0 \leq z_t \leq 1 \quad t \in [0, n].
 \end{aligned}$$

Note that inequality (29) is a special case of (ℓ, S) inequalities with $S = [k, \ell]$, inequality (30) is a special case of inequalities (3) with $S = [k, \ell_k]$ and inequality (31) is a special case of inequalities (8) with $S = [k, b_{k-1}]$.

5. COMPUTATIONAL RESULTS

In order to test the effectiveness of the inequalities described in Sections 3 and 4 in solving LSBI, we implement a branch-and-cut algorithm that incorporates these inequalities and perform computational experiments. All computations are done on a 2 GHz Pentium 4/Linux workstation with 1 GB main memory.

The data used in the experiments has the following properties: Demands are generated from integer uniform distribution between 0 and 30. Inventory upper bounds are generated from integer uniform distribution between 30 and $30(c + 1)$, where $c \geq 0$ is a parameter to determine tightness of capacity. Order costs are generated from integer uniform distribution between 4 and 24 and holding cost is equal to 1 for every period. Let f be the ratio of order fixed cost to variable inventory cost and g be the ratio of order fixed cost to inventory fixed cost. In order to observe the effect of varying capacity and cost parameters on the computations, we let $c \in \{2, 5, 10, 20\}$, $f \in \{1000, 2000, 5000\}$ and $g \in \{2, 5, 10\}$ and generate five random instances for each combination.

The first set of experiments is on solving LSBI with linear inventory costs. The problem instances are solved with the MIP solver of CPLEX¹ Version 8.1 Beta using first only the uncapacitated inequalities (1) (**Unc**) and then using inequalities (1), (3), and (5) (**Cap**) as cutting planes in the branch-and-cut tree. Given a fractional point, we find violated inequalities (1), (3), and (5) using the separation algorithms discussed in Section 3.4. CPLEX cuts are disabled in these experiments in order to isolate the impact due to the inequalities discussed in this paper. However, in order to see how CPLEX cuts would perform we also solve the same instances with the default settings of CPLEX (**Def**) without adding any user cuts.

A summary of these experiments for instances with 120 and 180 time periods is reported in Tables 1 and 2. In the second column of the tables we report the average integrality gap, which is $100 \times (\mathbf{zub} - \mathbf{zinit})/\mathbf{zub}$, where \mathbf{zinit} is the objective value of the initial LP relaxation and \mathbf{zub} is the objective value of the best integer solution. In the third column we compare the average percentage improvement of the integrality gap at the root node (**% gapimp**), which is $100 \times (\mathbf{zroot} - \mathbf{zinit})/(\mathbf{zub} - \mathbf{zinit})$, where \mathbf{zroot} is the objective value of the LP at the root node after the cuts are added. Columns **cuts** and **nodes** compare the average number of cuts added, and the average number of branch-and-cut nodes explored, respectively. In the last column we report the average CPU time elapsed (in seconds) if the problem is solved within one hour time limit. Otherwise, we also report, in parenthesis, the average percentage gap between the best lower bound and the best integer solution found in the search tree (**endgap**) and the number of instances that have positive end gap (**unslv**). Except for percentage gaps, the entries in the tables are rounded to a nearest integer.

In Tables 1 and 2 we observe that the initial integrality gap increases with the length of the planning horizon and the ratio of order fixed cost to linear inventory cost f . The capacitated cuts close almost all of the integrality gap for most instances and reduce the computational effort dramatically. Indeed, many instances are solved without the need for branching and the rest require only a few branches to prove optimality. While many of the instances cannot be solved to optimality in an hour of CPU time with default CPLEX and with uncapacitated lot-sizing inequalities (1), when the capacitated cuts (3) and (5) are added, all of them are solved within a few seconds. Note that uncapacitated lot-sizing inequalities (1) result in small gap improvement at the root node and have high **endgap** on average. As expected, the root gap improvement with uncapacitated inequalities increases with maximum capacity ratio c .

¹CPLEX is a trademark of ILOG, Inc.

TABLE 1. LSBI with linear inventory holding costs, $n = 120$.

f	c	gap	% gapimp			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
1000	2	17.8	73.5	22.1	95.2	195	42	482	93655	1220825	7	360	3574 (6.9:2)	2
	5	17.1	77.7	34.5	96.9	131	60	504	9109	579048	2	31	2285 (1.7:2)	3
	10	17.5	81.1	51.2	99.6	97	81	529	13495	65678	1	37	335	2
	20	17.8	83.5	67.4	99.7	72	111	525	1666	2793	1	5	17	2
2000	2	19.6	60.7	14.3	95.9	198	35	560	700111	1388269	3	2207 (1.6:3)	3614 (10.5:5)	2
	5	18.6	54.2	21.7	95.1	150	49	492	701982	1151008	6	1884 (0.9:2)	3613 (5.3:5)	3
	10	18.7	67.0	36.2	99.6	108	73	533	308372	675722	1	867	2637 (2.4:3)	3
	20	17.2	80.7	53.8	99.5	80	120	569	447	16085	1	2	87	2
5000	2	22.5	44.9	6.2	96.9	192	28	528	1182451	1554172	2	2936 (6.5:4)	3614 (15.7:5)	2
	5	21.0	60.3	10.7	94.8	194	35	493	501072	1225022	3	1218 (1.9:1)	3137 (9.3:4)	3
	10	20.9	53.2	19.3	100.0	384	58	591	666193	875933	0	2184 (1.7:3)	2938 (5.6:3)	3
	20	18.0	68.5	29.9	100.0	189	88	667	31493	166887	0	115	924 (0.9:1)	2
Average		18.9	67.1	28.5	97.8	166	65	539	350837	743454	2	988 (1.1:1)	2231 (4.9:3)	2

TABLE 2. LSBI with linear inventory costs, $n = 180$.

f	c	gap	% gapimp			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
1000	2	17.5	73.9	17.6	99.1	271	74	879	358306	701225	4	2404 (0.7:2)	3608 (14.2:5)	9
	5	17.1	78.7	30.3	99.7	201	85	852	140510	629417	2	854 (0.3:1)	3605 (8.9:5)	10
	10	17.4	82.5	47.5	99.3	182	114	818	28232	457648	2	181	3606 (4.2:5)	9
	20	18.5	77.7	67.6	99.6	128	152	839	106858	112079	1	559	1400	12
2000	2	19.1	55.1	10.0	99.8	230	58	972	620669	730623	1	2897 (6.4:4)	3612 (20.6:5)	9
	5	18.2	68.7	18.7	99.7	186	72	821	397850	705143	1	1527 (2.7:2)	3614 (13.8:5)	11
	10	18.8	64.8	32.1	99.7	190	100	881	402263	469990	1	1859 (1.6:2)	3612 (8.4:5)	11
	20	19.0	70.0	51.0	99.9	169	158	738	175411	314706	0	1043	3589 (2.5:4)	13
5000	2	22.2	44.4	5.4	100.0	237	46	956	818413	787667	1	3612 (9.6:5)	3613 (24.9:5)	9
	5	20.4	56.6	9.6	100.0	269	53	956	647729	747869	0	2656 (4.5:3)	3614 (17.8:5)	12
	10	21.2	56.4	19.7	100.0	195	75	1053	694361	660580	0	2890 (5.1:4)	3613 (12.4:5)	10
	20	19.7	52.8	22.0	100.0	347	103	1016	551326	537426	0	2901 (2.6:4)	3611 (8.3:5)	14
Average		19.1	65.1	27.6	99.7	217	91	898	411827	57119	1	1949 (2.8:2)	3425 (11.3:5)	11

The second set of computational experiments is on solving LSBI with linear and fixed inventory costs. In these experiments we test the marginal impact of the inequalities with fixed-charge variables (8), (9), and (19) over inequalities that do not make use of the fixed-charge variables (1), (3), and (5) as well as the marginal impact of inequalities that exploit the inventory bounds and fixed charges (9) and (19) over inequalities that only use inventory fixed charges (8). Violated inequalities (8), (9), and (19) are found as described in Section 4.3. A summary of these experiments for instances with 120 and 180 time periods is reported in Tables 3 and 4. The columns labelled as **Def** are for default CPLEX, **Unc** are for runs with uncapacitated inequalities (8), **Lin** are for runs with inequalities (1), (3), and (5), and finally **Cap** are for runs with inequalities (8), (9), and (19). For these instances we let the order fixed cost to inventory fixed cost ratio g equal to 10. The initial integrality gap of the LP relaxation of LSBI with inventory fixed costs is larger than for the case with linear inventory costs only. We observe that a significant effort is spent in strengthening the LP relaxations with the cutting planes; the numbers of cuts added are in the thousands.

A comparison of column **Unc** and **Lin** with **Cap** show that inequalities that ignore either the capacities or the inventory fixed-charge variables are not sufficient to solve the problem efficiently. We see that the addition of inequalities (9) and (19) close almost all of the integrality gap consistently for varying order fixed cost inventory ratios and capacities. Consequently, all of the instances that could not be solved within an hour of CPU time without inequalities (9) and (19), are solved in a few minutes when they are added. These experiments clearly demonstrate the positive impact of inequalities (9) and (19) over uncapacitated inequalities (8) that use fixed charge variables as well as inequalities (8), (9), and (19) over inequalities (1), (3), and (5) that do not incorporate fixed charge variables.

Table 5 summarizes the effect of changing the order fixed cost to inventory fixed cost ratio g . We observe that the smaller is the ratio, the bigger is the initial integrality gap. However, although the initial gap is higher, CPLEX cuts (especially flow cover cuts) lead to a better gap improvement at the root node for the instances with small g . It is remarkable that even with 97-98% gap improvement at the root node, thousands of branch-and-cut nodes are needed to obtain provably optimal solutions. Yet, for the case $g = 2$ and $c = 10$, default CPLEX finishes computations more quickly even when many nodes are explored. We also note that, the performance of our branch-and-cut algorithm is not affected significantly by the tightness of the inventory upper bounds, whereas the performance of the alternative branch-and-cut algorithms degrade with lower c .

Our experiments with problems that satisfy Wagner-Whitin nonspeculative cost structure show that the compact linear program in Theorem 11 is solved faster than the cutting plane algorithm that starts with the standard formulation (LSBI). Finally, we compare the performance of a branch-and-bound algorithm that solves the strengthened formulation given by (29)–(35) (**Compact**) with the performance of the branch-and-cut algorithm (**Cap**) that uses inequalities (8), (9), and (19) on instances that does not satisfy Wagner-Whitin nonspeculative cost structure. For this experiment, $n = 180$, $g = 10$ and order variable costs drawn from discrete uniform distribution between 4 and 104. We increase the variance in the order variable costs so that they are not approximately nonspeculative. Recall that the strengthened formulation contains a subset of the inequalities generated by the branch-and-cut algorithm. In Table 6 we report the initial integrality gap of the test problems (**gap**), the number of branch-and-cut nodes explored and the time to solve the problems. For the branch-and-cut algorithm, under columns **rgap** and **cuts**, we report the gap at the root node when all violated inequalities (8), (9), and (19) are added and the number of cuts added, respectively. We observe that the initial integrality gap at the root node for the compact formulation is much lower than that of the standard formulation. However, with the addition of violated inequalities (8), (9), and (19) to the standard formulation, the root integrality gap is almost closed. Consequently, the number of nodes explored is smaller for the branch-and-cut algorithm. We observe that the tighter the inventory upper bounds, the larger is the integrality gap and the longer is the solution time for the compact formulation. Also as the ratio of order fixed cost to variable inventory cost (f) increases, the initial integrality gap of the standard formulation as well as the number of explored nodes and the elapsed time increase for the branch-and-cut algorithm. The branch-and-cut algorithm is faster than the branch-and-bound algorithm for the compact formulation for smaller f .

TABLE 3. LSBI with fixed and linear inventory costs, $n = 120$ and $g = 10$.

f	c	gap	% gapimp			cuts			nodes			time(endgap:unslv)									
			Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap							
1000	2	24.5	82.8	49.6	67.4	99.5	373	1210	529	2395	116844	66587	758430	1	671	3615	(9.4:5)	3611	(3.0:5)	51	
	5	26.0	88.8	64.0	65.0	99.8	325	1334	590	2376	55381	49587	652147	1	294	3617	(4.8:5)	3610	(2.9:5)	57	
	10	28.1	88.1	76.6	64.1	99.9	288	1486	689	2531	192749	26742	548199	1	731	(0.4:1)	2693	(1.5:2)	3610	(3.7:5)	47
	20	30.6	90.1	87.2	62.6	99.9	265	1686	721	2030	25848	7231	460097	0	139	905	(0.2:1)	3610	(4.8:5)	52	
2000	2	27.6	65.2	44.1	65.6	100.0	391	1276	549	2108	827225	65867	753632	1	3612	(5.9:5)	3614	(14.8:5)	3612	(4.2:5)	54
	5	28.9	81.5	62.0	61.7	99.6	345	1327	543	2438	447568	50818	675077	4	2206	(0.4:2)	3613	(6.9:5)	3613	(4.8:5)	60
	10	31.0	82.6	72.9	60.9	100.0	322	1586	748	2320	404856	35623	475499	0	1455	(1.4:2)	2921	(3.7:4)	3612	(5.9:5)	56
	20	33.1	85.8	85.8	60.7	99.9	305	1812	921	2333	210006	9894	328056	1	961	(0.2:1)	1343		3610	(6.4:5)	58
5000	2	30.8	65.4	42.9	64.0	100.0	433	1345	512	2552	900144	62652	789896	1	3610	(6.8:5)	3613	(16.7:5)	3613	(6.0:5)	63
	5	31.4	63.9	57.2	59.3	99.9	356	1343	549	2755	919685	58119	820783	0	3612	(6.5:5)	3613	(10.5:5)	3613	(6.9:5)	59
	10	34.0	72.3	65.5	57.0	100.0	417	1484	641	2256	764752	46794	583127	0	2891	(4.9:3)	3613	(7.9:5)	3611	(8.6:5)	60
	20	35.1	78.1	80.1	56.4	100.0	525	1838	818	2113	560411	21298	337941	0	2893	(2.0:4)	2680	(1.9:3)	3611	(8.5:5)	57
Average		30.1	78.7	65.7	62.1	99.9	362	1477	651	2351	452122	41768	598574	1	1923	(2.4:2)	2978	(6.5:4)	3611	(5.5:5)	56

TABLE 4. LSBI with fixed and linear inventory costs, $n = 180$ and $g = 10$.

f	c	gap	% gapimp			cuts			nodes			time(endgap:unslv)									
			Def	Unc	Lin	Cap	Def	Unc	Lin	Cap	Def	Unc	Lin	Cap							
1000	2	24.7	84.5	45.6	70.8	99.4	592	2598	747	6388	279834	22117	505488	15	2909	(1.1:3)	3629	(13.9:5)	3613	(4.5:5)	418
	5	26.1	88.9	56.8	66.2	99.9	498	2778	885	7198	83519	20199	410016	1	753		3629	(10.6:5)	3614	(5.4:5)	391
	10	28.7	90.1	69.4	64.3	99.6	467	3074	940	7777	93670	16162	410060	3	926	(0.0:1)	3627	(7.0:5)	3612	(6.6:5)	359
	20	31.4	86.0	84.4	63.6	99.9	413	3546	1079	6454	345818	9770	323822	1	2780	(0.6:2)	3624	(1.9:5)	3612	(7.6:5)	410
2000	2	27.7	67.8	41.0	68.3	99.7	518	2851	680	4735	430593	21937	541781	4	2965	(7.7:4)	3630	(18.8:5)	3613	(6.1:5)	420
	5	29.1	76.5	54.6	63.9	99.9	476	2929	819	7081	368271	20142	405146	1	2424	(4.4:3)	3628	(13.3:5)	3610	(7.5:5)	433
	10	31.9	75.4	67.8	61.4	99.9	441	3365	956	5653	466267	12844	336865	2	3117	(4.7:4)	3631	(8.5:5)	3611	(8.7:5)	409
	20	34.8	74.3	76.6	60.1	99.9	375	3871	1109	4699	609782	9226	209676	2	2916	(4.9:4)	3632	(5.8:5)	3612	(10.1:5)	468
5000	2	31.0	64.6	41.3	66.7	99.9	536	2891	732	5134	410827	20719	481684	2	2930	(10.5:4)	3629	(20.3:5)	3614	(7.6:5)	432
	5	31.8	71.1	54.2	61.5	100.0	498	2914	824	3983	459227	21050	419505	0	2923	(7.3:4)	3629	(14.7:5)	3612	(9.1:5)	389
	10	34.5	71.5	61.9	57.9	100.0	386	3242	863	5768	562321	17503	385868	1	2898	(7.2:4)	3631	(12.4:5)	3613	(10.6:5)	378
	20	36.9	71.2	72.3	56.7	100.0	379	3755	1263	5720	595220	13545	219399	0	2930	(7.2:4)	3655	(8.4:5)	3613	(12.9:5)	437
Average		30.7	76.8	60.5	63.5	99.9	465	3151	908	5882	392115	17101	387443	3	2539	(4.6:3)	3631	(11.3:5)	3612	(8.1:5)	412

TABLE 5. LSBI with fixed and linear inventory costs, $n = 180$ and $f = 5000$.

g	c	gap	% gapimp			cuts			nodes			time (endgap:unslv)		
			Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap	Def	Unc	Cap
10	2	30.5	59.3	40.5	100.0	491	2998	5636	565010	21855	2	3610 (11.4:5)	3631 (20.8:5)	501
	5	32.3	78.7	49.0	100.0	462	2904	10692	457550	20259	1	2899 (4.5:4)	3630 (17.9:5)	490
	10	35.1	65.9	60.5	100.0	433	3329	4974	685967	18336	1	3610 (9.9:5)	3635 (13.3:5)	457
	20	39.0	70.5	71.9	100.0	542	4292	7763	649701	11277	1	3611 (8.5:5)	3660 (9.2:5)	484
5	2	40.3	70.2	62.2	100.0	539	4004	4762	554394	15671	2	3611 (10.7:5)	3630 (15.8:5)	471
	5	43.7	75.5	71.7	100.0	441	3899	6373	664511	16873	1	3612 (8.3:5)	3629 (12.0:5)	492
	10	47.8	73.9	77.8	100.0	434	4181	7647	719258	14012	2	3611 (9.4:5)	3654 (9.7:5)	545
	20	52.6	75.8	86.0	99.9	430	4771	6027	665906	9885	1	3599 (8.4:4)	3637 (5.8:5)	631
2	2	59.9	95.1	85.0	100.0	572	4817	6544	266314	11101	2	2329 (0.8:3)	3631 (8.5:5)	562
	5	64.5	95.8	91.9	100.0	485	4835	5416	174766	10936	1	1128 (0.6:1)	3629 (4.5:5)	572
	10	68.5	98.0	92.7	100.0	452	4934	5568	9169	12206	1	82	3631 (4.3:5)	602
	20	72.5	97.4	97.3	100.0	424	5282	8120	145534	8847	0	759 (0.2:1)	3631 (1.2:5)	682
Average		48.9	79.7	73.8	100.0	475	4187	6627	463173	14272	1	2705 (6.1:4)	3636 (10.3:5)	541

In summary, these computational experiments show that

1. inequalities that do not take into account inventory capacities result in a poor performance of the branch-and-cut algorithm;
2. similarly, inequalities that do not take into account inventory fixed-charge variables also result in a poor performance of the algorithm;
3. however, inequalities that exploit inventory bounds as well as fixed costs are very effective in strengthening the LP relaxations of the lot-sizing problem with inventory bounds and fixed costs consistently for a wide range of cost and capacity parameters;
4. consequently, they are very useful in solving the problem efficiently.

6. CONCLUDING REMARKS

In this paper, we study the facial structure of the polyhedron of lot-sizing with inventory upper bounds and fixed costs (LSBI). We first define facet-defining inequalities for the special case with linear inventory costs and then extend them to incorporate inventory fixed-charge variables. We give polynomial time separation algorithms and a linear programming formulation of LSBI under the Wagner-Whitin nonspeculative cost structure. The computational experiments suggest that the inequalities are very effective in solving the lot-sizing problem with inventory upper bounds and fixed costs.

The inequalities described here and extensions of them may be effective in solving more complicated production/order and inventory planning problems that contain inventory upper bounds and fixed costs as a substructure. Other classes of strong inequalities for LSBI are described in Küçüküyavuz (2004). Currently we are exploring how the results for LSBI can be used for solving general capacitated fixed-charge network flow problems.

TABLE 6. LSBI with fixed and linear inventory costs, compact formulation.

f	c	Compact			Cap				
		gap	nodes	time	gap	rgap	cuts	nodes	time
1000	2	2.0	12199	765	19.3	0.1	7112	9	66
	5	1.7	4649	401	20.0	0.1	3806	1	40
	10	1.2	1126	127	20.5	0.4	10192	1	17
	20	1.0	136	44	21.4	0.1	4315	1	16
2000	2	1.9	10637	791	22.9	0.3	10128	9	138
	5	1.7	4417	396	24.2	0.1	4472	13	131
	10	1.4	1297	156	25.3	0.1	8690	1	71
	20	1.2	278	60	26.6	0.1	13479	2	61
5000	2	1.2	6368	641	26.0	0.2	4956	23	414
	5	1.6	4201	417	28.5	0.3	5872	47	560
	10	1.5	701	119	30.0	0.3	5169	17	296
	20	0.9	129	51	31.4	0.3	20754	2	105
Average		1.4	3845	331	24.7	0.2	8245	10	159

7. APPENDIX: STRENGTH OF THE INEQUALITIES

In this section we give conditions under which the proposed inequalities are facet-defining. We introduce the following notation, which will be used throughout this section: For $1 \leq k \leq \ell \leq n$ and nonempty $S \subseteq [k, \ell]$ let

$$\begin{aligned} s &= \min\{t \in S\}, \\ s' &= \max\{t \in S\}, \\ \hat{s} &= \min\{t \in [s, \ell + 1] \setminus S\} \\ \bar{s} &= \max\{t \in S : t \leq p\} \quad (\bar{s} = k - 1 \text{ if } S \cap [k, p] = \emptyset). \end{aligned}$$

Also let e_t and r_t be the unit vectors corresponding to x_t and y_t for $t \in [1, n]$ respectively and q_t be the unit vector corresponding to z_t for $t \in [0, n]$, and r_0 and v_n be the unit vectors for i_0 and i_n . Finally, let $\epsilon > 0$ be an infinitesimally small number.

Under assumptions **(A1)** and **(A2)**, it is easy to check that the dimension of \mathcal{P} is $2n + 1$.

Proposition 12. *Inequality (1) is facet-defining for \mathcal{P} if and only if*

1. if $\ell < n$, then $S \neq \emptyset$,
2. $u_s > d_{(s+1)\ell}$, where $s = \min\{t \in S\}$,
3. $d_{t\ell} > 0$ for some $t \in S$ and if $d_{t\ell} = 0$ for $t \leq \ell$, then $t \in S$,
4. $u_{s-1} > d_{s(\hat{s}-1)}$, where $\hat{s} = \min\{t \in [s, \ell + 1] \setminus S\}$,
5. if $\ell < n$, then $u_{s'} \geq d_{(s'+1)(\ell+1)}$, where $s' = \max\{t \in S\}$.

Proof. (Necessity)

1. Since for $\ell < n$, $d_{\ell+1}(1 - x_{\ell+1}) \leq i_\ell$ and $x_{\ell+1} \leq 1$ dominate inequality (1) with $S = \emptyset$, we must have $S \neq \emptyset$.
2. Assume for contradiction that $u_s \leq d_{(s+1)\ell}$. Then adding the $(\ell, S \setminus \{s\})$ inequality, $\sum_{t \in S \setminus \{s\}} y_t \leq \sum_{t \in S \setminus \{s\}} d_{t\ell} x_t + i_\ell$, and the capacity inequality $y_s \leq (d_s + u_s)x_s$ gives an inequality at least as strong as (1) since $d_s + u_s \leq d_{s\ell}$. Therefore, from **(A3)**, we must have $u_t > d_{(t+1)\ell}$ for all $t \in [s, \ell - 1]$.
3. If $d_{t\ell} = 0$ for some $t \in [s, \ell] \setminus S$ then the $(\ell, S \cup \{t\})$ inequality dominates (1). If $d_{t\ell} = 0$ for all $t \in S$, then inequality (1) is implied by the aggregated demand constraint $i_{s-1} + \sum_{t \in [s, \ell]} y_t = i_\ell$.

4. If $u_{s-1} < d_{s(\hat{s}-1)}$, then there exists a period $j \in [s+1, \hat{s}-1]$ such that $u_{s-1} < d_{sj}$. Then, the cut-set inequality

$$(36) \quad \sum_{t \in [s, j]} x_t \geq 1$$

is valid. Now inequality (1)

$$\sum_{t \in [s, j]} y_t \leq \sum_{t \in [s, j]} d_{tj} x_t + i_j,$$

and the flow balance inequality

$$i_j + \sum_{t \in S \cap [j+1, \ell]} y_t \leq d_{(j+1)\ell} + i_\ell$$

imply

$$\begin{aligned} \sum_{t \in S} y_t &\leq \sum_{t \in [s, j]} d_{tj} x_t + d_{(j+1)\ell} + i_\ell \\ &\leq \sum_{t \in [s, j]} d_{tj} x_t + \sum_{t \in [s, j]} d_{(j+1)\ell} x_t + i_\ell \leq \sum_{t \in S} d_{t\ell} x_t + i_\ell, \end{aligned}$$

where the second to last inequality follows from inequality (36).

On the other hand if $u_{s-1} = d_{s(\hat{s}-1)}$, then summing inequality (3)

$$i_{s-1} + \sum_{t \in S} y_t \leq u_{s-1} + \sum_{t \in S} \min\{d_{s\ell} - u_{s-1}, d_{t\ell}\} x_t + i_\ell,$$

inequality (1)

$$\sum_{t \in [s, \hat{s}-1]} y_t \leq \sum_{t \in [s, \hat{s}-1]} d_{t(\hat{s}-1)} x_t + i_{\hat{s}-1}$$

and the balance equality

$$i_{\hat{s}-1} + d_{s(\hat{s}-1)} = i_{s-1} + \sum_{t \in [s, \hat{s}-1]} y_t$$

we obtain

$$\sum_{t \in S} y_t \leq \sum_{t \in S} d_{t\ell} x_t + i_\ell.$$

5. Since the facet induced by (1) is different from $x_{\ell+1} = 1$, it contains a point (x, y, i) such that $x_{\ell+1} = 0$ and, by feasibility, $i_\ell \geq d_{\ell+1}$. Since this point satisfies (1) at equality, $y_j = d_{j(\ell+1)}$ for some $j \in S$, which, by feasibility, implies that $u_{s'} \geq d_{(s'+1)(\ell+1)}$.

(*Sufficiency*) In order to prove sufficiency we exhibit $2n + 1$ affinely independent points on the face defined by (1). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities.) Let e_t and r_t be the unit vectors corresponding to x_t and y_t for $t \in [1, n]$ respectively and r_0 and v_n be the unit vectors for i_0 and i_n . If $\hat{s} \leq \ell$, then consider the point

$$w_0 = \sum_{t \in [1, s-2]} (d_t r_t + e_t) + d_{(s-1)(\hat{s}-1)} r_{s-1} + e_{s-1} + d_{\hat{s}\ell} r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [\ell+1, n]} (d_t r_t + e_t),$$

otherwise, consider the point

$$w_0 = \sum_{t \in [1, s-2]} (d_t r_t + e_t) + d_{(s-1)(\hat{s}-1)} r_{s-1} + e_{s-1} + \sum_{t \in [\ell+1, n]} (d_t r_t + e_t)$$

on the face defined by (1). First, we exhibit $2|S|$ affinely independent points. For each $j \in S$ consider the points

$$w_j = \begin{cases} w_0 - d_{j(\hat{s}-1)} r_{s-1} + d_{j\ell} r_j + e_j - d_{\hat{s}\ell} r_{\hat{s}} - e_{\hat{s}}, & \text{if } j \leq \hat{s} \leq \ell, \\ w_0 - d_{j(\hat{s}-1)} r_{s-1} + d_{j\ell} r_j + e_j & \text{if } j \leq \hat{s} = \ell + 1, \\ w_0 - d_{j\ell} r_{\hat{s}} + d_{j\ell} r_j + e_j, & \text{otherwise,} \end{cases}$$

$$\bar{w}_j = w_j + \epsilon r_j + \epsilon v_n.$$

Next, consider the following $2(\ell - s + 1 - |S|)$ points. For $j = \hat{s} \leq \ell$ let

$$w_j = w_s + e_{\hat{s}} \text{ and } \bar{w}_j = w_0 - \epsilon r_{\hat{s}} + \epsilon r_0.$$

For each $j \in [\hat{s} + 1, \ell] \setminus S$ consider the two points

$$w_j = w_0 + e_j \text{ and } \bar{w}_j = w_j + d_{j\ell} r_j - d_{j\ell} r_{\hat{s}}.$$

Finally, for each $j \in [1, s-1] \cup [\ell+1, n]$ let

$$w_j = \begin{cases} w_{s'} - e_j - d_j r_j + d_j r_{s'}, & \text{if } j = \ell + 1, \\ w_s - e_j - d_j r_j + d_j r_{j-1}, & \text{otherwise,} \end{cases}, \text{ and}$$

$$\bar{w}_j = \begin{cases} w_0 + \epsilon r_j + \epsilon v_n, & \text{if } j \geq \ell + 1, \\ w_0 + \epsilon r_j - \epsilon r_{\hat{s}}, & \text{if } j \leq s-1 \text{ and } \hat{s} \leq \ell, \\ w_0 - \epsilon r_j + \epsilon r_0, & \text{if } j = s-1 \text{ and } \hat{s} = \ell + 1, \\ w_0 + \epsilon r_j - \epsilon r_{s-1}, & \text{if } j \leq s-2 \text{ and } \hat{s} = \ell + 1. \end{cases}$$

These points are affinely independent feasible points satisfying (1) at equality (Küçükayavuz, 2004). \square

Proposition 13. *If $u_s \geq d_{(s+1)\ell}$, then inequality (3) is facet-defining for \mathcal{P} if and only if*

1. $u_{k-1} < d_{k\ell}$,
2. $u_{\tilde{s}} > d_{(\tilde{s}+1)\ell}$, where $\tilde{s} = \max\{t \in S : t \leq p\}$ ($\tilde{s} = k-1$ if $S \cap [k, p] = \emptyset$),
3. if $u_{s-1} < d_{sp}$, then either $S = [s, \ell] = [s, p]$ or $\hat{s} \in [s, p]$,
4. if $d_{t\ell} = 0$ for some $t \in [k, \ell]$, then $t \in S$,
5. if $k > 1$, then $u_{k-2} \geq d_{(k-1)(s-1)}$,
6. if $u_{k-1} = d_{k(p-1)}$, then $s < p$; else $s \leq p$,
7. if $\ell < n$, then $u_{s'} \geq d_{(s'+1)(\ell+1)}$,
8. if $u_{k-1} = d_k + u_k$, then $k \in S$.

Proof. Note that, by assumption **(A3)**, $u_s \geq d_{(s+1)\ell}$ if and only if $d_{kt} + u_t \geq d_{k\ell}$ for all $t \in S$. Then inequality (3) simplifies as

$$(37) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} \min\{d_{k\ell} - u_{k-1}, d_{t\ell}\} x_t + i_{\ell}.$$

If $u_s < d_{(s+1)\ell}$, then (3) is stronger than (37).

(Necessity)

1. If $u_{k-1} = d_{k\ell}$, then inequality (3) is implied by the aggregated flow balance equality

$$i_{k-1} + \sum_{j \in [k, \ell]} y_j = d_{k\ell} + i_{\ell}.$$

2. If $u_{\tilde{s}} \leq d_{(\tilde{s}+1)\ell}$, then inequality (3) is dominated by (5)

$$i_{k-1} + \sum_{j \in S: j \leq p} y_j \leq u_{k-1} + \sum_{j \in S: j \leq p} (d_{k\tilde{s}} - u_{k-1} + u_{\tilde{s}}) x_j$$

and inequality (1)

$$\sum_{j \in S: j > p} y_j \leq \sum_{j \in S: j > p} d_{j\ell} x_j + i_{\ell}.$$

So we must have $u_{\tilde{s}} > d_{(\tilde{s}+1)\ell}$. Note that, from **(A3)**, this condition implies that $u_t > d_{(t+1)\ell}$ for all $t \in [\tilde{s}, \ell]$.

3. Suppose that the condition is not true. Either $S = [s, p]$ and $k < s \leq p < \ell$ or $[s, p] \subset S$. Since $u_{s-1} < d_{sp}$, it follows that

$$\sum_{t \in [s, p]} x_t \geq 1.$$

Multiplying both sides of this inequality with $d_{k\ell} - u_{k-1}$ and adding $d_{k\ell} + i_\ell = i_{k-1} + \sum_{t \in [k, \ell]} y_t$, we obtain

$$\sum_{t \in [k, \ell]} y_t \leq \sum_{t \in [s, p]} (d_{k\ell} - u_{k-1})x_t + (u_{k-1} - i_{k-1}) + i_\ell,$$

which is stronger than (3) in either case.

4. Note that $d_{t\ell} > 0$ for all $t \in [k, p]$ since we must have $0 < u_{k-1} < d_{kp}$. If $d_{t\ell} = 0$ for some $t \in [p+1, \ell]$ and $t \notin S$, then inequality (3) where S augmented with t is stronger.
5. Since the facet defined by (3) is different from $x_{k-1} = 1$, there is a point (x, y, i) on the face with $x_{k-1} = 0$. Since (3) is satisfied at equality we have $i_{k-1} \geq d_{k(s-1)}$. As $y_{k-1} = 0$, this implies that $u_{k-2} \geq i_{k-2} \geq d_{(k-1)(s-1)}$.
6. If $s > p$, then inequality (3) is obtained by adding $i_{k-1} \leq u_{k-1}$ and the (ℓ, S) inequality. So we must have $s \leq p$. However, if $u_{k-1} = d_{k(p-1)}$, then $d_{k\ell} - u_{k-1} = d_{p\ell}$ and by the same argument, we need $s < p$. Note that this condition implies that $S \neq \emptyset$.
7. Follows from the same argument for item 5 of Proposition 12.
8. Suppose $u_{k-1} = d_k + u_k$ and $k \notin S$. Then inequality

$$i_k + \sum_{t \in S} y_t \leq u_k + \sum_{t \in S} \min\{d_{(k+1)\ell} - u_k, d_{t\ell}\}x_t + i_\ell,$$

is stronger than (3). This can be seen by subtracting flow balance equality

$$i_{k-1} + y_k = d_k + i_k$$

from (3).

(*Sufficiency*) We define $2n + 1$ affinely independent points on the face defined by (3). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities.) For consistency of notation, let d_0 be zero and e_0 be the zero vector. Consider the point $w_0 = \sum_{t \in [1, k-2]} (d_t r_t + e_t) + \sum_{t \in [\ell+1, n]} (d_t r_t + e_t) + (d_{k-1} + u_{k-1})r_{k-1} + e_{k-1}$. Note that w_0 is not feasible since $u_{k-1} < d_{k\ell}$. We perturb w_0 in order to define feasible points.

First, consider the following $2|S|$ points. For each $j \in S$ if $j \leq p$, then let

$$w_j = w_0 + e_j + (d_{k\ell} - u_{k-1})r_j,$$

otherwise let

$$w_j = \begin{cases} w_0 + e_j + d_{j\ell}r_j + (d_{k(j-1)} - u_{k-1})r_{\hat{s}} + e_{\hat{s}}, & \text{if } \hat{s} \leq p, \\ w_0 + e_j + d_{j\ell}r_j + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + e_{s-1} + d_{\hat{s}(j-1)}r_{\hat{s}} + e_{\hat{s}}, & \text{if } s > k, \hat{s} > p, \\ w_0 + e_j + d_{j\ell}r_j + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + d_{\hat{s}(j-1)}r_{\hat{s}} + e_{\hat{s}}, & \text{if } s = k, \hat{s} > p. \end{cases}$$

Also for $j \in S$ let

$$\bar{w}_j = \begin{cases} w_j + \epsilon r_j + \epsilon v_n, & \text{if } j \geq \bar{s}, \\ w_j + \epsilon r_j - \epsilon r_{k-1}, & \text{otherwise.} \end{cases}$$

For $j = k-1$ and $k > 1$ let

$$\begin{aligned} w_j &= w_0 - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} + d_{(k-1)(s-1)}r_{k-2} + d_{s\ell}r_s + e_s \text{ and} \\ \bar{w}_j &= w_{\bar{s}} - \epsilon r_{k-1} + \epsilon r_{\bar{s}}. \end{aligned}$$

For $j \in [k, \ell] \setminus S$ consider

$$w_j = w_s + e_j \text{ and}$$

$$\bar{w}_j = \begin{cases} w_0 + (d_{k\ell} - u_{k-1})r_j + e_j, & \text{if } j \in [\hat{s}, p] \text{ or } j = s-1, \hat{s} > p, \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}}, & \text{if } j \in [p+1, \ell] \setminus \{\hat{s}\}, \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}}, & \text{if } j \in [k, \hat{s}-1] \text{ and } \hat{s} \leq p, \\ w_{s-1} + \epsilon r_j + e_j - \epsilon r_{s-1}, & \text{if } j \in [k, s-2] \text{ and } \hat{s} \geq p, \\ w_0 + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + e_{s-1} + d_{\hat{s}\ell}r_{\hat{s}} + e_{\hat{s}}, & \text{if } j = \hat{s} > p \text{ and } s > k, \\ w_0 + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + d_{\hat{s}\ell}r_{\hat{s}} + e_{\hat{s}}, & \text{if } j = \hat{s} > p \text{ and } s = k. \end{cases}$$

For $j \in [\ell+1, n]$ and $\ell < n$ consider

$$w_j = \begin{cases} w_{s'} - e_j - d_j r_j + d_j r_{s'}, & \text{if } j = \ell+1, \\ w_{s'} - e_j - d_j r_j + d_j r_{j-1}, & \text{if } j \in [\ell+2, n], \end{cases} \quad \text{and } \bar{w}_j = w_{s'} + \epsilon r_j + \epsilon v_n.$$

For $j \in [1, k-2]$ and $k > 2$ let

$$w_j = w_{s'} - e_j - d_j r_j + d_j r_{j-1} \text{ and } \bar{w}_j = w_{s'} + \epsilon r_j - \epsilon r_{k-1}.$$

Finally let

$$\bar{w}_0 = \begin{cases} w_s + \epsilon r_0 - \epsilon r_{k-1}, & \text{if } k > 1, \\ w_{\hat{s}} - \epsilon r_{k-1} + \epsilon r_{\hat{s}}, & \text{if } k = 1. \end{cases}$$

The $2n+1$ points given above are affinely independent points on the face (3) (Küçükayavuz, 2004). \square

Proposition 14. *Let $S \subseteq [k, \ell]$ for $1 \leq k \leq \ell \leq n$. If $u_s \geq d_{(s+1)\ell} + u_\ell$, then inequality (5) is facet-defining for \mathcal{P} if and only if*

1. $u_{k-1} < d_{k\ell} + u_\ell$,
2. $S \subseteq [k, q]$,
3. if $u_{s-1} < d_{sq}$, then $S \neq [s, q]$,
4. if $k > 1$, then $u_{k-2} \geq d_{(k-1)(s-1)}$ if $S \neq \emptyset$ and $u_{k-2} \geq d_{k-1} + u_{k-1}$ if $S = \emptyset$,
5. if $u_{k-1} = d_k + u_k$, then $k \in S$.

Proof. Note that, by assumption **(A3)**, $u_s \geq d_{(s+1)\ell} + u_\ell$ if and only if $d_{kt} + u_t = d_{k\ell} + u_\ell$ for all $t \in S$. Then inequality (3) simplifies as

$$(38) \quad i_{k-1} + \sum_{t \in S} y_t \leq u_{k-1} + \sum_{t \in S} (d_{k\ell} - u_{k-1} + u_\ell)x_t.$$

If $u_s < d_{(s+1)\ell} + u_\ell$, then (5) is stronger than (38).

(Necessity)

1. From assumption **(A3)**, $u_{k-1} \leq d_{k\ell} + u_\ell$. If $u_{k-1} = d_{k\ell} + u_\ell$, then inequality (5) is implied by the aggregated flow balance equality

$$i_{k-1} + \sum_{t \in [k, \ell]} y_t = d_{k\ell} + i_\ell \leq d_{k\ell} + u_\ell.$$

2. The statement holds trivially if $q = \ell$. Otherwise, from the definition of q , we have $d_{k\ell} - u_{k-1} + u_\ell \geq d_{(q+1)\ell} + u_\ell \geq d_t + u_t$ for all $t \in [q+1, \ell]$. Then if $t \in S \cap [q+1, \ell]$, the inequality is dominated by the one with $S' = S \setminus \{t\}$ and $y_t \leq (d_t + u_t)x_t$.
3. Suppose $u_{s-1} < d_{sq}$ and $S = [s, q]$. Then

$$\sum_{t \in [s, q]} x_t \geq 1$$

is valid. Multiplying both sides of this inequality with $d_{kq} - u_{k-1} + u_q$, and adding $i_q \leq u_q$ and $i_{k-1} + \sum_{t \in [k, q]} y_t = d_{kq} + i_q$ gives

$$(39) \quad i_{k-1} + \sum_{t \in [k, q]} y_t \leq u_{k-1} + \sum_{t \in [s, q]} (d_{kq} - u_{k-1} + u_q)x_t.$$

However, from condition 2 and assumption **(A3)** we have $u_q = d_{(q+1)\ell} + u_\ell$. Thus (39) equals

$$i_{k-1} + \sum_{t \in [k, q]} y_t \leq u_{k-1} + \sum_{t \in [s, q]} (d_{k\ell} - u_{k-1} + u_\ell)x_t.$$

4. If $S \neq \emptyset$, the proof is the same as for condition 5 of Proposition 13. If $S = \emptyset$ and $u_{k-2} < d_{k-1} + u_{k-1}$, then $i_{k-1} \leq u_{k-1}$ is dominated by the valid inequality $(u_{k-1} + d_{k-1} - u_{k-2})(1 - x_{k-1}) + i_{k-1} \leq u_{k-1}$.

5. If $u_{k-1} = d_k + u_k$ and $k \notin S$, then adding inequality (5)

$$i_k + \sum_{t \in S} y_t \leq u_k + \sum_{t \in S} (d_{(k+1)\ell} - u_k + u_\ell)x_t,$$

and the flow balance equality

$$i_{k-1} + y_k = d_k + i_k,$$

we obtain a stronger inequality.

(*Sufficiency*) By assumption **(A3)**, we have $u_q = d_{(q+1)\ell} + u_\ell$, and by condition 2 we have $S \subseteq [k, q]$ where $q \leq \ell$. Therefore, we assume below that $\ell = q$. We give $2n + 1$ affinely independent points on the face (5). For simplicity, we represent the points in the variable space $(i_0, (y, x), i_n)$. (The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities.) Let $w_0 = \sum_{j \in [1, k-1]} (e_j + d_j r_j) + \sum_{j \in [\ell+1, n]} (e_j + \bar{d}_j r_j) + e_{k-1} + (d_{k-1} + u_{k-1})r_{k-1}$, where \bar{d}_j is the remaining demand for period j when $i_\ell = u_\ell$ and there is no production in $[\ell, j]$, i.e., $\bar{d}_j = d_j - (u_\ell - d_{(\ell+1)(j-1)})^+$. (Let d_0 be zero and e_0 be the zero vector for $k = 1$.) Note that w_0 is not feasible since $u_{k-1} < d_{k\ell} + u_\ell$. Similarly, let d'_j be the remaining demand for period j when $i_{k-1} = u_{k-1}$ and there is no production in $[k, j]$, i.e., $d'_j = d_j - (u_{k-1} - d_{k(j-1)})^+$. Let $\ell = k$ and $s = k - 1$ if $S = \emptyset$. Consider the following feasible point of \mathcal{P}

$$\tilde{w} = \begin{cases} w_0 + (d_{k\ell} - u_{k-1})r_{\hat{s}} + e_{\hat{s}} + \sum_{j \in [\ell+1, n]} (d_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset \text{ and } \hat{s} \leq \ell, \\ w_0 + (d_{k\ell} - u_{k-1})r_{s-1} + e_{s-1} + \sum_{j \in [\ell+1, n]} (d_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset, s > k, \hat{s} > \ell, \\ w_0 + \sum_{j \in [\ell+1, n]} (d'_j - \bar{d}_j)r_j & \text{if } S \neq \emptyset, s = k, \hat{s} > \ell, \\ w_0 + \sum_{j \in [\ell+1, n]} (d'_j - \bar{d}_j)r_j & \text{if } S = \emptyset. \end{cases}$$

For $j \in S$ consider

$$w_j = w_0 + e_j + (d_{k\ell} + u_\ell - u_{k-1})r_j \text{ and } \bar{w}_j = w_j + \epsilon r_j - \epsilon r_{k-1}.$$

For $j = k - 1$ and $k > 1$ if $S \neq \emptyset$, then consider

$$w_j = w_0 - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} + d_{(k-1)(s-1)}r_{k-2} + (u_\ell + d_{s\ell})r_s + e_s,$$

otherwise consider

$$w_j = \tilde{w} - e_{k-1} - (d_{k-1} + u_{k-1})r_{k-1} + (d_{k-1} + u_{k-1})r_{k-2}.$$

Also let

$$\bar{w}_{k-1} = \tilde{w} - \epsilon r_j + \epsilon r_0.$$

For $j \in [k, \ell] \setminus S$ consider

$$w_j = \begin{cases} w_s + e_j & \text{if } S \neq \emptyset \\ w_0 + e_j + \sum_{t \in [\ell+1, n]} (d'_t - \bar{d}_t) & \text{if } S = \emptyset \end{cases} \text{ and}$$

$$\bar{w}_j = \begin{cases} w_0 + e_j + (d_{k\ell} + u_\ell - u_{k-1})r_j, & \text{if } j \in [s+1, \ell], \\ \bar{w}_{\hat{s}} + \epsilon r_j + e_j - \epsilon r_{\hat{s}}, & \text{if } j \in [k, s] \text{ and } \hat{s} \leq \ell, \\ \tilde{w} + \epsilon r_j + e_j - \epsilon r_{s-1}, & \text{if } j \in [k, s-2] \text{ and } \hat{s} = \ell + 1, \\ \tilde{w} + \epsilon r_j + \epsilon v_n & \text{if } j = s-1 \text{ and } \hat{s} = \ell + 1 \end{cases}$$

For $j \in [1, k-2]$ and $k > 2$ let

$$w_j = \tilde{w} - e_j - d_j r_j + d_j r_{j-1} \text{ and } \bar{w}_j = \tilde{w} + \epsilon r_j - \epsilon r_{k-1}.$$

For $j = \ell + 1$ let

$$w_j = \begin{cases} w_s - e_j & \text{if } S \neq \emptyset \\ \bar{w}_\ell - e_j & \text{otherwise,} \end{cases} \quad \text{and} \\ \bar{w}_j = \tilde{w} + \epsilon r_j + \epsilon v_n.$$

For $j \in [\ell + 2, n]$ let $w_j = \tilde{w} - e_j - d_j r_j + d_j r_{j-1}$ and $\bar{w}_j = \tilde{w} + \epsilon r_j + \epsilon v_n$.

These $2n+1$ points are affinely independent points satisfying inequality (5) at equality (Küçükyavuz, 2004). \square

Under assumptions **(A1)** and **(A2)**, it is easy to verify that the dimension of \mathcal{Q} is $3n + 2$.

Proposition 15. *Inequality (8) is facet-defining for \mathcal{Q} under the conditions of Proposition 12 if $u_{k-1} > d_{k\ell}$ and either $S = [k, b_k] = [k, \ell]$ or $[k, b_k] \setminus S \neq \emptyset$ where $b_k = \min\{t \in [k, \ell] : d_t > 0\}$.*

Proof. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. (The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities.) Let q_t be the unit vector corresponding to z_t for $t \in [0, n]$. For the case that $k = 0$ we use $2n + 1$ points described in Proposition 12 with $q_j = 1$ for all $j \in [0, n]$. For $j \in [0, s-2] \cup [\ell+1, n]$ consider $\tilde{w}_j = w_s + \sum_{t \in [0, n] \setminus \{j\}} q_t$. For $j \in [s-1, \ell]$ let $\tilde{w}_j = w_j + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $j = n$. If $j < n$ then consider $\tilde{w}_j = w_{j+1} + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $\{j+1\} \in S$, $\tilde{w}_j = w_0 + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $j+1 = \hat{s}$, and $\tilde{w}_j = \bar{w}_{j+1} + \sum_{t \in [0, n] \setminus \{j\}} q_t$ if $\{j+1\} \notin S$ and $j+1 \neq \hat{s}$. These $3n + 2$ points are affinely independent.

Now for $k > 0$ let w_j for all $j \in [1, n]$ be as described in sufficiency proof of Proposition 12.

Let $\hat{w}_{k-1} = \sum_{t \in [1, k-2] \cup [\ell+1, n]} (d_t r_t + e_t) + d_{(k-1)\ell} r_{k-1} + e_{k-1} + \sum_{t \in [k-1, \ell-1]} q_t$, and

$$\tilde{w}_{k-1} = \hat{w}_{k-1} + \epsilon r_{k-1} + \epsilon v_n + \sum_{t \in [\ell, n]} q_t.$$

Also if $k > 1$ let $\bar{w}_{k-1} = \hat{w}_k - d_{k-1} r_{k-1} - e_{k-1} + d_{k-1} r_{k-2} + q_{k-2}$.

If $[k, b_k] \setminus S \neq \emptyset$, then let $\bar{s} \in [k, b_k] \setminus S$. For $j \in S$ consider the points

$$\hat{w}_j = \begin{cases} \sum_{t \in [1, k-1] \cup [\ell+1, n]} (d_t r_t + e_t) + d_j \epsilon r_j + e_j + \sum_{t \in [j, \ell-1]} q_t, & \text{if } j \leq b_k, \\ \sum_{t \in [1, k-1] \cup [\ell+1, n]} (d_t r_t + e_t) + d_j \epsilon r_j + e_j + d_{\bar{s}(j-1)} r_{\bar{s}} + e_{\bar{s}}, & \text{otherwise,} \end{cases} \\ \bar{w}_j = \hat{w}_j + \epsilon r_j + \epsilon v_n + \sum_{t \in [\ell, n]} q_t, \text{ and}$$

$$\tilde{w}_j = \begin{cases} \hat{w}_{j+1} + q_j, & \text{if } j < n, \\ \hat{w}_j + q_j, & \text{otherwise.} \end{cases}$$

For $j \in [k, b_k] \setminus S$ consider

$$\hat{w}_j = \sum_{t \in [1, k-1] \cup [\ell+1, n]} (d_t r_t + e_t) + d_j \epsilon r_j + e_j + \sum_{t \in [j, \ell-1]} q_t,$$

and for $j \in [b_k + 1, \ell] \setminus S$ consider

$$\hat{w}_j = \sum_{t \in [1, k-1] \cup [\ell+1, n]} (d_t r_t + e_t) + d_j \epsilon r_j + e_j + d_{\bar{s}(j-1)} r_{\bar{s}} + e_{\bar{s}} + \sum_{t \in [\bar{s}, j-2] \cup [j, \ell-1]} q_t.$$

Also, for $j \in [k, \ell] \setminus S$ consider

$$\bar{w}_j = \hat{w}_{k-1} + e_j, \text{ and}$$

$$\tilde{w}_j = \begin{cases} \hat{w}_{j+1} + q_j, & \text{if } j < n, \\ \hat{w}_j + q_j, & \text{otherwise.} \end{cases}$$

For $j \in [1, k-2] \cup [\ell+1, n]$ consider

$$\hat{w}_j = \begin{cases} \hat{w}_{s'} - d_j r_j - e_j + d_j r_{s'} + q_\ell, & \text{if } j = \ell + 1 \\ \hat{w}_s - d_j r_j - e_j + d_j r_{j-1} + q_{j-1}, & \text{otherwise,} \end{cases} \\ \bar{w}_j = \begin{cases} \hat{w}_{k-1} + \epsilon r_j - \epsilon r_{k-1} + \sum_{t \in [j, k-2]} q_t, & \text{if } j \leq k-2 \\ \hat{w}_{k-1} + \epsilon r_j + \epsilon v_n + \sum_{t \in [j, n]} q_t, & \text{otherwise,} \end{cases} \\ \tilde{w}_j = \hat{w}_{k-1} + q_j.$$

Finally for $k > 1$ consider the following two points

$$\bar{w}_0 = \hat{w}_{k-1} + \epsilon r_0 - \epsilon r_{k-1} + \sum_{t \in [0, k-2]} q_t, \text{ and}$$

$$\tilde{w}_0 = \hat{w}_{k-1} + q_0.$$

These $3n+2$ points are affinely independent feasible points that satisfy (8) at equality (Küçükyavuz, 2004). \square

Proposition 16. *Inequality (9) is facet-defining for \mathcal{Q} if $\{k-1\} \cup ([k, p-1] \cap S) \subseteq T$ and the conditions of Proposition 13 are satisfied.*

Proof. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities. Consider the $2n+1$ affinely independent points described in Proposition 13 with $q_t = 1$ for all $t \in [0, n]$. For each $j \in [0, k-2] \cup [\ell, n]$ consider the point $\hat{w}_j = w_s + \sum_{t \in [0, n] \setminus \{j\}} q_t$, where w_s is as defined in Proposition 13. For each $j \in [p, \ell-1]$, if $\hat{s} \leq p$, then consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [\ell+1, n]} (d_t r_t + e_t) + (d_{kj} - u_{k-1})r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [k-1, \ell-1] \setminus \{j\}} q_t + (d_{k-1} + u_{k-1})r_{k-1} + e_{k-1} + d_{(j+1)\ell}r_{j+1} + e_{j+1}$, otherwise let $\hat{w}_j = \sum_{t \in [1, k-2] \cup [\ell+1, n]} (d_t r_t + e_t) + (d_{k(\hat{s}-1)} - u_{k-1})r_{s-1} + e_{s-1} + d_{\hat{s}j}r_{\hat{s}} + e_{\hat{s}} + \sum_{t \in [k-1, \ell-1] \setminus \{j\}} q_t + (d_{k-1} + u_{k-1})r_{k-1} + e_{k-1} + d_{(j+1)\ell}r_{j+1} + e_{j+1}$. For each $j \in [k-1, p-1]$ with $s(j) \leq \ell$, letting $t(j) = \max\{t \in T : t \leq j\}$, consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1] \cup [\ell+1, n]} (d_t r_t + e_t) + \sum_{t \in [k-1, \ell-1] \setminus (\{t(j), s(j)\} \cap (T \cup \{j\}))} q_t + d_{(k-1)t(j)}r_{k-1} + e_{k-1} + d_{s(j)\ell}r_{s(j)} + e_{s(j)}$. For each $j \in [k-1, p-1]$ with $s(j) > \ell$, consider $\hat{w}_j = \sum_{t \in [1, k-2]} (d_t r_t + e_t) + d_{(k-1)t(j)}r_{k-1} + e_{k-1} + \sum_{t \in [k-1, \ell-1] \setminus (\{t(j), s(j)\} \cap (T \cup \{j\}))} q_t + \sum_{t \in [t(j)+1, s(j)-1] \cup [\ell+1, n]} (d_t r_t + e_t)$. These $3n+2$ points are affinely independent points on the face (9) (Küçükyavuz, 2004). \square

Proposition 17. *Inequality (19) is facet-defining for \mathcal{Q} if $\{k-1\} \cup ([k, p-1] \cap S) \subseteq T$ and the conditions of Proposition 14 are satisfied.*

Proof. For simplicity, we represent the points in the variable space $(i_0, (y, x, z), i_n)$. The values of the variables i_t for $t \in [1, n-1]$ can be obtained via the flow balance equalities. Consider the $2n+1$ affinely independent points described in Proposition 14 with $q_j = 1$ for all $j \in [0, n]$. For each $j \in [0, k-2]$ consider $\hat{w}_j = \tilde{w} + \sum_{t \in [0, n] \setminus \{j\}} q_t$ where \tilde{w} is as defined in Proposition 14. For each $j \in [t_\tau, n]$, let $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t_\tau+1, n]} (d_t r_t + e_t) + d_{(k-1)t_\tau}r_{k-1} + e_{k-1} + \sum_{t \in [k-1, t_\tau-1] \cup [t_\tau+1, n] \setminus \{j\}} q_t$. Finally for $j \in [k-1, t_\tau-1]$, letting $t(j) = \max\{t \in T : t \leq j\}$, if $s(j) \leq \ell$ then consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1]} (d_t r_t + e_t) + \sum_{t \in [\ell+1, n]} (\bar{d}_t r_t + e_t) + d_{(k-1)t(j)}r_{k-1} + e_{k-1} + \sum_{t \in [k-1, \ell-1] \setminus (\{t(j), s(j)-1\} \cap (T \cup \{j\}))} q_t + (d_{s(j)\ell} + u_\ell)r_{s(j)} + e_{s(j)} + \sum_{t \in [\ell+1, n]} q_t$, where \bar{d}_t is the effective demand in period t if $i_\ell = u_\ell$. If $s(j) > \ell$ then consider $\hat{w}_j = \sum_{t \in [1, k-2] \cup [t(j)+1, s(j)-1]} (d_t r_t + e_t) + d_{(k-1)t(j)}r_{k-1} + e_{k-1} + \sum_{t \in [k-1, \ell-1] \setminus (\{t(j), s(j)-1\} \cap (T \cup \{j\}))} q_t + \sum_{t \in [\ell+1, n]} (d_t r_t + e_t)$. These $3n+2$ points are affinely independent points on the face (19) (Küçükyavuz, 2004). \square

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