A Note on "Lot-sizing with fixed charges on stocks: the convex hull"

Dinakar Gade and Simge Küçükyavuz
Department of Integrated Systems Engineering
The Ohio State University
Columbus, OH 43210
{gade.6,kucukyavuz.2}@osu.edu

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Abstract

We update and complete the proof of Proposition 7 in Van Vyve and Ortega (2004), which states that the projection of a facility location reformulation of an uncapacitated lot sizing problem with fixed charges on stocks (ULSW) to the original space is equivalent to that of the tight shortest path reformulation of ULSW. Their proof is interesting and consists of two cases, only first of which is analyzed in detail. We show that the second case exhibits several challenges not present in the first one and necessitates an updated proof.

Keywords: Tight extended formulations; lot-sizing; fixed charges.

1 Introduction

Van Vyve and Ortega (2004) consider an uncapacitated lot sizing problem with fixed charges on stocks and no backlogging (ULSW) over a finite horizon n. They propose the following tight extended formulation (SPW) based on a shortest path formulation:

$$\min \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n-1} g_i w_i \tag{1}$$

$$\sum_{i=1}^{j} \rho_j^i = 1, \qquad j \in [1, n] \tag{2}$$

$$x_i = \sum_{j=i}^{n} d_j \rho_j^i, \qquad i \in [1, n] \tag{3}$$

$$y_t \geq \rho_t^t, \qquad t \in [1, n] \tag{4}$$

$$y_t \leq 1, \qquad t \in [1, n] \tag{5}$$

$$w_t \geq \sum_{i=1}^{t} \rho_{t+1}^i, \qquad t \in [1, n-1] \tag{6}$$

$$w_t \leq 1, \qquad t \in [1, n-1] \tag{7}$$

$$\rho_i^i \geq \rho_{i+1}^i \geq \rho_{i+2}^i \geq \cdots \geq \rho_n^i \geq 0, \qquad i \in [1, n] \tag{8}$$

where, y_i, w_t are the setup variables for production and inventory at period $i \in [1, n] := \{1, \ldots, n\}$, and $t \in [1, n-1]$, respectively; x_i is the amount produced in period $i \in [1, n]$, and ρ_j^t is the fraction of demand in period j, d_j, satisfied from production in period t, where $1 \leq t \leq j \leq n$. The costs associated with y_i, w_i and x_i are f_i, g_i and c_i, respectively. (Note that unit production cost c_i includes the unit holding costs after projecting the inventory variables.)

The authors relax the SPW formulation by replacing constraints (8) by the following weaker valid constraints

$$y_i \geq \rho_j^i, \qquad j \in [1, n], i \in [1, j] \tag{9}$$

$$w_s \geq \sum_{i=1}^{s} \rho_j^i, \qquad j \in [2, n], s \in [1, j-1] \tag{10}$$

$$\rho_j^i \geq 0, \qquad j \in [1, n], i \in [1, j]. \tag{11}$$

They refer to the the formulation (1)–(3), (7), (9)–(11) as the facility location reformulation (FLW). Let the feasible sets associated with SPW and FLW be X^{SPW} and X^{FLW}, respectively. The projection of X^{FLW} to the original (x, y, w) space, $\text{proj}_{x,y,w}\left(X^{FLW}\right)$, provides valid inequalities for ULSW. The next proposition states that $\text{proj}_{x,y,w}\left(X^{FLW}\right)$ is equivalent to $\text{proj}_{x,y,w}\left(X^{SPW}\right)$, and as SPW is shown to be a tight formulation, $\text{proj}_{x,y,w}\left(X^{FLW}\right)$ gives the convex hull of solutions to ULSW (Van Vyve and Ortega, 2004).

Proposition 1. *(Van Vyve and Ortega, 2004)* $\text{proj}_{x,y,w}\left(X^{SPW}\right) = \text{proj}_{x,y,w}\left(X^{FLW}\right)$.

The proof in their paper is as follows:

FLW is a relaxation of SPW, thus $\text{proj}_{x,y,w}\left(X^{SPW}\right) \subseteq \text{proj}_{x,y,w}\left(X^{FLW}\right)$. Let any $(x, y, w) \in \text{proj}_{x,y,w}$ (X^{FLW}) be given. $\text{proj}_{x,y,w}\left(X^{SPW}\right) \supseteq \text{proj}_{x,y,w}\left(X^{FLW}\right)$ is proved by showing that there exists ρ such that $(x, y, w, \rho) \in X^{FLW}$ and $\rho_j^i \geq \rho_{j+1}^i$ for $j \in [1, n-1], i \in [1, j]$. For contradiction, suppose that for every ρ such that $(x, y, w, \rho) \in X^{FLW}$, there exist k and l such that $\rho_{k+1}^l > \rho_k^l$. For any given ρ, let l_ρ be the minimum such l and k_ρ be the minimum such k when $l = l_\rho$. Also let u_ρ be the minimum u which satisfies $\rho_u^{l_\rho} = \rho_{u+1}^{l_\rho} = \cdots = \rho_{k_\rho}^{l_\rho}$ and let $\lambda_\rho = \rho_{u_\rho - 1}^{l_\rho} - \rho_{u_\rho}^{l_\rho} > 0$ (taking $\rho_{l-1}^l = 1$). Define also $\gamma_\rho = \rho_{k_\rho+1}^{l_\rho} - \rho_{k_\rho}^{l_\rho} > 0$. Because ρ satisfies (2) for

$j = k$ and $k+1$,

$$\sum_{i=1}^{k} \left(\rho_k^i - \rho_{k+1}^i\right) = \rho_{k+1}^{k+1} \geq 0. \tag{12}$$

Hence, for any given ρ, equation (12) for $k = k_\rho$ implies that there exists a $t \neq l_\rho$ such that $\rho_{k_\rho+1}^t < \rho_{k_\rho}^t$. Let t_ρ be the minimum such t and $\delta_\rho = \rho_{k_\rho}^{t_\rho} - \rho_{k_\rho+1}^{t_\rho} > 0$. Now partition all ρ for which $(x, y, w, \rho) \in X^{FLW}$ according to the values $(l_\rho, k_\rho, u_\rho, t_\rho)$, and order the partitions by increasing values of l_ρ, k_ρ, then decreasing values of u_ρ and finally increasing value of t_ρ in this order of priority. Let α be a member of the last partition. The subscript α on $l_\alpha, k_\alpha, u_\alpha, t_\alpha, \gamma_\alpha, \lambda_\alpha$ and δ_α is dropped for notational convenience. The authors construct a new vector β such that $(x, y, w, \beta) \in X^{FLW}$, and β is a member of a partition after that of α, which leads to a contradiction. The vector β is constructed as follows:

$$\beta_j^i = \begin{cases} \alpha_j^i, & \text{if } i \neq t, l \text{ or } j \notin [u, k+1], \\ \alpha_j^i + \frac{\varepsilon}{d_{u,k}}, & \text{if } i = l, j \in [u, k], \\ \alpha_j^i + \frac{\varepsilon}{d_j}, & \text{if } i = t, j = k+1, \\ \alpha_j^i - \frac{\varepsilon}{d_{u,k}}, & \text{if } i = t, j \in [u, k] \\ \alpha_j^i - \frac{\varepsilon}{d_j}, & \text{if } i = l, j = k+1, \end{cases}$$

where $d_{i,j} := \sum_{t=i}^{j} d_t$, and ε satisfies

$$\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} = \min\left(\gamma, \lambda, \delta\right). \tag{13}$$

The authors prove that β constructed in this manner lies in a partition after that of α when $t < l$. They claim, without proof, that the same β also lies in a partition after that of α when $t > l$. However, when $t > l$, it is possible that $u \leq j < t \leq k$ and hence the construction of $\beta_j^t = \alpha_j^t - \frac{\varepsilon}{d_{u,k}}, \forall j \in [u, k]$ becomes invalid since α_j^t is not defined for $j \in [u, t-1]$ and thus the proof cannot hold. In the next section, we provide the complete proof of Proposition 1.

2 Proof of Proposition 1

Proof. Recall that $\text{proj}_{x,y,w}\left(X^{SPW}\right) \subseteq \text{proj}_{x,y,w}\left(X^{FLW}\right)$ since FLW is a relaxation of SPW. Let any $(x, y, w) \in \text{proj}_{x,y,w}\left(X^{FLW}\right)$ be given. We prove that $\text{proj}_{x,y,w}\left(X^{SPW}\right) \supseteq \text{proj}_{x,y,w}\left(X^{FLW}\right)$ by showing that there exists ρ such that $(x, y, w, \rho) \in X^{FLW}$ and $\rho_j^i \geq \rho_{j+1}^i$ for $j \in [1, n-1], i \in [1, j]$. For contradiction, suppose that for every ρ such that $(x, y, w, \rho) \in X^{FLW}$, there exist k and l such that $\rho^l_{k+1} > \rho^l_k$. For any given ρ, let l_ρ, k_ρ, u_ρ, λ_ρ and γ_ρ be as defined in §1. We make the following claim:

Claim 1. *For any $(x, y, w, \rho) \in X^{FLW}$ and $j \in [u_\rho, k_\rho]$, there exists an index $\tau_{j_\rho} \neq l_\rho$, such that $\rho_j^{\tau_{j_\rho}} > \rho_{k_\rho+1}^{\tau_{j_\rho}}$.*

Proof. Let $j \in [u_\rho, k_\rho]$. Suppose the claim is not true, i.e., $\rho^i_{k_\rho+1} \geq \rho^i_j$ for all $i \in [1, j], i \neq l_\rho$. This implies that $\sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_{k_\rho+1} \geq \sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_j$. Let $\rho^{l_\rho}_{k_\rho+1} = 1 - \xi$. We have $\sum_{i\in[1,k_\rho+1]\setminus\{l_\rho\}} \rho^i_{k_\rho+1} = \xi$ and so $\sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_{k_\rho+1} \leq \xi$. We get, $\xi \geq \sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_{k_\rho+1} \geq \sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_j$. But $\rho^{l_\rho}_{k_\rho+1} > \rho^{l_\rho}_{k_\rho} = \rho^{l_\rho}_j$ and we must have $1 - \rho^{l_\rho}_{k_\rho+1} < 1 - \rho^{l_\rho}_j$, i.e., $\xi < \sum_{i\in[1,j]\setminus\{l_\rho\}} \rho^i_j$, a contradiction. □

Let τ_{j_ρ} be the smallest index t such that $\rho^t_j > \rho^t_{k_\rho+1}$ for a given ρ and $j \in [u_\rho, k_\rho]$. We partition all ρ for which $(x, y, w, \rho) \in X^{FLW}$ according to the values (l_ρ, k_ρ) and order the partitions by increasing values of l_ρ and k_ρ in that order of preference. (Note that our partitioning scheme is different than that of Van Vyve and Ortega (2004) as necessitated by cases when $t > l$.) Let α be a member of the last partition with the largest $\rho^{l_\rho}_{k_\rho}$ value. Note that α is well-defined for a given $(x, y, w) \in \text{proj}_{x,y,w}(X^{FLW})$, and partition (l, k), and can be found by the (bounded) linear program

$$\{\max \rho^l_k : \rho \text{ satisfies } (2) - (3), (9) - (11), \rho^i_j \geq \rho^i_{j+1}, \text{ for } i < l \text{ and } j \in [i, n-1]; \text{ or } i = l \text{ and } j \in [1, k-1]\}.$$

We construct a new vector β such that $(x, y, w, \beta) \in X^{FLW}$ and β is a member of a partition in or after the partition containing α, and $\beta^l_k > \alpha^l_k$ leading to a contradiction. Henceforth, we drop the subscript α on $l_\alpha, k_\alpha, u_\alpha, \tau_{j_\alpha}$ and the indices l, k, u, τ_j will be in reference to α unless otherwise stated. We observe that α can fall under any of the following cases:

1. $\tau_k < l$.

2. $\tau_k > l$ and there exists $j \in [u, k-1]$, such that $\tau_j < l$.

3. $\tau_j > l, \forall j \in [u, k]$.

In each of the cases we construct a new vector β such that $l_\beta \geq l_\alpha, k_\beta \geq k_\alpha$ and $\beta^l_k > \alpha^l_k$ which contradicts the assumption that α falls in the last partition, (l, k), or that α has the largest ρ^l_k value among all members in the last partition.

Case 1. $\tau_k < l$ (see Table 1 for an example). In this case, our construction of β is the same as that of Van Vyve and Ortega (2004). For completeness, we provide the details of the proof of Case 1, with respect to our partitioning scheme.

Table 1: Example of α, $t < l$

				u		k	$k+1$
	1.00	0.75	0.15	0.00	0.00	0.00	0.00
$t = \tau_k$		0.25	0.20	0.20↓	0.10↓	0.10↓	0.00↑
l			0.65	0.60↑	0.60↑	0.60↑	0.75↓
				0.20	0.25	0.30	0.00
					0.05	0.00	0.25
						0.00	0.00
							0.00

Let $\delta_k = \alpha_k^{\tau_k} - \alpha_{k+1}^{\tau_k} > 0$. We construct β as follows:

$$\beta_j^i = \begin{cases} \alpha_j^i, & \text{if } i \neq \tau_k, l \text{ or } j \notin [u, k+1], \\ \alpha_j^i + \frac{\varepsilon}{d_{u,k}}, & \text{if } i = l, j \in [u, k], \\ \alpha_j^i + \frac{\varepsilon}{d_j}, & \text{if } i = \tau_k, j = k+1, \\ \alpha_j^i - \frac{\varepsilon}{d_{u,k}}, & \text{if } i = \tau_k, j \in [u, k], \\ \alpha_j^i - \frac{\varepsilon}{d_j}, & \text{if } i = l, j = k+1. \end{cases}$$

The value of ε is chosen such that it is the largest number that satisfies (14)–(16)

$$\gamma \geq \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} \tag{14}$$

$$\lambda \geq \frac{\varepsilon}{d_{u,k}} \tag{15}$$

$$\delta_k \geq \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}}. \tag{16}$$

In Table 1, ↑ (↓) indicates an increase (decrease) in the value of an entry when the vector α is updated as β.

Note that our definition of ε is different than that of Van Vyve and Ortega (2004). This choice of ε guarantees the following:

(1.a) $\beta_j^i \geq \beta_{j+1}^i, i < l, j \in [i, n-1]$. The statement is trivially true for $i \neq \tau_k$ or for $j \notin [u, k+1]$, because $\beta_j^i = \alpha_j^i$ for such i and $j \geq i$, due to the choice of l. For $i = \tau_k, j \in [u, k-1]$ we have $\beta_j^{\tau_k} \geq \beta_{j+1}^{\tau_k}$. For $i = \tau_k, j = k$, we have $\beta_j^i = \alpha_j^i - \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^i + \delta_k - \frac{\varepsilon}{d_{u,k}} \geq \beta_{k+1}^i$ from (16).

(1.b) $\beta_j^l \geq \beta_{j+1}^l, j \leq u-1$. This is true for $j \neq u-1$. For $j = u-1$, we have $\beta_{u-1}^l = \alpha_{u-1}^l = \alpha_u^l + \lambda = \alpha_u^l + \frac{\varepsilon}{d_{u,k}} + \lambda - \frac{\varepsilon}{d_{u,k}} = \beta_u^l + \lambda - \frac{\varepsilon}{d_{u,k}} \geq \beta_u^l$, using (15).

(1.c) $\beta^l_u = \cdots = \beta^l_k$. This is true since $\alpha^l_u = \cdots = \alpha^l_k$ and $\beta^l_j = \alpha^l_j + \frac{\varepsilon}{d_{u,k}}, j \in [u, k]$.

(1.d) $\beta^l_k \leq \beta^l_{k+1}$. This is because $\beta^l_k = \alpha^l_k + \frac{\varepsilon}{d_{u,k}} = \alpha^l_{k+1} - \gamma + \frac{\varepsilon}{d_{u,k}} = \alpha^l_{k+1} - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} \leq \beta^l_{k+1}$ from (14).

(1.e) Because $\gamma, \lambda, \delta_k > 0$, we have $\varepsilon > 0$. As a result, we have $\beta^l_k > \alpha^l_k$.

From (1.a)–(1.c), we ensure that $l_\beta \geq l_\alpha$ and $k_\beta \geq k_\alpha$. If (14) is satisfied at equality, then from (1.d), it follows that $\beta^l_k = \beta^l_{k+1}$, so either $l_\beta = l_\alpha$ and $k_\beta > k_\alpha$ or $l_\beta > l_\alpha$ and hence β falls in a partition after that of α, a contradiction. If (14) is not satisfied at equality, then either (15) or (16) is satisfied at equality. As a result, $\beta^l_k < \beta^l_{k+1}$ and $l_\alpha = l_\beta$, $k_\alpha = k_\beta$, but $\beta^l_k > \alpha^l_k$ from (1.e), which contradicts the assumption that α^l_k has the largest value among all members in the last partition, (l, k). For proof of feasibility of β, we refer the reader to the proof in Van Vyve and Ortega (2004). Note that $\tau_k = t$, where t is as defined in §1. When $t < l$, Van Vyve and Ortega (2004) are able to use the row t in their construction of vector β from α using the non-increasing order of $\alpha^i_j, i < l$ to maintain feasibility and reach a contradiction (see example in Table 1). However, when $t > l$, the order is not available and a different update scheme must be used, which we address in Cases 2 and 3.

Case 2. $\tau_k > l$ and there exists $j \in [u, k-1]$, such that $\tau_j < l$ (see Table 2 for an example).

Table 2: Example of α when $t > l$, there exists $j \in [u, k] : \tau_j < l$

				u	p	k	$k+1$
	1.00	0.75	0.15	0.00	0.00	0.00	0.00
τ_p		0.25	0.20	0.20↓	0.20↓	0.10↑	0.10↑
l			0.65	0.60↑	0.60↑	0.60↑	0.75↓
				0.20	0.15	0.00	0.00
					0.05	0.00	0.15
τ_k						0.30↓	0.00↑
							0.00

We define $p = \max\{j \in [u, k], \tau_j < l\}$. Using this definition of p, we have,

$$\alpha^{\tau_p}_{p+1} = \cdots = \alpha^{\tau_p}_k = \alpha^{\tau_p}_{k+1}, \tag{17}$$

since otherwise the definitions of τ_j, k and l will be violated. In addition, we define $\delta_j = \alpha^{\tau_j}_j - \alpha^{\tau_j}_{k+1} > 0, j \in [p, k]$. For $i \in [l+1, k]$ we let $T_i := \{j \in [i, k] : \tau_j = i\}$ and $\mathcal{D}_i := \sum_{j \in T_i} d_j$. Using α, we construct a vector β as follows:

$$\beta_j^i = \begin{cases} \alpha_j^i + \frac{\varepsilon}{d_{u,k}}, & \text{if } i = l, j \in [u,k] \\ \alpha_j^i - \frac{\varepsilon}{d_{k+1}}, & \text{if } i = l, j = k+1 \\ \alpha_j^i - \frac{\varepsilon}{d_{u,k}}, & \text{if } i = \tau_p, j \in [u,p] \\ \alpha_j^i + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}}, & \text{if } i = \tau_p, j \in [p+1,k] \\ \alpha_j^i - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}}, & \text{if } j \in [p+1,k], i = \tau_j \\ \alpha_j^i + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right), & \text{if } j = k+1, i \in [l+1,k] : T_i \neq \emptyset \\ \alpha_j^i, & \text{otherwise.} \end{cases} \tag{18}$$

The value of ε is chosen such that it is the largest number that satisfies (19)–(23),

$$\lambda \geq \frac{\varepsilon}{d_{u,k}} \tag{19}$$

$$\delta_p \geq \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} + \frac{\varepsilon}{d_{u,k}} \tag{20}$$

$$\delta_j \geq \frac{\mathcal{D}_{\tau_j}}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right), j \in [p+1,k] \tag{21}$$

$$\gamma \geq \frac{\varepsilon}{d_{p+1,k}} \tag{22}$$

$$\gamma \geq \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}}. \tag{23}$$

This guarantees the following:

(2.a) $\beta_j^i \geq \beta_{j+1}^i$ for $i < l, j \in [i, n-1]$. The result trivially holds for $i \neq \tau_p, j \in [i, n-1]$. The result also holds for $i = \tau_p, j \in [u, p-1]$ since $\beta_j^i = \alpha_j^i - \frac{\varepsilon}{d_{u,k}} \geq \alpha_{j+1}^i - \frac{\varepsilon}{d_{u,k}} = \beta_{j+1}^i$. Similarly, the result holds for $i = \tau_p, j \in [p+1, k-1]$. For $i = \tau_p, j = p, \beta_j^i = \alpha_j^i - \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^i + \delta_p - \frac{\varepsilon}{d_{u,k}} = \alpha_{j+1}^i + \delta_p - \frac{\varepsilon}{d_{u,k}} = \alpha_{j+1}^i + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} + \delta_p - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \beta_{j+1}^i + \delta_p - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \geq \beta_{j+1}^i$ using (17) and (20). For $i = \tau_p, j = k, \beta_j^i = \alpha_j^i + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \alpha_{j+1}^i + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} > \alpha_{j+1}^i = \beta_{j+1}^i$, from (17).

(2.b) $\beta_j^l \geq \beta_{j+1}^l, j \leq u-1$. This is true for $j \neq u-1$. For $j = u-1$, we have $\beta_{u-1}^l = \alpha_{u-1}^l = \alpha_u^l + \lambda = \alpha_u^l + \frac{\varepsilon}{d_{u,k}} + \lambda - \frac{\varepsilon}{d_{u,k}} = \beta_u^l + \lambda - \frac{\varepsilon}{d_{u,k}} \geq \beta_u^l$, from (19).

(2.c) $\beta_u^l = \cdots = \beta_k^l$. This is true since $\alpha_u^l = \cdots = \alpha_k^l$ and $\beta_j^l = \alpha_j^l + \frac{\varepsilon}{d_{u,k}}, j \in [u,k]$.

(2.d) $\beta_k^l \leq \beta_{k+1}^l$. This is because $\beta_k^l = \alpha_k^l + \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^l - \gamma + \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^l - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} \leq \beta_{k+1}^l$, from (23).

(2.e) Since $\varepsilon > 0$, we have $\beta_k^l > \alpha_k^l$.

From (2.a)–(2.c), we ensure that $l_\beta \geq l_\alpha$ and $k_\beta \geq k_\alpha$. If (23) is satisfied at equality, then from (2.d), it follows that $\beta_k^l = \beta_{k+1}^l$, so either $l_\beta = l_\alpha$ and $k_\beta > k_\alpha$ or $l_\beta > l_\alpha$, so β falls in a partition after that of α, a contradiction. If (23) is not satisfied at equality, then one or more of (19)–(22) are satisfied at equality. As a result, $\beta_k^l < \beta_{k+1}^l$ and $l_\alpha = l_\beta$, $k_\alpha = k_\beta$, but $\beta_k^l > \alpha_k^l$ from (2.e), which contradicts the assumption that α_k^l has the largest value among all members in the last partition, (l, k). Note that in Case 2, τ_j for $j \in [u, k]$ could increase or decrease, unlike in Case 1, where $t = \tau_k$ increases with every update. This motivates our redefinition of a partition. Next, we show that β is feasible in FLW.

Constraint (2) is trivially satisfied for $j \notin [u, k+1]$. For $j \in [u, p]$ we have $\sum_{i=1}^{j} \beta_j^i = \sum_{i \in [1,j] \setminus \{l, \tau_p\}} \alpha_j^i + \alpha_j^{\tau_p} - \frac{\varepsilon}{d_{u,k}} + \alpha_j^l + \frac{\varepsilon}{d_{u,k}} = \sum_{i=1}^{j} \alpha_j^i = 1$. For $j \in [p+1, k]$ we have

$$\sum_{i=1}^{j} \beta_j^i = \sum_{i \in [1,j] \setminus \{l, \tau_p, \tau_j\}} \alpha_j^i + \alpha_j^{\tau_p} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} + \alpha_j^l + \frac{\varepsilon}{d_{u,k}} + \alpha_j^{\tau_j} - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \sum_{i=1}^{j} \alpha_j^i = 1.$$

For $j = k+1$, we have

$$\begin{aligned} \sum_{i=1}^{j} \beta_j^i &= \sum_{i \in [1,l-1] \text{ or } i \in [l+1,k]: T_i = \emptyset} \alpha_j^i + \alpha_j^l - \frac{\varepsilon}{d_{k+1}} + \sum_{i \in [l+1,k]: T_i \neq \emptyset} \left(\alpha_j^i + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) \right) \\ &= \sum_{i=1}^{k+1} \alpha_{k+1}^i - \frac{\varepsilon}{d_{k+1}} + \frac{d_{p+1,k}}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) = 1. \end{aligned}$$

Constraint (3) is trivially satisfied for $i < l$ and $i \neq \tau_p$; or $i \in [l+1, k] : T_i = \emptyset$. For $i = l$ we have,

$$\sum_{j=i}^{n} d_j \beta_j^i = \sum_{j \in [i,n] \setminus [u,k+1]} d_j \alpha_j^i + \sum_{j=u}^{k} d_j \left(\alpha_j^i + \frac{\varepsilon}{d_{u,k}} \right) + d_{k+1} \left(\alpha_{k+1}^i - \frac{\varepsilon}{d_{k+1}} \right) = \sum_{j=i}^{n} d_j \alpha_j^i = x_i.$$

For $i = \tau_p$,

$$\sum_{j=i}^{n} d_j \beta_j^i = \sum_{j \in [i,n] \setminus [u,k]} d_j \alpha_j^i + \sum_{j=u}^{k} d_j \alpha_j^i - \sum_{j=u}^{p} d_j \frac{\varepsilon}{d_{u,k}} + \sum_{j=p+1}^{k} d_j \frac{\varepsilon d_{up}}{d_{p+1,k} d_{u,k}} = \sum_{j=i}^{n} d_j \alpha_j^i = x_i.$$

For $i \in [l+1, k] : T_i \neq \emptyset$, we have

$$\begin{aligned} \sum_{j=i}^{n} d_j \beta_j^i = &\sum_{j \in [i,n] \setminus (T_i \cup \{k+1\})} d_j \alpha_j^i + \sum_{j \in T_i} d_j \alpha_j^i - \sum_{j \in T_i} d_j \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) \\ &+ d_{k+1} \alpha_{k+1}^i + \frac{d_{k+1} \mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) = \sum_{j=i}^{n} d_j \alpha_j^i = x_i. \end{aligned}$$

Constraint (9) is trivially satisfied for $i < l$ and $i \neq \tau_p$; or $i \in [l+1, k] : T_i \neq \emptyset$ and $j \notin T_i \cup \{k+1\}$; or $i \in [l+1, k] : T_i = \emptyset$; or $j \notin [u, k+1]$; or $i = \tau_p$ and $j = k+1$. For $i = l, j = k+1$, $\beta^i_j = \alpha^i_j - \frac{\varepsilon}{d_{k+1}} < \alpha^i_j \leq y_i$. For $i = l, j \in [u, k], \beta^i_j = \alpha^i_j + \frac{\varepsilon}{d_{u,k}} = \alpha^i_{u-1} - \lambda + \frac{\varepsilon}{d_{u,k}} \leq \alpha^i_{u-1} \leq y_i$ from (19). For $i = \tau_p, j \in [u, p], \beta^i_j = \alpha^i_j - \frac{\varepsilon}{d_{u,k}} < \alpha^i_j \leq y_i$. For $i = \tau_p, j \in [p+1, k], \beta^i_j = \alpha^i_j + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \alpha^i_p - \delta_p + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \leq \alpha^i_p \leq y_i$, from (20). For $i \in [l+1, k] : T_i \neq \emptyset, j = k+1, \beta^i_j = \alpha^i_j + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) = \alpha^i_q - \delta_q + \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right) \leq \alpha^i_q \leq y_i$, where $q \in T_i$, from (21). For $j \in [p+1, k], i = \tau_j, \beta^i_j = \alpha^i_j - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{up}}{d_{p+1,k} d_{u,k}} < \alpha^i_j \leq y_i$.

Constraint (10) is trivially satisfied for $s < \tau_p$; or $j \notin [u, k+1]$; or $j = k+1$ and $s < l$. For $j \in [u, p], \tau_p \leq s < l$, we have $\sum_{i=1}^s \beta^i_j = \sum_{i \in [1,s] \setminus \{\tau_p\}} \alpha^i_j + \alpha^{\tau_p}_j - \frac{\varepsilon}{d_{u,k}} < \sum_{i=1}^s \alpha^i_j \leq w_s$. For $j \in [u, p], l \leq s, \sum_{i=1}^s \alpha^i_j = \sum_{i \in [1,s] \setminus \{\tau_p, l\}}^s \alpha^i_j + \alpha^{\tau_p}_j - \frac{\varepsilon}{d_{u,k}} + \alpha^l_j + \frac{\varepsilon}{d_{u,k}} = \sum_{i=1}^s \alpha^i_j \leq w_s$. For $j \in [p+1, k], \tau_p \leq s < l, \sum_{i=1}^s \beta^i_j = \sum_{i \in [1,s] \setminus \{\tau_p\}} \alpha^i_j + \alpha^{\tau_p}_j + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \leq \sum_{i \in [1,s] \setminus \{\tau_p\}} \alpha^i_p + \alpha^{\tau_p}_p - \delta_p + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \leq \sum_{i=1}^s \alpha^i_p \leq w_s$, from the definition of l and (20). For $j \in [p+1, k], l \leq s < \tau_j, \sum_{i=1}^s \beta^i_j = \sum_{i \in [1,s] \setminus \{l, \tau_p\}} \alpha^i_j + \alpha^l_j + \frac{\varepsilon}{d_{u,k}} + \alpha^{\tau_p}_j + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \leq \sum_{i \in [1,s] \setminus \{\tau_p, l\}} \alpha^i_{k+1} + \alpha^l_{k+1} - \gamma + \frac{\varepsilon}{d_{u,k}} + \alpha^{\tau_p}_{k+1} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \sum_{i=1}^s \alpha^i_{k+1} - \gamma + \frac{\varepsilon}{d_{p+1,k}} \leq w_s$, from the definition of τ_j, (17) and (22). For $j \in [p+1, k], \tau_j \leq s, \sum_{i=1}^s \beta^i_j = \sum_{i \in [1,s] \setminus \{l, \tau_p, \tau_j\}} \alpha^i_j + \alpha^{\tau_p}_j + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} + \alpha^l_j + \frac{\varepsilon}{d_{u,k}} + \alpha^{\tau_j}_j - \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} = \sum_{i=1}^s \alpha^i_j \leq w_s$. For $j = k+1, l \leq s$ we have,

$$\sum_{i=1}^s \beta^i_j = \sum_{i \in [1, l-1] \text{ or } i \in [l+1, s] : T_i = \emptyset} \alpha^i_j + \alpha^l_j - \frac{\varepsilon}{d_{k+1}} + \sum_{i \in [l+1, s] : T_i \neq \emptyset} \alpha^i_j + \sum_{i \in [l+1, s] : T_i \neq \emptyset} \frac{\mathcal{D}_i}{d_{k+1}} \left(\frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon d_{u,p}}{d_{p+1,k} d_{u,k}} \right).$$

Note that the last term in the above expression is no larger than $\frac{\varepsilon}{d_{k+1}}$ and thus for this case $\sum_{i=1}^s \beta^i_j \leq \sum_{i=1}^s \alpha^i_j \leq w_s$.

Case 3. $\tau_j > l, \forall j \in [u, k]$ (see Table 3 for an example).

Table 3: Example of α when $t > l, \tau_j > l, \forall j \in [u, k]$

				u	$u+1$	k	$k+1$
	1.00	0.75	0.15	0.00	0.00	0.00	0.00
		0.25	0.20	0.10	0.10	0.10	0.10
l			0.65	0.60↑	0.60↑	0.60↑	0.75↓
τ_u, τ_{u+1}				0.30↓	0.15↓	0.00	0.00↑
					0.15	0.00	0.15
τ_k						0.30↓	0.00↑
							0.00

Recall that $\delta_j = \alpha^{\tau_j}_j - \alpha^{\tau_j}_{k+1} > 0, j \in [u, k]$, and for $i \in [l+1, k]$, we let $T_i := \{j \in [i, k] : i = \tau_j\}$ and $\mathcal{D}_i := \sum_{j \in T_i} d_j$. Let l, k, u, λ, γ be as they were defined earlier. We construct a new vector, β, using α as follows,

$$\beta_j^i = \begin{cases} \alpha_j^i, & \text{if } i < l;\ \text{ or } i \in [l+1,k] : T_i = \emptyset;\ \text{ or } j \notin [u,k+1] \\ \alpha_j^i + \frac{\varepsilon}{d_{u,k}}, & \text{if } i = l, j \in [u,k] \\ \alpha_j^i - \frac{\varepsilon}{d_{k+1}}, & \text{if } i = l, j = k+1 \\ \alpha_j^i - \frac{\varepsilon}{d_{u,k}}, & \text{if } j \in [u,k], i = \tau_j \\ \alpha_j^i + \frac{\varepsilon \mathcal{D}_i}{d_{k+1} d_{u,k}}, & \text{if } j = k+1, i \in [l+1,k] : T_i \neq \emptyset. \end{cases} \tag{24}$$

The value of ε is chosen such that it is the largest number that satisfies (25)–(27),

$$\lambda \geq \frac{\varepsilon}{d_{u,k}} \tag{25}$$

$$\gamma \geq \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} \tag{26}$$

$$\delta_j \geq \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon \mathcal{D}_{\tau_j}}{d_{u,k} d_{k+1}}, j \in [u,k]. \tag{27}$$

This guarantees the following:

(3.a) $\beta_j^i \geq \beta_{j+1}^i$ for $i < l, j \in [u, k+1]$ since $\beta_j^i = \alpha_j^i$.

(3.b) $\beta_j^l \geq \beta_{j+1}^l, j \leq u-1$. This is true for $j \neq u-1$. For $j = u-1$, we have $\beta_{u-1}^l = \alpha_{u-1}^l = \alpha_u^l + \lambda = \alpha_u^l + \frac{\varepsilon}{d_{u,k}} + \lambda - \frac{\varepsilon}{d_{u,k}} = \beta_u^l + \lambda - \frac{\varepsilon}{d_{u,k}} \geq \beta_u^l$ from (25).

(3.c) $\beta_u^l = \cdots = \beta_k^l$. This is true since $\alpha_u^l = \cdots = \alpha_k^l$ and $\beta_j^l = \alpha_j^l + \frac{\varepsilon}{d_{u,k}}, j \in [u,k]$.

(3.d) $\beta_k^l \leq \beta_{k+1}^l$. This is because $\beta_k^l = \alpha_k^l + \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^l - \gamma + \frac{\varepsilon}{d_{u,k}} = \alpha_{k+1}^l - \frac{\varepsilon}{d_{k+1}} - \gamma + \frac{\varepsilon}{d_{u,k}} + \frac{\varepsilon}{d_{k+1}} \leq \beta_{k+1}^l$, from (26).

(3.e) Since $\varepsilon > 0$, we have $\beta_k^l > \alpha_k^l$.

From (3.a)–(3.c), we ensure that $l_\beta \geq l_\alpha$ and $k_\beta \geq k_\alpha$. If (26) is satisfied at equality, then from (3.d), it follows that $\beta_k^l = \beta_{k+1}^l$, so either $l_\beta = l_\alpha$ and $k_\beta > k_\alpha$, or $l_\beta > l_\alpha$, so β falls in a partition after that of α, a contradiction. If (26) is not satisfied at equality, then either (25) or (27) is satisfied at equality. As a result, $\beta_k^l < \beta_{k+1}^l$ and $l_\alpha = l_\beta$, $k_\alpha = k_\beta$, but $\beta_k^l > \alpha_k^l$ from (3.e), which contradicts the assumption that α_k^l has the largest value among all members in the last partition, (l,k). We now show that β is feasible in X^{FLW}.

Constraint (2) is trivially satisfied for any $j \notin [u, k+1]$. For $j \in [u,k]$, we have, $\sum_{i=1}^{j} \beta_j^i = \sum_{i \in [1,j] \setminus \{l, \tau_j\}} \alpha_j^i +$

$\alpha^l_j + \frac{\varepsilon}{d_{u,k}} + \alpha^{\tau_j}_j - \frac{\varepsilon}{d_{u,k}} = \sum_{i=1}^{j} \alpha^i_j = 1$. For $j = k+1$, we have,

$$\begin{aligned}\sum_{i=1}^{j} \beta^i_j &= \sum_{i\in[1,j]\setminus\{l\}:T_i=\emptyset} \alpha^i_j + \alpha^l_j - \frac{\varepsilon}{d_{k+1}} + \sum_{i\in[l+1,k]:T_i\neq\emptyset} \alpha^i_j + \sum_{i\in[l+1,k]:T_i\neq\emptyset} \frac{\varepsilon\mathcal{D}_i}{d_{u,k}d_{k+1}}\\ &= \sum_{i=1}^{j} \alpha^i_j - \frac{\varepsilon}{d_{k+1}} + \sum_{i\in[l+1,k]:T_i\neq\emptyset} \frac{\varepsilon\mathcal{D}_i}{d_{u,k}d_{k+1}}\\ &= \sum_{i=1}^{j} \alpha^i_j - \frac{\varepsilon}{d_{k+1}} + \frac{\varepsilon}{d_{k+1}} = 1.\end{aligned}$$

Constraint (3) is trivially satisfied for $i < l$ and for $i \in [l+1,k] : T_i = \emptyset$.

$$\sum_{j=i}^{n} \beta^i_j = \sum_{j\in[i,n]\setminus[u,k+1]} \alpha^i_j + \sum_{j=u}^{k} d_j\left(\alpha^i_j + \frac{\varepsilon}{d_{u,k}}\right) + d_{k+1}\left(\alpha^i_{k+1} - \frac{\varepsilon}{d_{k+1}}\right) = \sum_{j=i}^{n} d_j\alpha^i_j = x_i.$$

For $i \in [l+1,k] : T_i \neq \emptyset$, we have,

$$\begin{aligned}\sum_{j=i}^{n} \beta^i_j &= \sum_{j\in[i,n]\setminus T_i} d_j\alpha^i_j + \sum_{j\in T_i} d_j\left(\alpha^i_j + \frac{\varepsilon}{d_{u,k}}\right) + d_{k+1}\left(\alpha^i_{k+1} - \frac{\varepsilon\mathcal{D}_i}{d_{k+1}d_{u,k}}\right)\\ &= \sum_{j=i}^{n} d_j\alpha^i_j + \frac{\varepsilon\mathcal{D}_i}{d_{u,k}} - \frac{\varepsilon\mathcal{D}_i}{d_{u,k}} = x_i.\end{aligned}$$

Constraint (9) is trivially satisfied for $i < l$; or $i \in [l+1,k] : T_i = \emptyset$; or $j \notin [u,k+1]$. For $i = l, j = k+1, \beta^l_{k+1} = \alpha^l_{k+1} - \frac{\varepsilon}{d_{k+1}} < \alpha^l_{k+1} \leq y_l$. For, $i = l, j \in [u,k], \beta^i_j = \alpha^i_j + \frac{\varepsilon}{d_{u,k}} = \alpha^l_{k+1} - \gamma + \frac{\varepsilon}{d_{u,k}} < \alpha^l_{k+1} \leq y_i$, from (26). For $j \in [u,k], i = \tau_j, \beta^i_j = \alpha^i_j - \frac{\varepsilon}{d_{u,k}} < \alpha^i_j \leq y_i$. For $i \in [l+1,k] : T_i \neq \emptyset$ and $j \in T_i$, we have, $\beta^i_{k+1} = \alpha^i_{k+1} + \frac{\varepsilon\mathcal{D}_i}{d_{k+1}d_{u,k}} = \alpha^i_j - \delta_j + \frac{\varepsilon\mathcal{D}_i}{d_{k+1}d_{u,k}} \leq \alpha^i_j \leq y_i$, from (27).

Constraint (10) is trivially satisfied for $s < l$ or $j \notin [u,k+1]$. For $l \leq s, j = k+1$, we have, $\sum_{i=1}^{s} \beta^i_j = \sum_{i=1}^{s} \alpha^i_j - \frac{\varepsilon}{d_{k+1}} + \sum_{i\in[l+1,s]:T_i\neq\emptyset} \frac{\varepsilon\mathcal{D}_i}{d_{u,k}d_{k+1}} \leq \sum_{i=1}^{s} \alpha^i_j \leq w_s$. For $\tau_j > s \geq l, j \in [u,k]$, from the definition of τ_j and l, $\beta^i_j = \alpha^i_j = \alpha^i_{k+1}$ for $i < l$ and $\beta^i_j \leq \beta^i_{k+1}$ for $i \in [l,s]$, hence, $\sum_{i=1}^{s} \beta^i_j \leq \sum_{i=1}^{s} \beta^i_{k+1} \leq w_s$. Finally, for $j \in [u,k], l < \tau_j \leq s$, we have, $\sum_{i=1}^{s} \beta^i_j = \sum_{i\in[1,s]\setminus\{l,\tau_j\}} \alpha^i_j + \alpha^{\tau_j}_j + \alpha^l_j + \frac{\varepsilon}{d_{u,k}} - \frac{\varepsilon}{d_{u,k}} = \sum_{i=1}^{s} \alpha^i_j \leq w_s$.

In all cases, we have constructed a vector β, feasible for X^{FLW}, such that it falls in a partition in or after that of α with $\beta^l_k > \alpha^l_k$, thereby contradicting the assumption that α is a member of the last partition, (l,k), with the largest ρ^l_k value. As a result, for a given $(x,y,w) \in \text{proj}_{x,y,w}\ (X^{FLW})$, there exists ρ such that $(x,y,w,\rho) \in X^{FLW}$ and inequality (8) is satisfied. Hence, $\text{proj}_{x,y,w}(X^{SPW}) \supseteq \text{proj}_{x,y,w}(X^{FLW})$ and the proof is complete. □

In conclusion, *FLW* is indeed a tight extended formulation for ULSW. Van Vyve and Ortega (2004) fully characterize the projection of this formulation onto the original space of variables and hence provide a complete linear description of the convex hull of solutions to ULSW in its original space.

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