MIXED-INTEGER OPTIMIZATION APPROACHES FOR DETERMINISTIC AND STOCHASTIC INVENTORY MANAGEMENT

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Abstract. A fundamental problem arising in most supply chain or production operations is to determine order or production lot sizes and inventory levels to maintain high service levels and low costs. These problems are challenging both theoretically and computationally as they include concave costs representing economies-of-scale; they are dynamic; and they are subject to uncertainty in demands, costs and lead times. In addition, service level restrictions may result in chance constraints. In this tutorial, we survey recent mixed-integer optimization models and methods for various lot sizing and inventory control problems. We consider problems both when demand is dynamic and deterministic; and when demand is random following a discrete and finite non-stationary distribution over a finite planning horizon. We use polyhedral combinatorics to develop cutting planes to tighten the original mixed-integer formulations. In addition, we show how a polynomial dynamic program to solve a subproblem can be used to develop a strong extended formulation. We summarize computational experiments that illustrate the effectiveness of these methods in solving difficult capacitated multi-item order lot-sizing problems.

Keywords: Lot sizing, service levels, extended formulations, cutting planes, convex hull

1. Introduction

At the core of most supply chain or production operations is the problem of determining order or production lot sizes and inventory levels so that customer demand can be satisfied at high service levels and low costs. Realistic representations of this problem are inherently complicated as costs are concave due to economies-of-scale, and demand, costs and lead times are subject to uncertainty. As a result, addressing these challenging problems has attracted the attention of a significant body of research over the years.

In the first part of this tutorial, we consider inventory management problems under dynamic, deterministic demand, which are referred to as lot-sizing problems. The natural mixed-integer formulations of even the polynomially solvable cases of the lot-sizing problem have weak linear programming relaxation solutions. We describe mixed-integer optimization models and show how to develop strong valid inequalities using the structure of the problem. In some cases, we give the complete linear programming description of the convex hull of solutions. The cutting planes developed using certain structures remain valid for classes of $\mathcal{NP}$-hard lot-sizing problems containing such substructures. In addition, our computational experiments show that they are highly effective

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in obtaining optimal solutions to problems that cannot be solved with the state-of-the-art optimization software within a reasonable time limit. We also describe methods to obtain alternative formulations that have tighter linear programming relaxations. When such formulations have a polynomial number of new variables and they solve the problem as a linear program instead of a mixed-integer program, we call such reformulations *tight and compact extended formulations.*

While demand may be considered deterministic in some settings, in most cases demand is uncertain. Thus, in the second part of this tutorial, we consider stochastic inventory management problems. Classical stochastic inventory management models typically assume a stationary demand distribution that is not correlated from one time period to the next. Under such an assumption, lot size reorder point policies are known to be optimal [60, 41]. When these assumptions are violated, the lot sizes can be determined by dynamic programming with a large state space, which suffers from the curse of dimensionality. In this tutorial, we consider stochastic programming and mixed-integer optimization approaches to model and solve such problems effectively. We assume that the uncertainty set is finite and can be represented by a finite set of scenarios.

Finally, we consider problems where service level contracts with customers place constraints on orders that are allowed to be shipped late. Late shipments can be extremely disruptive to downstream businesses and frequent stock-outs can result in significant loss of customer goodwill. One way to model the effects of backlogs is to add a penalty for stock-outs. However, suppliers often find it difficult to estimate such intangible components of their backorder costs. Thus, constraints such as the Type 1 service level, which limits the proportion of time demand is backlogged, are increasingly used in practice [41]. In this tutorial, we consider both the models where a reliable backorder penalty is available, referred to as lot sizing with backlogging; and models with service level constraints instead of estimates on intangible costs on backorders, referred to as lot sizing with service levels.

2. Deterministic Models

2.1. **No Backlogging.** We begin our discussion with the classical uncapacitated lot-sizing (ULS) problem first introduced by Wagner and Whitin [52]. In this problem, demand for a single item over the finite planning horizon of length n is known with certainty. The fixed and unit ordering costs in period t are given by f_t and g_t, respectively. In addition, the unit holding cost in period t, is given by c_t. The goal is to determine the order lot size in each period so that the total ordering and holding costs are minimized while meeting the demand on time (i.e., no backlogging). Wagner and Whitin [52] gave an $\mathcal{O}(n^2)$ algorithm for this problem, which was later improved to run in $\mathcal{O}(n \log n)$ time [1, 13, 51]. The algorithm relies on the structure of the optimal solution, known as the *zero inventory ordering* property. According this property, there exists an optimal solution in which an order takes place in a period only if the incoming inventory to that period is zero. In other words, there exists an optimal solution which is a concatenation of *regeneration intervals* of the form $\{i, j\}$ with $i \leq j$, where production occurs in period i to satisfy demands until period j.

Let x_t and s_t be the amount ordered and amount of ending inventory at period t, respectively. Also let $w_t = 1$ if an order takes place in period t, and $w_t = 0$ otherwise. Figure 1 depicts the fixed-charge network representation of ULS. Throughout, we let $[i, j] := \{t \in \mathbb{Z} : i \leq t \leq j\}$ ($[i, j] := \emptyset$ if $i > j$). Then the mixed-integer programming (MIP) formulation of ULS is:

$$
\begin{aligned}
&\min \textstyle\sum_{t=1}^{n} \left(f_t w_t + g_t x_t + c_t s_t\right) \\
&\quad s_{t-1} + x_t = d_t + s_t, \qquad t \in [1,n] \qquad (1)\\
&\quad x_t \le d_{tn} w_t, \qquad t \in [1,n] \qquad (2)\\
&\quad s_0 = s_n = 0 \qquad (3)\\
&\quad w \in \{0,1\}^n, \qquad (4)\\
&\quad (x,s) \in \mathbb{R}_+^{2n+1}, \qquad (5)
\end{aligned}
$$

where $d_{kt} := \sum_{i=k}^{t} d_i$ for $k \le t$ ($d_{kt} = 0$ otherwise). Constraints (1) are the flow balance equations. The variable upper bound constraints (2) ensure that $w_t = 1$ whenever $x_t > 0$. As the problem is uncapacitated, the coefficient of the variable upper bound constraint is d_{tn} for the order size in period t. Without loss of generality, we assume that the starting and ending stocks are zero in (3).

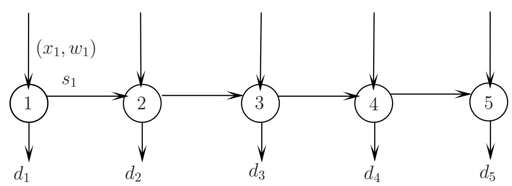

FIGURE 1. Fixed-charge network for ULS

The linear programming (LP) relaxation of the ULS formulation (1)–(5) is very weak. Krarup and Bilde [30] give an alternative *extended* formulation, called the facility location formulation, by introducing new variables u_{ij}, $1 \le i \le j \le n$ that represent production in period i to satisfy demand in period j. Using these variables, the variable upper bound constraint can be tightened as $u_{ij} \le d_j w_i$. In fact, Krarup and Bilde [30] show that the facility location formulation is tight, because it solves ULS as a linear program even when constraints (4) are relaxed. In addition, it is compact, as only a polynomial number ($\mathcal{O}(n^2)$) of new variables are added. Eppen and Martin [12] also give an alternative compact and tight extended formulation based on the shortest path algorithm that solves ULS.

Barany et al. [8] give the complete linear description of ULS in its original space of (x, w, s) variables, by the (ℓ, S) inequalities:

$$\sum_{j \in S} x_j \le \sum_{j \in S} d_{j\ell} w_j + s_\ell, \qquad (6)$$

where $\ell \in [1,n]$, $S \subseteq [1,\ell]$. It is easy to prove the validity of these inequalities. Let $\hat{x}_j$ be the portion of production in period j to satisfy demands in $[j,\ell]$ only, and let $\bar{x}_j$ be the portion of production in period j that is used to satisfy demands in $[\ell+1,n]$. Clearly, $x_j = \hat{x}_j + \bar{x}_j$. For each $j \in S$, we have $\hat{x}_j \le d_{j\ell} w_j$. In addition, $\sum_{j\in S} \bar{x}_j \le s_\ell$. Therefore, $\sum_{j \in S} x_j = \sum_{j\in S}(\hat{x}_j + \bar{x}_j) \le \sum_{j\in S} d_{j\ell} w_j + s_\ell$. Note that there are exponentially many inequalities (6), but the separation problem of finding the most violated (ℓ, S) inequality by a fractional point is trivial. Therefore, from polynomial equivalence of optimization and separation [20], a cutting plane method to solve ULS runs in polynomial time.

Obviously, if our main goal was to solve ULS, then the existing $\mathcal{O}(n \log n)$ algorithms are much more efficient than solving the extended formulations or the cutting plane method based on (ℓ, S)

inequalities. However, both the extended formulations and the valid inequalities remain valid for harder problems which have the ULS substructure, such as multi-item capacitated lot sizing [48, 53, 40]. In contrast, the polynomial algorithms for ULS fail to extend to these $\mathcal{NP}$-hard problems.

2.2. **Backlogging.** In most cases, it might be unrealistic or unprofitable to ensure that all demand is delivered on time. In this section, we consider a lot-sizing problem where not all demand has to be delivered on time. The only requirement is that all demand is delivered by the end of the planning horizon. In this case, it is assumed that a unit penalty p_t is incurred for the amount backlogged in period t. This problem is known as uncapacitated lot sizing with backlogging (ULSB). It is also polynomially solvable [57, 58].

Introducing the variables r_t as the total amount backlogged in period t, the MIP formulation of ULSB is:

$$\min \textstyle\sum_{t=1}^{n} (f_t w_t + g_t x_t + c_t s_t + p_t r_t)$$
$$s_{t-1} + x_t - r_{t-1} = d_t + s_t - r_t, \qquad t \in [1, n] \tag{7}$$
$$x_t \leq d_{1n} w_t, \qquad t \in [1, n] \tag{8}$$
$$s_0 = r_0 = s_n = r_n = 0 \tag{9}$$
$$(x, s, r) \in \mathbb{R}_+^{3n+2} \tag{10}$$
$$w \in \{0, 1\}^n. \tag{11}$$

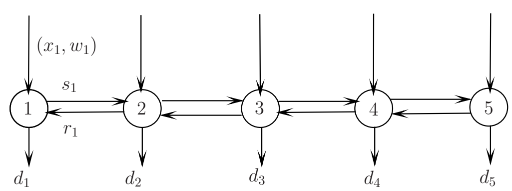

Figure 2. Fixed-charge network for ULSB

Pochet and Wolsey [43] study the polyhedral structure of ULSB and give a class of valid inequalities. However, these inequalities are not enough to give a complete linear description of ULSB. Next, we give a recent class of inequalities for ULSB that subsumes that in [43].

Theorem 1. *[32] For $S \subseteq [1, n]$, $L, R \subseteq [0, n]$, the inequality*

$$\sum_{t \in S} u_t x_t \leq \sum_{t \in S} (\sum_{i=1}^{u_t} d_{(k(t,i)+1)k'(t,i)}) w_t + \sum_{t \in L} \gamma_t r_t + \sum_{t \in R} \beta_t s_t, \tag{12}$$

is valid for ULSB, where

(i) $\gamma_t \in \mathbb{Z}_+$, $t \in L$, *and* $\beta_t \in \mathbb{Z}_+$, $t \in R$,
(ii) $u_t \in [1, q_t]$, $t \in S$ *with* $q_t = \min\{\sum_{i \in L: i < t} \gamma_i, \sum_{i \in R: i \geq t} \beta_i\}$,
(iii) $k(t, i) = \max\{k_i \in L \cap [0, t-1] : \sum_{j \in L \cap [k_i, t-1]} \gamma_j \geq i\}$, $t \in S$ *and* $i \in [1, u_t]$,
(iv) $k'(t, i) = \min\{k'_i \in R \cap [t, n] : \sum_{j \in R \cap [t, k'_i]} \beta_j \geq i\}$, $t \in S$ *and* $i \in [1, u_t]$.

The proof of Theorem 1, though more involved, is similar to the validity proof of inequalities (6). We illustrate it on the next example; the full proof is in [32].

Example 1. Consider inequality (12) with $L = [1, 2], R = [4, 5]$ and $S = [3, 5]$:

$$x_3 + 2x_4 + x_5 \le d_{34}w_3 + (d_{34} + d_{25})w_4 + d_{35}w_5 + r_1 + r_2 + s_4 + s_5.$$

Let $\bar{x}_j$ be the portion of production in period j that is used to satisfy the demands in $[k(j,1)+1, k'(j,1)]$ and $\tilde{x}_j$ be the portion of production in period j that goes through $r_{k(j,1)}$ and $\hat{x}_j$ be the portion of production in period j that goes through $s_{k'(j,1)}$. Also let $\bar{x}_4^2$ be the portion of production in period 4 to satisfy demands in [2,5]; $\tilde{x}_4^2$ be the portion of production in period 4 that goes through r_1 (the backlog quantity in the second largest period in L before period 4); and $\hat{x}_4^2$ be the portion of production in period 4 that goes through s_5 (the inventory quantity in the second smallest period in R on or after period 4). Therefore, $x_4 = \bar{x}_4 + \tilde{x}_4 + \hat{x}_4 = \bar{x}_4^2 + \tilde{x}_4^2 + \hat{x}_4^2$. Observe that $\bar{x}_4 \le d_{34}w_4$, $\bar{x}_4^2 \le d_{25}w_4$, $\bar{x}_3 \le d_{34}w_3$, $\bar{x}_5 \le d_{35}w_5$, $r_2 \ge \tilde{x}_3 + \tilde{x}_4 + \tilde{x}_5$, $s_4 \ge \hat{x}_3 + \hat{x}_4$, $r_1 \ge \tilde{x}_4^2$ and $s_5 \ge \hat{x}_4^2 + \hat{x}_5$. Therefore,

$$\begin{aligned} x_3 + 2x_4 + x_5 &= \sum_{j=3}^{5}(\bar{x}_j + \tilde{x}_j + \hat{x}_j) + \bar{x}_4^2 + \tilde{x}_4^2 + \hat{x}_4^2 \\ &\le d_{34}w_3 + (d_{34} + d_{25})w_4 + d_{35}w_5 + r_1 + r_2 + s_4 + s_5, \end{aligned}$$

is valid for ULSB.

What is perhaps surprising with inequalities (12) is that some of the continuous variables x, s and r may have coefficients that are greater than 1. In fact, if these coefficients are restricted to be 0 or 1, then the resulting inequalities are the inequalities proposed by Pochet and Wolsey [43]. Indeed, next result shows that inequalities (12) are enough to give the convex hull of solutions to ULSB.

Theorem 2. *[32] Inequalities* (7)–(10) *and* (12) *give the complete linear description of ULSB in its original space. Furthermore, there exists a polynomial separation LP to find the most violated inequality* (12) *by a given fractional point.*

This theorem is proved by projecting the tight extended formulation proposed in Pochet and Wolsey [43] onto the original space of variables. The separation LP is also based on the facility location extended formulation of ULSB [43].

Next, we illustrate the effectiveness of inequalities (12) in solving a difficult class of k-item lot sizing instances with a single setup per period using a branch-and-cut algorithm. These problems are called *small bucket* problems [44] and consist of k single-item problems linked by the mode constraint:

$$\sum_{i=1}^{m} w_t^j \le 1, t \in [1, n]$$

where the superscript j refers to item j, $j = 1, \ldots, k$. Therefore, the inequalities we developed for the single-item problem are valid for the multi-item problem.

The problem instances are solved with the MIP solver of CPLEX Version 10.0. CPLEX cuts are disabled in the experiments with the branch-and-cut algorithm using inequalities (12) (denoted by `LSB`) to underline the impact of the proposed inequalities. However, to see how CPLEX cuts would perform, we also solve the same instances with the default settings of CPLEX (`Cpx`) without adding any user cuts. We report our experience with a set of 60 randomly generated problem instances with $n = 50$ and $k = 5$ in Table 1. These computations are done under 3600 CPU seconds time and 512 MB memory limits. In Table 1, column `initgap` indicates the average percentage integrality gap between the LP relaxation and the MIP solutions; column `%gapimp` indicates the average percentage improvement in the integrality gap after adding cuts at the root node; column `cuts` indicates the average number of cuts added; column `nodes` is the average number of branch-and-cut nodes explored; column `time` is the average solution time in seconds; and the column `%endgap` is the average remaining integrality gap at termination.

TABLE 1. Multi-item lot-sizing with backlogging.

	initgap	%gapimp	cuts	nodes	time	%endgap
Cpx	60.66	58.21	730.03	1345743.18	1387.78	6.77
LSB	60.66	**86.22**	4187.62	4220.65	**597.85**	**0.08**

All sets of instances, but one, can be solved to optimality well within an hour time limit with our inequalities, whereas many problem instances cannot be solved within the time and memory limits with the default settings of CPLEX. A more detailed explanation of these experiments is available in Küçükyavuz and Pochet [32].

2.3. **Service level constraints.** In this section, we study a deterministic uncapacitated lot-sizing problem with a Type 1 service level constraint, which we refer to as lot sizing with service levels (LS-SL) [15]. The duration of the service contract (i.e., the planning horizon) is n. The problem is to determine the lowest cost ordering plan that meets the demand over the horizon at a predetermined service level, τ, where τ is defined as the proportion of time the demand is met over the horizon. In other words, letting $\kappa = \lfloor (1-\tau)n \rfloor$, the service level constraint enforces that backorders occur in no more than κ out of the n periods.

Let z_t be 1 if backlogging takes place in period t, and 0 otherwise. A 'natural' formulation of LS-SL is given by the constraints (7)–(11) and

$$r_t \le d_{1t} z_t, \quad t \in [1,n] \tag{13}$$
$$\textstyle\sum_{t=1}^{n} z_t \le \kappa \tag{14}$$
$$z \in \{0,1\}^n .$$

Constraints (13) are setup constraints for backlogs, and inequality (14) is the service level constraint.

Gade and Küçükyavuz [15] characterize the structure of the extreme point solutions to LS-SL, which are a concatenation of *regeneration intervals* (c.f. [58]) of the form $\{i,j,k\}$ with $i \le j \le k$, where the incoming stock and outgoing backorder in period i and the outgoing stock and incoming backorder in period k are zero. The production in period j satisfies its own demand and the demand backlogged in periods i through $j-1$, and the demand in periods $j+1$ through k from stocks. Furthermore, the total number of backorder periods cannot exceed κ.

Based on this extreme point structure, Gade and Küçükyavuz [15] give an $\mathcal{O}(n^2\kappa)$ shortest path algorithm for its solution as depicted in Figure 3 for $n=3$. In this shortest path network, there are three types of nodes for each time period i and for each possible value of remaining backorders allowed j: (i,j); (i',j); and (i'',j), for $i \in [1,n], j \in [(\kappa-i+1)^+,\kappa]$. The source node is $(1,\kappa)$, representing that κ backorder periods are allowed starting from time period 1. In addition, there is a dummy (sink) node $n+1$, which is a conglomeration of nodes $(n+1,j)$ for all $j \in [0,\kappa]$. There exists a *backlogging* arc from (i,j) to $(k', j-k+i)$ for all $k \in [i,n]$, and $i \in [1,n], j \in [\max\{k-i,(\kappa-i+1)^+\},\kappa]$, which represents producing in period k to satisfy demands in periods $[i,k-1]$. There exists a *production* arc from (i',j) to (i'',j) for all $i \in [1,n], j \in [(\kappa-i+1)^+,\kappa]$. A path visiting this arc represents production in period i. Finally, there exists an *inventory* arc from

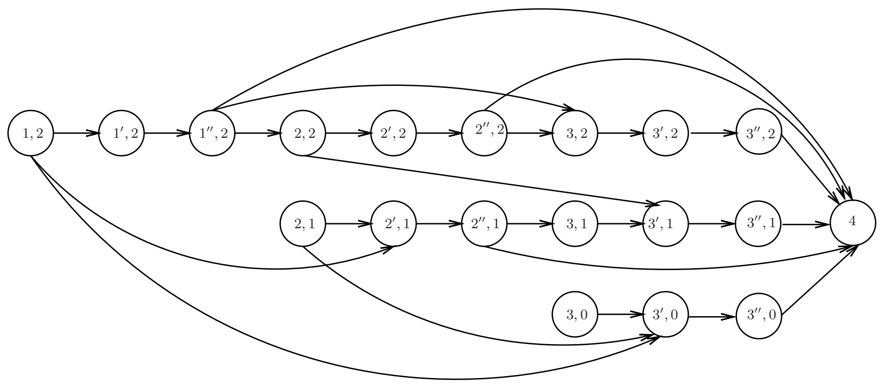

FIGURE 3. Shortest Path Representation of LS-SL for $n = 3, \kappa = 2$.

(i'', j) to (k, j) for $i \in [1, n-1], j \in [(\kappa - i + 1)^+, \kappa]$ and $k \in [i+1, n+1]$. Such an arc represents producing in period i to satisfy demands in periods $[i, k-1]$. For example, in Figure 3, the path $(1, 2) \to (2', 1) \to (2'', 1) \to (4, 1) =: 4$ represents a regeneration interval $\{1, 2, 3\}$, where production occurs in period 2 to satisfy demands in periods $[1, 3]$.

Next, we show how to obtain a compact and tight extended formulation for LS-SL based on the shortest path network. Let $\psi_{ijk} = 1$, for $i \in [1, n], j \in [0, \kappa]$ and $k \in [i, n]$ such that $j - k + i \geq 0$, if the shortest path visits the arc from (i, j) to $(k', j-k+i)$, and 0 otherwise. Let $\omega_{ij} = 1$ if the shortest path visits the arc from (i', j) to (i'', j) for all $i \in [1, n], j \in [0, \kappa]$, and 0 otherwise. Finally, let $\rho_{ijk} = 1$ if the shortest path visits the arc from (i'', j) to (k, j) for $i \in [1, n], j \in [0, \kappa], k \in [i+1, n+1]$.

The extended formulation includes the flow balance equations for the variables (ψ, ρ, z) in the shortest path network. In addition, constraints (15)–(19) define the relationship between the original variables and the new variables:

$$x_t = \sum_{j=0}^{\kappa} d_t \omega_{tj} + \sum_{j=0}^{\kappa} \sum_{k=t+2}^{n+1} d_{t+1,k-1} \rho_{tjk} + \sum_{i=1}^{t-1} \sum_{j=t-i}^{\kappa} d_{i,t-1} \psi_{ijt}, \qquad t \in [1, n] \tag{15}$$

$$s_t = \sum_{i=1}^{t} \sum_{j=0}^{\kappa} \sum_{k=t+2}^{n+1} d_{t+1,k-1} \rho_{ijk}, \qquad t \in [1, n-1] \tag{16}$$

$$r_t = \sum_{k=t+1}^{n} \sum_{i=1}^{t} \sum_{j=k-i}^{\kappa} d_{i,t} \psi_{ijk}, \qquad t \in [1, n-1] \tag{17}$$

$$w_t \geq \sum_{j=0}^{\kappa} \omega_{tj}, \qquad t \in [1, n] \tag{18}$$

$$z_t \geq \sum_{i=1}^{t} \sum_{k=t+1}^{n} \sum_{j=k-i}^{\kappa} \psi_{ijk}, \qquad t \in [1, n-1]. \tag{19}$$

For example, in inequalities (19), the backorder setup variable z_t, when zero, blocks any flow on arcs corresponding to the variables $\psi_{ijk}, i \in [1, t], j \in [k-i, \kappa], k \in [t+1, n]$.

In addition, Gade and Küçükyavuz [15] study the relaxation of LS-SL when the service level constraint (14) is dropped. The resulting model generalizes the uncapacitated lot sizing problem with backlogging studied in [43] and the uncapacitated lot sizing problem with stock fixed costs studied in [50, 16]. The extended formulation for this relaxation is tested on computational experiments on multi-item problems similar to the one described in Section 2.2 with an additional service level constraint. A summary of these experiments for the average of 60 randomly generated instances with $n = 60$ periods and $k = 5$ items is given in Table 2. These computations are done under a 3600 CPU second time limit.

TABLE 2. Comparison of the natural and extended formulations for $n = 60$

	initgap	nodes	time	% endgap
Natural	58.54	228914.0	3411.2	4.89
Extended	0.16	12.3	1.0	0.00

As can be seen, 5-item problems with service levels that cannot be solved within an hour with CPLEX 12.1 can be solved within a second with the extended formulation. More detailed experiments presented in Gade and Küçükyavuz [15] also indicate that the extended formulation scales well as the number of items or time periods increases.

2.4. **Concluding Remarks.** In the first part of this survey, we presented recent developments in the polyhedral study of various order lot-sizing problems. The book by Pochet and Wolsey [44] is an excellent review of earlier results on various lot-sizing problems.

The models presented here assume that the order lot size is not capacitated. The capacitated lot-sizing (CLS) problem with time-varying order capacities is $\mathcal{NP}$-hard [10]. Valid inequalities are proposed in Atamtürk and Muñoz [7], Leung et al. [33], Loparic et al. [34], Pochet [42] and Hartman et al. [26]. In contrast, the constant capacity lot-sizing problem (CCLS) is polynomially-solvable [14]. Pochet and Wolsey [45] study the polyhedral structure of CCLS and give a class of valid inequalities. However, the convex hull of solutions to CCLS in its original space is an open problem. Van Vyve [49] gives a linear programming extended formulation for CCLS with backlogging.

When there are capacities on the inventory levels but not on order quantities, the problem is referred to as lot sizing with bounded inventory (ULSBI) [36]. This problem is polynomially solvable [6]. Atamtürk and Küçükyavuz [5] study the polyhedral structure of ULSBI and give a class of valid inequalities. However, the complete linear description of ULSBI is an open problem.

If unmet demand is lost instead of backordered, then the problem is known as lot-sizing with lost sales. Loparic et al. [35] present a polyhedral study of this problem. There is limited work on the structure of multi-item lot-sizing problems with the exceptions of Yaman [56] and Miller and Wolsey [40]. Finally, most work assumes a single echelon order planning environment. Recently, Melo and Wolsey [39] gave an extended formulation for the 2-echelon lot-sizing problem where there is external demand only in the last echelon. Geunes [18], Akartunalı and Miller [3], Wu and Shi [54] and Wu et al. [55] propose heuristics for the multi-item multi-echelon lot-sizing problem with complex bill-of-material structures based on the associated facility location and shortest path reformulations. Finally, Zhang et al. [59] give an exponential class of inequalities for the serial 2-echelon problem where there are demands in the intermediate echelon, and establish a hierarchy between various extended formulations. Gaglioppa et al. [17] give a polynomial class of valid inequalities for the multi-echelon lot-sizing problem with more complex assembly structures.

The valid inequalities (6) and (12) presented in this section can be extended to hold for more general fixed-charge network flow problems such as those arising in global distribution planning.

Indeed commercial software packages such as CPLEX use a generalization of inequalities (6) known as network (or flow path) inequalities [47], and routinely add these cuts to improve the solution times of fixed-charge network flow problems. Extension of the network inequalities to more complex network structures such as that of ULSB, which contains cycles; or multi-echelon lot sizing, which contains grids, is an interesting research problem.

3. Stochastic Models

In this section, we consider lot-sizing problems where the costs, demands and order lead times follow a discrete-time stochastic process with finite probability space. Traditionally, the methods developed for such problems assume that the underlying stochastic process is stationary and homogeneous and/or that the costs, demands and lead times are independent random variables [4, 11, 29]. In this section, we assume that the stochastic process is very general, i.e., cost, demand and lead time distributions are non-stationary and are correlated.

3.1. **No Backlogging.** Let ω_t be the random variable representing demand in period t. Suppose that the distribution of demand is discrete and has finite support. Then the demand distribution can be represented by a finite number, m, of scenarios, with probabilities $\pi_1, \ldots, \pi_m$. Let g_{ti} and f_{ti} be the variable and fixed cost of ordering, c_{ti} be the variable holding cost, and d_{ti} be the demand in period t under scenario i. Also let x_{ti} be the decision variable representing order quantity in period t, s_{ti} be the inventory level at the end of period t under scenario i, and w_{ti} be 1 if an order setup is made, and 0 otherwise. In stochastic lot sizing (SLS), the non-anticipativity of the decisions must be enforced, i.e., the scenarios which have the same set of past outcomes (demands) until time t, should have the same action in period t. Let $\mathcal{S}_{t\ell} = \{k \in [1,m] \setminus \{\ell\} : h_{j\ell} = h_{jk}, j \in [1, t-1]\}$, be the set of scenarios that share the same demand history with scenario ℓ until time t. Finally, let M_{ti} be the order capacity in period t under scenario i (M_{ti} is a large enough number if the problem is uncapacitated).

The deterministic equivalent formulation of SLS is:

$$\min \textstyle\sum_{i=1}^m \sum_{t=1}^n \pi_i(g_{ti}x_t + f_{ti}w_{ti} + c_{ti}s_{ti})$$

$$s_{(t-1)i} + x_{ti} = d_{ti} + s_{ti}, \qquad t \in [1,n], i \in [1,m] \tag{20}$$

$$x_{ti} \le M_{ti}w_{ti} \qquad t \in [1,n], i \in [1,m]$$

$$s_{0i} = 0, \qquad i \in [1,m] \tag{21}$$

$$x_{t\ell} = x_{tk} \qquad t \in [1,n], \ell \in [1,m], k \in \mathcal{S}_{t\ell} \tag{22}$$

$$(x,s) \in \mathbb{R}_+^{2nm+m} \tag{23}$$

$$w \in \{0,1\}^{nm}. \tag{24}$$

Proposition 3. *[27, 28] There exists a dynamic program that solves the uncapacitated SLS in polynomial time with respect to the input size, n, m, when there are random lead times that do not cross over time.*

Guan and Miller [23] give an algorithm for stochastic lot sizing with zero lead times that runs in polynomial time with respect to n and m. Ahmed et al. [2] show that the zero inventory ordering policy, valid for the deterministic lot-sizing problem, does not hold for the stochastic case. Guan et al. [22] propose a branch-and-cut algorithm to solve the stochastic lot-sizing problem with zero lead times. Lulli and Sen [38] give a branch-and-price algorithm for the stochastic batch-sizing problem. The stochastic lot-sizing problem with backlogging (SLSB) can be modeled similarly.

Guan [21] gives a dynamic program that solves SLSB in polynomial time in the size of the scenario tree.

3.2. **Service level constraints.**

In this section, we consider the stochastic lot-sizing problem with a Type 1 service level constraint. We propose two models (*static* and *dynamic*) depending on when the order decisions are made.

3.2.1. *Static probabilistic lot sizing.*

In this section, we describe the *static* probabilistic lot-sizing problem (SPLS) with demand uncertainty and service level constraints [31]. In SPLS, the order quantities, x_t, are determined at the beginning of the finite planning horizon of length n and cannot be changed during the planning horizon as some of the demands are revealed. The objective is to minimize the expected total ordering and holding costs so that the joint probability of not stocking out over the planning horizon is at least τ, a predetermined service level. Let $\xi_t := \sum_{j=1}^{t} \omega_j$ be the random variable representing cumulative demand until time t. The chance-constrained model for SPLS is:

$$\min \quad \mathbb{E}\left(\sum_{t=1}^{m}(g_t x_t + c_t s_t(\xi_t))\right)$$

$$\text{s.t.} \quad \mathbb{P}\begin{pmatrix} x_1 & \geq & \xi_1 \\ x_1 + x_2 & \geq & \xi_2 \\ x_1 + x_2 + x_3 & \geq & \xi_3 \\ \vdots & \vdots & \vdots \\ x_1 + x_2 + x_3 + \cdots + x_n & \geq & \xi_n \end{pmatrix} \geq \tau \tag{25}$$

$$\sum_{j=1}^{n} x_j \geq \xi_n \tag{26}$$

$$x_t \leq M_t w_t \qquad t \in [1, n] \tag{27}$$

$$s_t(\xi_t) \geq \sum_{j=1}^{t} x_j - \xi_t \qquad t \in [1, n] \tag{28}$$

$$w \in \{0, 1\}^n \tag{29}$$

$$s_t(\xi_t), x_t \geq 0, \qquad t \in [1, n] \tag{30}$$

where $s_t(\xi_t)$ is the inventory level at the end of period t, g_t is the unit ordering cost at time t, c_t is the inventory holding cost per unit item at the end of time t and M_t is the order capacity (M_t is a large enough number if the problem is uncapacitated). Note, from constraints (28) and (30), that $s_t(\xi_t)$ is given by $s_t(\xi_t) = (\sum_{j=1}^{t} x_j - \xi_t)^+$, where $(a)^+ := \max\{a, 0\}$. Constraint (26) ensures that all demand is met by the end of the planning horizon. Even for the probabilistic *linear* program, without the discrete variables w, the feasible region is non-convex (cf.[46]).

Suppose that the distribution of demand is discrete and has finite support. Then the demand distribution can be represented by a finite number, m, of scenarios, with probabilities $\pi_1, \ldots, \pi_m$.

The deterministic equivalent formulation of SPLS without order setup costs is:

$$
\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^m \sum_{t=1}^n \pi_i (g_{ti} x_t + c_{ti} s_{ti}) \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^t x_j \geq h_{ti}(1 - z_i), \qquad t \in [1, n], i \in [1, m] && (31)\\
& \textstyle\sum_{j=1}^n x_j \geq h_{ni} \qquad i \in [1, m] && (32)\\
& s_{ti} \geq \textstyle\sum_{j=1}^t x_j - h_{ti} \qquad t \in [1, n], i \in [1, m] && (33)\\
& x_t \leq M_t w_t \qquad t \in [1, n] && (34)\\
& \textstyle\sum_{i=1}^m \pi_i z_i \leq 1 - \tau && (35)\\
& s \in \mathbb{R}_+^{nm}, z \in \{0, 1\}^m && (36)\\
& w \in \{0, 1\}^n && (37)\\
& x \in \mathbb{R}_+^n, && (38)
\end{aligned}
$$

where $h_{ti} := \sum_{j=1}^t d_{ji}$ is the cumulative demand in period t under scenario i, $i \in [1, m]$, x_t is the decision variable representing order quantity in period t, s_{ti} is the inventory level at the end of period t under scenario i, and $z_i = 0$ if demand in some period is not backlogged under scenario i, $i \in [1, m]$, and 1 otherwise, as ensured by constraints (31), following the convention of Luedtke et al. [37]. Beraldi and Ruszczyński [9] also consider a similar model, but ignore the inventory variables and holding costs, which results in an underestimation of the total cost and a potentially suboptimal solution. Constraints (33) and (36) ensure that $s_{ti} = (\sum_{j=1}^t x_j - h_{ti})^+$. Constraint (35) ensures that the desired service level is met.

Letting $y_t = \sum_{j=1}^t x_j$ denote the cumulative order quantity in period t, inequalities (31) have the so-called *mixing set* as its substructure, for which strong valid inequalities are proposed in Günlük and Pochet [25], Luedtke et al. [37] and Küçükyavuz [31]. Küçükyavuz [31] illustrates the effectiveness of a branch-and-cut algorithm using mixing inequalities and their extensions to include the knapsack constraint (35) to solve probabilistic lot-sizing problems.

Example 2. Consider an example with 5 time periods and 5 scenarios. Let the demand and unit order cost be as indicated in Table 3. Let the holding cost be 10 percent of the unit order cost and the target service level τ be 0.85. Suppose that the fixed costs are zero. The optimal solution to the static PLS problem is given in Table 4. The optimal expected cost is 29493.3, and stock-outs occur only in scenario 3.

TABLE 3. Test case demand and cost structure

	Scenario									
	1		2		3		4		5	
Time	Demand	Cost	Demand	Cost	Demand	Cost	Demand	Cost	Demand	Cost
1	69	170	69	170	12	129	12	129	35	192
2	73	48	80	108	15	60	15	60	59	67
3	68	85	18	87	87	77	80	161	79	155
4	30	63	67	79	55	48	72	154	36	48
5	80	95	100	19	86	26	16	38	47	23
π_s	0.30		0.10		0.05		0.20		0.35	

Next, we give a model under the assumption that order lot sizes are dynamically determined after partially observing the uncertain demand.

TABLE 4. Order quantities for the SPLS model

Time	1	2	3	4	5
Order	69	141	0	30	94

3.2.2. *Dynamic probabilistic lot sizing.* This section is based on Goel and Küçükyavuz [19]. The probabilistic lot sizing model in Section 3.2.1 is *static* in nature. In the formulation given by (31)–(38), the assumption is that the order decisions, x_t for all time periods t, will be made now, and these decisions will not change over time as the uncertain demand is revealed. However, a more flexible and efficient planning model would be to allow the order decision in period t to take the observed demands in periods $1, \ldots, t-1$ into account. This gives rise to what we refer to as the dynamic probabilistic lot-sizing problem (DPLS).

Let $x_t(\xi_1, \xi_2, \ldots, \xi_{t-1})$ be the decision variable at stage $t \in [1, n]$, whose value is determined after the random variables $\xi_1, \xi_2, \ldots, \xi_{t-1}$ are observed. Then the chance constraint is updated as:

$$\mathbb{P}\begin{pmatrix} x_1 & \geq & \xi_1 \\ x_1 + x_2(\xi_1) & \geq & \xi_2 \\ x_1 + x_2(\xi_1) + x_3(\xi_1, \xi_2) & \geq & \xi_3 \\ \vdots & \vdots & \vdots \\ x_1 + x_2(\xi_1) + x_3(\xi_1, \xi_2) + \cdots + x_n(\xi_1, \xi_2, \ldots, \xi_{n-1}) & \geq & \xi_n \end{pmatrix} \geq \tau,$$

where $x_t(\xi_1, \xi_2, \ldots, \xi_{t-1})$ is the amount to be ordered at time period t depending on the cumulative demands, ξ_i, observed in periods $i \in [1, t-1]$.

Let x_{ti} represent the quantity ordered in period t in scenario i to allow order decisions to be dependent on the demand realizations until time t. Also let $w_{ti} = 1$ if an order setup is made, and $w_{ti} = 0$ otherwise. In dynamic probabilistic lot sizing (DPLS), the non-anticipativity of the decisions must be enforced, i.e., the scenarios which have the same set of past outcomes (demands) until time t, should have the same action in period t. Let $\mathcal{S}_{t\ell} = \{k \in [1, m] \setminus \{\ell\} : h_{j\ell} = h_{jk}, j \in [1, t-1]\}$, be the set of scenarios that share the same demand history with scenario ℓ until time t. The deterministic equivalent of DPLS is

$$\begin{aligned}
\min \quad & \textstyle\sum_{i=1}^{m} \pi_i \sum_{t=1}^{n} (g_{ti} x_{ti} + f_{ti} w_{ti} + c_{ti} s_{ti}) & & \\
\text{s.t.} \quad & (35) - (36) & & \\
& \textstyle\sum_{i=j}^{t} x_{ji} \geq h_{ti}(1 - z_i) & t \in [1, n], i \in [1, m] & \qquad (39) \\
& \textstyle\sum_{j=1}^{n} x_{ji} \geq h_{ni} & i \in [1, m] & \qquad (40) \\
& x_{ti} \leq M_{ti} w_{ti} & t \in [1, n], i \in [1, m] & \\
& s_{ti} \geq \textstyle\sum_{j=1}^{t} x_{ji} - h_{ti} & t \in [1, n], i \in [1, m] & \\
& x_{t\ell} = x_{tk} & t \in [1, n], \ell \in [1, m], k \in \mathcal{S}_{t\ell} & \qquad (41) \\
& x \in \mathbb{R}_+^{nm}. & &
\end{aligned}$$

Constraints (41) enforce non-anticipativity.

Example 2 (cont.) The optimal solution to the example problem, when decisions are updated based on the demands observed is given in Table 5. The optimal expected cost is 24094.505, which is 18% lower than the static model. For this model, stock-outs occur under scenarios 2 and 3.

Lulli and Sen [38] consider a similar probabilistic batch-sizing problem. However, in their model, non-anticipativity is not enforced for scenarios in which demand is not met. For Example 2, if

TABLE 5. Order quantities for the DPLS model

	Scenario				
Time	1	2	3	4	5
1	69	69	69	69	69
2	80	80	110	110	104
3	61	0	0	0	0
4	110	0	0	0	36
5	0	185	76	16	47

non-anticipativity is not enforced for scenarios with stock-outs, the optimal solution is to order nothing under scenarios 2 and 3 over the entire horizon, which may not be implementable. If non-anticipativity is enforced, but the end of horizon demand constraint (40) is dropped, then the optimal solution is to order 69 and 110 units in periods 1 and 2, respectively, for both scenarios 2 and 3, so that the non-anticipativity is met. However, after period 3 no order takes place and all demand is lost under these scenarios.

The natural deterministic equivalent model is weak as its linear programming relaxation is highly fractional. Next, we give valid inequalities using the lot-sizing structure of the problem to tighten this formulation. Here, we let r_{ti} denote the quantity backordered in period t under scenario i, i.e., $r_{ti} = (h_{ti} - \sum_{j=1}^{t} x_{ji})^+$. Also, let $d_{jti} := \sum_{k=j}^{t} d_{ki}$ for $j \leq t$.

Theorem 4. *[19] For a given scenario i, $1 \leq t \leq \ell \leq n$, and $S \subseteq [t, \ell]$, the inequalities*

$$\sum_{j \in S} x_{ji} \leq \sum_{j \in S} d_{j\ell i} w_{ji} + s_{\ell i} + r_{(t-1)i} + d_{t(j^*-1)i} z_i \tag{42}$$

where $j^* = \max\{j \in S\}$, are valid.

Note that when $z_i = 0$, backlogging is not allowed and $r_{ji} = 0$ for all $j \in [1, n]$. Hence, inequality (42) reduces to the (ℓ, S) inequalities (6). When $z_i = 1$, the maximum amount backordered in periods $[t, \ell]$ to be fulfilled by the orders in periods $j \in S$ is $d_{t(j^*-1)i}$. Hence, inequality (42) is valid. In Goel and Küçükyavuz [19], we show the effectiveness of a branch-and-cut algorithm using the mixing inequalities of Küçükyavuz [31] and inequalities (42) to solve DPLS-FC.

3.3. **Concluding Remarks.** Much less is known about the polyhedral structure of stochastic inventory management problems. For example, for SLS with a polynomial number of scenarios, the complete linear description in the original space is an open problem, except when $n = 2$ [24]. It would be interesting to study type 2 service level models (also known as *fill rate*) [41]. Finally, any realistic stochastic programming representation of lot sizing problems under uncertainty will require a large number of scenarios. For such problems, cutting plane methods should be used in concert with decomposition methods. We are currently exploring these research topics.

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