A Polyhedral Study of Production Ramping

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Abstract

We give strong formulations of ramping constraints — used to model the maximum change in production level for a generator or machine from one time period to the next — and production limits. For the two-period case, we give the first complete description of the convex hull of the feasible solutions. The two-period inequalities can be readily used to strengthen ramping formulations without the need for separation. For the general case, we define exponential classes of multi-period variable upper bound and multi-period ramping inequalities, and give conditions under which these inequalities define facets of ramping polyhedra. Finally, we present exact polynomial separation algorithms for the inequalities and report computational experiments on using them in a branch-and-cut algorithm to solve unit commitment problems in power generation.

Keywords: Ramping, unit commitment, co-generation, production smoothing, convex hull, polytope, valid inequalities, facets, computation

1 Introduction

In this paper, we study a polyhedral structure common in production or machining environments where there are significant start-up and fixed operating costs for generators (or machines) in addition to physical constraints on the production capacity. Of particular interest are the ramping constraints — used to model the maximum change in production level from one time period to the next — and production limits. Our main motivation to study this structure is its application in unit commitment problems in power generation; therefore, throughout the paper we use the term generator to also represent machines. However, in Section 1.1 we highlight other applications that have the same structure.

Let n be the length of the planning horizon. Throughout the paper, we let $[a, b] := \{j \in \mathbb{Z} : a \leq j \leq b\}$ ($[a, b] = \emptyset$ if $a > b$). First, we describe the physical constraints of a generator. The maximum production level when a generator is started up (and before it is shut-down) is given by $\bar{u}$. In addition, the maximum change in production from one operating period to the next, in absolute value, is limited by $\delta > 0$, and the minimum and maximum production levels in any period are limited by ℓ and u, respectively.

Next we describe the decision variables. For $t \in [1, n]$, let p_t be the production level in period t, and x_t be 1 if the generator is operating in period t and 0 otherwise. Let s_t be 1 if the generator is started in period t and 0 otherwise, and z_t be 1 if the generator is stopped in period t and 0 otherwise, for $t \in [2, n]$. Then we define the *ramp-up polytope* as the convex hull of ramp-up constraints and the production limits for a single generator given by

$$\ell x_t \leq p_t, \qquad t \in [1, n] \tag{1a}$$

$$p_t \leq u x_t, \qquad t \in [1, n] \tag{1b}$$

$$x_{t+1} - x_t \leq s_{t+1}, \qquad t \in [1, n-1] \tag{1c}$$

$$s_{t+1} \leq 1 - x_t, \qquad t \in [1, n-1] \tag{1d}$$

$$s_t \leq x_t, \qquad t \in [2, n] \tag{1e}$$

$$p_{t+1} - p_t \leq \bar{u} s_{t+1} + \delta x_t, \qquad t \in [1, n-1] \tag{1f}$$

$$s \in \{0, 1\}^{n-1}, x \in \{0, 1\}^n, p \in \mathbb{R}^n_+. \tag{1g}$$

We refer to the feasible set defined by constraints (1a)-(1g) as $\mathcal{U}$. We also define the *ramp-down polytope* as the convex hull of ramp-down constraints and the production limits for a single generator given by constraints (1a), (1b), and by

$$x_t - x_{t+1} \leq z_{t+1}, \qquad t \in [1, n-1] \tag{2c}$$

$$z_t \leq 1 - x_t, \qquad t \in [2, n] \tag{2d}$$

$$z_t \leq x_{t-1}, \qquad t \in [2, n] \tag{2e}$$

$$p_t - p_{t+1} \leq \bar{u} z_{t+1} + \delta x_{t+1}, \qquad t \in [1, n-1] \tag{2f}$$

$$z \in \{0, 1\}^{n-1}, x \in \{0, 1\}^n, p \in \mathbb{R}^n_+. \tag{2g}$$

We refer to the feasible set defined by constraints (1a), (1b), and (2c)-(2g) as $\mathcal{D}$. Constraints (1f) and (2f) ensure that the production level in the first (last) period the generator is started up (shut down) is at most $\bar{u}$, and that the absolute value of the difference in production levels from one period to the next is at most δ. Observe that we replace inequality (1f) in formulation $\mathcal{U}$ with inequality

(2f), and represent the relationship between the shut-down variables z and on/off variables x with inequalities (2c)-(2e), to obtain the formulation $\mathcal{D}$ in the space of (p, x, z).

For simplicity of notation, we define $\bar{u}$ as the maximum production level for both the start-up and shut-down of a generator and similarly, the maximum change for both ramp-up and ramp-down is represented by δ. However, these values can be different for start-up, shut-down, ramp-up and ramp-down and all the results in this study will still hold, since we study the ramp-up and ramp-down polytopes ($\mathcal{U}$ and $\mathcal{D}$, respectively) separately.

Throughout the paper, we make the following assumptions to ignore uninteresting cases:

(A1) $u \geq \bar{u} \geq \ell$,

(A2) $u - \ell \geq \delta > 0$.

Note that if Assumption (A1) does not hold and the maximum production level when a generator is started up/shut down $\bar{u}$ is strictly smaller than the minimum production level ℓ, then we cannot start up or shut-down this generator (from inequalities (1a)). In other words, we can fix variables s_t and z_t to 0 for all $t \in [2, n]$, and eliminate them from the formulation. Additionally, if for a generator the maximum production level when it is started up or shut down $\bar{u}$ is strictly greater than maximum production level u, then we can strengthen the original ramping inequalities (1f) and (2f) in formulation $\mathcal{U}$ and $\mathcal{D}$, respectively, by replacing $\bar{u}$ with u. Similarly, if Assumption (A2) does not hold and the maximum ramping rate δ is greater than the difference between maximum production and minimum production levels $u - \ell$, then we can let $\delta = u - \ell$, because we cannot ramp more than this difference. Note that Assumption (A2) also implies that $u > \ell$. It is easy shown that the ramp-up and ramp-down polytopes $conv(\mathcal{U})$ and $conv(\mathcal{D})$ are full-dimensional under Assumptions (A1)-(A2).

Fan et al. (2002) and Frangioni and Gentile (2006) show that a unit commitment problem with a single generator is solvable in $O(n^3)$. Damcı-Kurt (2014) gives a different $O(n^3)$ algorithm to optimize over the ramping polytope, which is a special case of the unit commitment problem considered in Fan et al. (2002) and Frangioni and Gentile (2006).

1.1 Motivation

With deregulation of the energy industry and higher penetration of wind and other intermittent power supplies, the problem of scheduling power generators to meet the load (demand for energy) over large geographical regions has become increasingly challenging. At the crux of most power system operations is the so-called unit commitment problem (UC), which seeks to determine a minimum cost production schedule of a set of power generators to meet the load while satisfying a host of operational constraints (Sheble and Fahd, 1994). Two main sets of decisions are made in a UC problem. The first set of decisions determines which generators to turn on/off at each time period, whereas the second set determines the amount of output of each online generator at each time period so that the load is met. Operational constraints on the generators include spinning reserves, min/max electricity output levels, minimum up/down time and ramping up/down limits, among others.

The combinatorial nature of the operational constraints makes the UC problem particularly difficult to solve to optimality for practical large-scale instances. Even small improvements in the quality of solutions affect the price of electricity over large geographical regions and lead to millions of dollars of savings per day for the consumers. Therefore, independent system operators

(ISOs) are keen to find provably optimal solutions to the UC problem. Over the years, various algorithmic approaches such as dynamic programming (Lowery, 1966, Snyder et al., 1987), branch-and-bound (Cohen and Yoshimura, 1983), Benders decomposition (Baptisella and Geromel, 1980), Lagrangian relaxation (Hobbs et al., 2001), unit decommitment (Tseng et al., 2000), genetic algorithms (Kazarlis et al., 1996, Orero and Irving, 1998), simulated annealing (Zhuang and Galiana, 1990) and tabu search (Mantawy et al., 1998) have been proposed to find near-optimal feasible solutions to the UC problem. The reader is referred to Saravanan et al. (2013) for a recent review of solution approaches that consider both deterministic and stochastic loads.

Recent advances in mixed-integer programming (MIP) software have made it possible to solve larger instances of the UC problem to optimality. MIP formulations also offer additional modeling flexibility to handle challenging operational constraints. Garver (1962) describes the first MIP formulation for the UC problem, which has been used extensively. However, there has been limited research on the polyhedral structure of the UC problem to strengthen the MIP formulations in order to leverage the advances in the state-of-the-art optimization software. One of the exceptions is the work of Lee et al. (2004), which considers a relaxation of the UC problem with only the minimum up/down time constraints. The authors propose alternating up/down inequalities that are valid for this relaxation and show that the formulation is tight. Subsequently, Rajan and Takriti (2005) give a compact extended formulation of this relaxation, which includes the additional start-up and shut-down variables.

In this paper, we consider a different relaxation of the unit commitment problem with ramping constraints and production limits. In addition to the unit commitment problem faced by the utilities, ramping constraints also appear in the cogeneration of electrical and thermal power in commercial and large-scale residential buildings, where the heating and cooling rates of the boilers, and the ramping rates of the turbines need to be kept within safe limits (Havel and Šimovič, 2013, Rong and Lahdelma, 2007). Ramping constraints also arise in production or order lot-sizing problems in supply chain management (Pochet and Wolsey, 1995, Pochet, 2001, Pochet and Wolsey, 2006). Constantino (1996, 1998), Agra and Constantino (1999) study the polyhedral structure of lot sizing with start-ups, production lower and upper bounds. In addition, Silver (1967) and Pekelman (1975) observe that in certain environments, production smoothing constraints $|p_{t+1} - p_t| \leq \delta$ are necessary to avoid large fluctuations in the production floor. However, to the best of our knowledge, there is no polyhedral study that considers production smoothing in the lot-sizing context. Note that lot-sizing polytope also includes inventory variables and constraints, which may not be present in unit commitment without energy storage. On the other hand, the minimum up and down-time constraints of unit commitment generally do not apply in lot-sizing. Hence the structure we study captures the common elements in these different production environments.

Carrion and Arroyo (2006), Frangioni et al. (2009) and Wang et al. (2012) consider alternative formulations to represent the ramping constraints. These formulations can be strengthened with the addition of the start-up and shut-down variables (Ostrowski et al., 2012, Morales-España et al., 2013a,b). Ostrowski et al. (2012) give polynomial classes of upper bound, and two- and three-period ramp-up and ramp-down inequalities to strengthen their formulation. Our polyhedral study complements that of Ostrowski et al. (2012) by providing several exponential classes of multi-period ramping and multi-period variable upper bound inequalities.

The organization of the paper is as follows. In Section 2, we study the ramping relaxation of the UC problem with two time periods ($n = 2$) and develop new inequalities that give the complete convex hull description of this relaxation. This allows us to understand the structure in a simpler variant, before we extend our results to the general case. In Section 3, we generalize these inequalities to multiple periods and propose several exponential classes of valid inequalities for ramp-up and ramp-down relaxations ($\mathcal{U}$ and $\mathcal{D}$, respectively). Furthermore, we prove the strength of these inequalities and describe exact polynomial separation algorithms for them. Finally, in Section 4, we provide computational results that show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the unit commitment problem with ramping constraints.

2 Two-Period Ramping Polytope

In this section, we study the two-period ramping polytope in detail. The purpose of focusing on the two-period case is to find the ideal ramping constraints for formulating ramping. The linear number of inequalities defined for this case can be readily used in strengthening UC formulations without the need for separation.

For $t \in [1, n-1]$, let the corresponding two-period ramp-up polytope be given by $\mathcal{U}_t^2 = \{(p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1}) \in \mathbb{R}_+^2 \times \{0,1\}^3 : x_{t+1} - x_t \leq s_{t+1}; s_{t+1} \leq 1 - x_t; s_{t+1} \leq x_{t+1}; p_{t+1} - p_t \leq \bar{u}s_{t+1} + \delta x_t; \ell x_j \leq p_j \leq u x_j, j = t, t+1\}$. The two-period ramp-down polytope $\mathcal{D}_t^2$, for $t \in [1, n-1]$ is defined similarly.

2.1 Inequalities for Two-Period Ramp-Up Polytope

Consider the ramping constraint (1f), which was first introduced by Ostrowski et al. (2012). Analyzing the left hand side (LHS) of the ramping constraint, in any integral feasible solution, we can see that $p_{t+1} - p_t$ can be bounded from above based on the values of x_{t+1}, x_t, s_{t+1}, as illustrated in Table 1. Any inequality that has a right hand side (RHS) that is no smaller than this upper bound for each feasible value of x_{t+1}, x_t, s_{t+1} is valid. We see from Table 1 that (1f) is valid. The strongest possible inequality would attain the upper bounds on LHS as its RHS for all possible feasible values of x_{t+1}, x_t, s_{t+1}. First, we present new valid inequalities for the two-period ramp-up polytope ($\mathcal{U}_t^2$) for any $t \in [1, n-1]$. In Section 2.2, we present the convex hull of the two-period ramp-up polytope, which includes the new inequalities. Finally, in Section 2.3. we present the analogous inequalities for the two-period ramp-down polytope $\mathcal{D}_t^2$.

Table 1: Comparison of two-period ramp-up inequalities.

			Upper bound on LHS	RHS	
x_t	x_{t+1}	s_{t+1}	$p_{t+1} - p_t$	(1f)	(3)
0	0	0	0	0	0
0	1	1	$\bar{u}$	$\bar{u}$	$\bar{u}$
1	0	0	$-\ell$	δ	$-\ell$
1	1	0	δ	δ	δ

Proposition 1. *For $t \in [1, n-1]$, the two-period ramp-up inequality*

$$p_{t+1} - p_t \leq (\bar{u} - \ell - \delta)s_{t+1} + (\ell + \delta)x_{t+1} - \ell x_t \tag{3}$$

is valid and defines a facet of $conv(\mathcal{U}_t^2)$*.*

Proof. From the column labeled (3) in Table 1, we see that this inequality is valid. From the following five affinely independent points $(p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1})$ in $\mathcal{U}_t^2$, we see that (3) defines a facet of $conv(\mathcal{U}_t^2)$: $(0,0,0,0,0), (0,\bar{u},0,1,1)$, $(\ell,0,1,0,0), (\ell,\ell+\delta,1,1,0), (u-\delta,u,1,1,0)$. □

From the RHS columns in Table 1, we see that (3) is the strongest ramping inequality one can derive for the left hand side, $p_{t+1} - p_t$, and therefore dominates the two-period ramping inequality (1f) used in the definition of $\mathcal{U}_t^2$.

We now consider the upper bound constraint (1b) for period $t+1$. This inequality can be strengthened if the generator starts up in period $t+1$, as follows:

Proposition 2. *For* $t \in [1, n-1]$*, the two-period variable upper bound (VUB) ramp-up inequality*

$$p_{t+1} \leq u x_{t+1} - (u - \bar{u}) s_{t+1} \tag{4}$$

is valid and defines a facet of $conv(\mathcal{U}_t^2)$*.*

Proof. When $s_{t+1} = 0$, inequality (4) is the same as (1b), which is valid. When $s_{t+1} = 1$, the production level in period $t+1$ can be no greater than $\bar{u}$, which is the right hand side of (4). From the following five affinely independent points $(p_t, p_{t+1}, x_t, x_{t+1}, s_{t+1})$ in $\mathcal{U}_t^2$, we see that (4) defines a facet of $conv(\mathcal{U}_t^2)$: $(0,0,0,0,0), (0,\bar{u},0,1,1)$, $(\bar{u},0,1,0,0), (u,u,1,1,0), (u-\delta,u,1,1,0)$. □

Note that inequalities (3) and (4) are new ramping and upper bound inequalities that can be used to strengthen the existing MIP formulations of the UC problem with start-up variables. Ostrowski et al. (2012) present a polynomial class of VUB inequalities for a related problem which also includes minimum up (down) time constraints which impose a constraint on the number of periods the generator needs to stay on (off) before it can be shut down (started up). For this more restricted problem, the inequalities proposed by Ostrowski et al. (2012) are the same as two-period VUB inequality (4) when minimum up time, denoted by $\overline{v}$, equals to one, but they are not valid for $\mathcal{U}$ when $\overline{v} \geq 2$.

2.2 Convex Hull for Two-Period Ramp-Up Polytope

In this section, we present the convex hull of the two-period ramp-up polytope ($conv(\mathcal{U}_t^2)$), and the convex hull of its projection on to the space without start variables for any $t \in [1, n-1]$. For conciseness, we do not give the analogous results on $conv(\mathcal{D}_t^2)$. We will see that $conv(\mathcal{U}_t^2)$ can be described using trivial inequalities and the new ramp-up inequalities (3) and (4). (No other inequalities are needed.) By completely describing this simpler two-period polytope, we start from the strongest possible two-period ramping inequalities before generalizing them to the multi-period setting, deriving strong valid inequalities for $\mathcal{U}$ and $\mathcal{D}$ in Section 3.

First, we give the convex hull of the two-period ramp-up polytope ($conv(\mathcal{U}_t^2)$) for any given

$t \in [1, n-1]$. Consider the LP

$$s_{t+1} \leq x_{t+1} \tag{5a}$$
$$s_{t+1} \geq x_{t+1} - x_t \tag{5b}$$
$$s_{t+1} \geq 0 \tag{5c}$$
$$p_{t+1} \geq \ell x_{t+1} \tag{5d}$$
$$p_t \geq \ell x_t \tag{5e}$$
$$p_{t+1} - p_t \leq (\ell + \delta)x_{t+1} + (\bar{u} - \ell - \delta)s_{t+1} - \ell x_t \tag{5f}$$
$$p_{t+1} \leq u x_{t+1} - (u - \bar{u})s_{t+1} \tag{5g}$$
$$p_t \leq u x_t \tag{5h}$$
$$s_{t+1} \leq 1 - x_t. \tag{5i}$$

Observe that (5f) is the same as (3), and that (5g) is the same as (4). Note that we drop the integer restrictions on x and s. We also drop the bounds on variables x_t and x_{t+1} because inequalities (5a) and (5b) imply that $x_t \geq 0$, inequalities (5c) and (5i) imply that $x_t \leq 1$, inequalities (5a) and (5c) imply that $x_{t+1} \geq 0$, and inequalities (5b) and (5i) imply that $x_{t+1} \leq 1$.

Theorem 1. $conv(\mathcal{U}_t^2) = \{(p, x, s) \in \mathbb{R}^5 : (5a) - (5i)\}$.

Proof. Note that $conv(\mathcal{U}_t^2)$ is bounded because all the variables are bounded. We will prove this theorem by showing that every extreme point of the polytope defined by (5a)-(5i) is integral. To do this, we will consider the intersection of five linearly independent inequalities among the inequalities (5a)-(5i) ($\binom{9}{5} = 126$ possible points). However, all inequalities except inequality (5i) intersect at the origin. Therefore, we will let inequality (5i) hold at equality (i.e., $s_{t+1} = 1 - x_t$) and choose four out of the remaining eight inequalities (5a)-(5h) to be tight to obtain extreme points different than the origin. Hence, $\binom{8}{4} = 70$ points have to be considered. Throughout the proof we make use of the following observations.

Observation 1. If $\delta = u - \ell$, then the ramping inequality (5f) becomes $p_{t+1} - p_t \leq u x_{t+1} - (u - \bar{u})s_{t+1} - \ell x_t$, and it is not a facet because it is dominated by inequalities (5e) and (5g). Because it cannot be a facet, inequalities (5a)-(5e) and (5g)-(5i) are enough to give $conv(\mathcal{U}_t^2)$ in this case. In the following proof, we assume that $u > \ell + \delta$ if we are considering inequality (5f) as an inequality that holds at equality in an extreme point.

Observation 2. If inequalities (5e) and (5h) are both satisfied at equality in any solution, then we must have $x_t = 0$ because $u > \ell$ from Assumption (A2).

Next we consider the different cases where, in addition to inequality (5i), four inequalities from inequalities (5a)-(5h) are satisfied at equality.

1. Assume that inequality (5a) is satisfied at equality.

   In this case we obtain $s_{t+1} = 1 - x_t = x_{t+1}$. We need three more inequalities from seven remaining inequalities (5b)-(5h) to be satisfied at equality, i.e., $\binom{7}{3} = 35$ cases to consider. Next we show the simplifications in these inequalities, if there are any.

   - Inequality (5b) reduces to $x_t \geq 0$ and $x_{t+1} \leq 1$. Thus, if inequality (5b) holds at equality, then $x_t = 0$, $x_{t+1} = s_{t+1} = 1$, which is integral. Therefore, we do not need to consider this case.

- Inequality (5c) reduces to $x_{t+1} \geq 0$ and $x_t \leq 1$. Thus, if inequality (5c) holds at equality, then $s_{t+1} = x_{t+1} = 0$ and $x_t = 1$, which is integral. Therefore, we do not need to consider this case.
- Inequality (5f) reduces to $p_{t+1} - p_t \leq \bar{u}x_{t+1} - \ell x_t$, which is dominated by inequalities (5a), (5e) and (5g). Thus, we do not need to consider this case.
- Inequality (5g) reduces to $p_{t+1} \leq \bar{u}x_{t+1}$.

We are only left with the case where inequalities (5a), (5i) and three out of inequalities (5d), (5e), (5g) and (5h) are satisfied at equality. If inequalities (5d) and (5g) are both satisfied at equality, then we obtain $(u - \ell)x_{t+1} = (u - \bar{u})s_{t+1}$. If $\bar{u} > \ell$, then $x_{t+1} = 0 = s_{t+1}$ and $x_t = 1$ at this extreme point, which is integral. If $\bar{u} = \ell$, then (5a), (5i), (5d) and (5g) are linearly dependent so this case cannot correspond to an extreme point. Inequalities (5e) and (5h) can both be satisfied at equality only if $x_t = 0$ (from Observation 2). In this case, we have $x_{t+1} = 1 = s_{t+1}$, which is also an integral point.

2. Assume that inequality (5b) is satisfied at equality (and inequality (5a) is not).

   In this case we obtain $s_{t+1} = x_{t+1} - x_t = 1 - x_t$ and $s_{t+1} < x_{t+1}$. Thus, $x_{t+1} = 1$, $x_t > 0$ and $s_{t+1} < 1$. We need three more inequalities from six remaining inequalities (5c)-(5h) to hold at equality, i.e., $\binom{6}{3} = 20$ cases to consider. Next, we show the simplifications in inequalities, if there are any.

   - Inequality (5c) reduces to $x_t \leq 1$. Thus, if inequality (5c) holds at equality, then $x_t = 1 = x_{t+1}$, $s_{t+1} = 0$, which is integral. Therefore, we do not need to consider this case.
   - Inequality (5d) reduces to $p_{t+1} \geq \ell$.
   - Inequality (5f) reduces to $p_{t+1} - p_t \leq \bar{u} - (\bar{u} - \delta)x_t$.
   - Inequality (5g) reduces to $p_{t+1} \leq \bar{u} + (u - \bar{u})x_t$.

   So we have to choose three out of inequalities (5d)-(5h) to hold at equality. There are $\binom{5}{3} = 10$ cases to consider. Note that we cannot have inequalities (5e) and (5h) both hold at equality, because we assume that $x_t > 0$ (from Observation 2). Hence, there are only seven cases left. Also, if we let inequalities (5d) and (5g) hold at equality, then $\ell = \bar{u} + (u - \bar{u})x_t = ux_t + \bar{u}(1 - x_t)$ which cannot hold because $0 < x_t \leq 1$, $\bar{u} \geq \ell$ and $u > \ell$ from Assumption (A2). So there are only four cases left.

   - If we choose inequalities (5e) and (5f) to hold at equality, then we have $p_t = \ell x_t$ and $p_{t+1} = \bar{u} - (\bar{u} - \ell - \delta)x_t$. We need to consider two cases. If inequality (5g) is chosen as the third inequality that holds at equality, then $(\ell + \delta - \bar{u})x_t = (u - \bar{u})x_t$, which is infeasible because $x_t > 0$ and, from Observation 1, $u > \ell + \delta$. If inequality (5d) is chosen as the third inequality that holds at equality, then $(\bar{u} - \ell - \delta)x_t = \bar{u} - \ell$. Because $1 \geq x_t > 0$ in this case and $\delta > 0$, this extreme point is infeasible.
   - If we choose inequalities (5f) and (5h) to hold at equality, then we have $p_t = ux_t$ and $p_{t+1} = \bar{u} + (u - \bar{u} + \delta)x_t$. If inequality (5d) is chosen as the third inequality that holds at equality, then $p_{t+1} = \ell = \bar{u} + (u - \bar{u} + \delta)x_t$. So we obtain $(u - \bar{u} + \delta)x_t = \ell - \bar{u}$. Because $x_t > 0$, $\delta > 0$ and $u > \ell$ (from Assumption (A2)), this point is not feasible. If inequality

(5g) is chosen as the third inequality that holds at equality, then $p_{t+1} = \bar{u} + (u - \bar{u})x_t = \bar{u} + (u - \bar{u} + \delta)x_t$. Because $x_t, \delta > 0$, this solution is not feasible.

3. Assume that inequality (5c) is satisfied at equality (and inequalities (5a) and (5b) are not).

   In this case, we obtain $s_{t+1} = 0 = 1 - x_t$, $s_{t+1} < x_{t+1}$ and $s_{t+1} > x_{t+1} - x_t$. Thus, $x_t = 1$, $x_{t+1} > 0$ and $x_{t+1} < 1$. We need three more inequalities from five remaining inequalities (5d)-(5h) to hold at equality, i.e., $\binom{5}{3} = 10$ cases to consider. Next, we show the simplifications in inequalities, if there are any.

   - Inequality (5e) reduces to $p_t \geq \ell$.
   - Inequality (5f) reduces to $p_{t+1} - p_t \leq (\ell + \delta)x_{t+1} - \ell$.
   - Inequality (5g) reduces to $p_{t+1} \leq u x_{t+1}$.
   - Inequality (5h) reduces to $p_t \leq u$.

   We divide the cases as follows:

   - If we let inequalities (5d) and (5g) hold at equality, then $p_{t+1} = u x_{t+1} = \ell x_{t+1}$. Because $x_{t+1} > 0$ and $u > \ell$ (from Assumption (A2)), this point is infeasible (from Observation 2).
   - If we let inequalities (5e) and (5h) hold at inequality, then $p_t = \ell = u$ which is infeasible because $u > \ell$ (from Assumption (A2)).
   - If we choose inequalities (5f) and (5h) to hold at equality, then $p_t = u$. If inequality (5g) is chosen as the third inequality to hold at equality, then $(u - \ell - \delta)x_{t+1} = (u - \ell)$. Because $x_{t+1} < 1$ and $\delta > 0$, this point is infeasible. If inequality (5d) is chosen as the third inequality to hold at equality, then $\delta x_{t+1} = \ell - u$, which is infeasible because $x_{t+1} > 0$ and from Observation 1 we have $u > \ell + \delta$.
   - If we choose inequalities (5e) and (5f) to hold at inequality, then $p_t = \ell$ and $p_{t+1} = (\ell + \delta)x_{t+1}$. We need to consider two cases. If inequality (5g) is chosen as the third inequality that is satisfied at equality, then we have $p_{t+1} = u x_{t+1} = (\ell + \delta)x_{t+1}$. This point is infeasible because $x_{t+1} > 0$ and $u > \ell + \delta$ from Observation 1. If inequality (5d) is chosen as the third inequality that holds at equality, then we have $p_{t+1} = \ell x_{t+1} = (\ell + \delta)x_{t+1}$. Because $x_{t+1}, \delta > 0$, this point is infeasible.

4. Assume that inequality (5d) is satisfied at equality (and inequalities (5a),(5b) and (5c) are not).

   In this case, $p_{t+1} = \ell x_{t+1}$, $s_{t+1} = 1 - x_t$, $s_{t+1} < x_{t+1}$, $s_{t+1} > x_{t+1} - x_t$, and $s_{t+1} > 0$. Hence, $0 < x_t < 1$ and $0 < x_{t+1} < 1$. We need three more inequalities from four remaining inequalities (5e)-(5h) to hold at equality, i.e., there are 4 cases to consider. Note, from Observation 2, that we cannot have inequalities (5e) and (5h) both hold at equality because $x_t > 0$. Therefore, inequalities (5f) and (5g) must hold at equality and one of inequalities (5e) or (5h) must hold at equality. In this case, inequality (5g) becomes $(u - \ell)x_{t+1} + (u - \bar{u})x_t \geq (u - \bar{u})$. Because inequality (5g) holds at equality, $x_{t+1} = \frac{(u-\bar{u})}{(u-\ell)}(1 - x_t) = \frac{(u-\bar{u})}{(u-\ell)} s_{t+1}$. Recall that $\bar{u} \geq \ell$, from (A1), so we obtain $x_{t+1} \leq s_{t+1}$, which contradicts our assumption that $x_{t+1} > s_{t+1}$.

5. Assume that inequality (5e) is satisfied at equality (and inequalities (5a),(5b), (5c) and (5d) are not).

   In this case, $p_t = \ell x_t$, $s_{t+1} = 1 - x_t$, $s_{t+1} < x_{t+1}$, $s_{t+1} > x_{t+1} - x_t$, $s_{t+1} > 0$ and $p_{t+1} > \ell x_{t+1}$. Hence, $0 < x_t < 1$ and $0 < x_{t+1} < 1$. We need all three of the remaining inequalities (5f)-(5h) to hold at equality. However, this point is not feasible from Observation 2 and the assumption that $x_t > 0$.

We have showed that all intersections of five linearly independent constraints among inequalities (5a)-(5i), if feasible, give integral extreme points. Hence, the proof is complete. □

The unit commitment problem can also be formulated without the start-up variables (see Carrion and Arroyo (2006) and Frangioni et al. (2009)). The ramp-up polytope without start variables is given by

$$\ell x_t \le p_t, \qquad t \in [1, n] \tag{6a}$$
$$p_t \le u x_t, \qquad t \in [1, n] \tag{6b}$$
$$p_{t+1} - p_t \le \bar{u} - (\bar{u} - \delta) x_t, \qquad t \in [1, n-1] \tag{6c}$$
$$x \in \{0, 1\}^n, p \in \mathbb{R}^n_+. \tag{6d}$$

We refer to the feasible set defined by constraints (6a)-(6d) as $\mathcal{UNS}$. By projecting out the start-up variables in the convex hull definition of $\mathcal{U}^2_t$, for $t \in [1, n-1]$, we obtain new inequalities for this more compact formulation as well. For a given $t \in [1, n-1]$, let the two-period ramp-up polytope without start variables be $\mathcal{UNS}^2_t = \{(p, x) \in \mathbb{R}^2_+ \times \{0, 1\}^2 : p_{t+1} - p_t \le \bar{u} - (\bar{u} - \delta)x_t; \ell x_j \le p_j \le u x_j, j = t, t+1\}$. Next, we describe the constraints that we use to describe $conv(\mathcal{UNS}^2_t)$:

$$p_t \le u x_t \tag{7}$$
$$p_t \ge \ell x_t \tag{8}$$
$$p_{t+1} \ge \ell x_{t+1} \tag{9}$$
$$p_{t+1} \le \bar{u} x_{t+1} + (u - \bar{u}) x_t \tag{10}$$
$$x_{t+1} \le 1 \tag{11}$$
$$p_{t+1} \le u x_{t+1} \tag{12}$$
$$x_t \le 1, \tag{13}$$

and if $\bar{u} \le \ell + \delta$, then

$$p_{t+1} - p_t \le \bar{u} x_{t+1} - (\bar{u} - \delta) x_t \tag{14}$$
$$p_{t+1} - p_t \le (\ell + \delta) x_{t+1} - \ell x_t, \tag{15}$$

else if $\bar{u} > \ell + \delta$, then

$$p_{t+1} - p_t \le (\bar{u} - \ell - \delta) + (\ell + \delta) x_{t+1} - (\bar{u} - \delta) x_t \tag{16}$$
$$(u - \ell - \delta) p_{t+1} - (u - \bar{u}) p_t \le \bar{u}(u - \ell - \delta) x_{t+1} - \ell(u - \bar{u}) x_t. \tag{17}$$

Corollary 1. *For $\bar{u} \le \ell + \delta$, $conv(\mathcal{UNS}^2_t) = \{(p, x) \in \mathbb{R}^{2n} : (7) - (15)\}$, and for $\bar{u} > \ell + \delta$, $conv(\mathcal{UNS}^2_t) = \{(p, x) \in \mathbb{R}^{2n} : (7) - (13), (16), (17)\}$.*

Proof. See Appendix A. □

Note that inequalities (10) and (14)-(17) are upper bound and ramping inequalities that can be used to strengthen the existing formulations of the UC problem without start variables. In particular, inequality (17) has an interesting structure, because the coefficients of the production variables p_t and p_{t+1} are not necessarily -1 and 1, respectively, as is the case in all known inequalities representing ramping.

2.3 Inequalities for Two-Period Ramp-Down Polytope

Using the symmetry between ramping up and ramping down constraints, we can derive the ramp-down analogues of the ramp-up inequality (3) and the variable upper bound inequality (4).

Proposition 3. *For $t \in [1, n-1]$, the two-period ramp-down inequality*

$$p_t - p_{t+1} \leq (\bar{u} - \ell - \delta)z_{t+1} + (\ell + \delta)x_t - \ell x_{t+1} \tag{18}$$

is valid and defines a facet of $conv(\mathcal{D}_t^2)$.

Proof. Similar to the proof of Proposition 1 using analogous arguments, and substituting for z_{t+1}. □

Proposition 4. *For $t \in [1, n-1]$, the two-period VUB ramp-down inequality*

$$p_t \leq ux_t - (u - \bar{u})z_{t+1} \tag{19}$$

is valid and defines a facet of $conv(\mathcal{D}_t^2)$.

Proof. Similar to the proof of Proposition 2. □

3 Facets of Multi-Period Ramping Polytope

3.1 Ramp-Up Polytope

In this section, we present multi-period variable upper bound (VUB) and two classes of multi-period ramp-up inequalities for $\mathcal{U}$ for any $n \geq 2$. In Section 3.2, we study the strength of these inequalities, and present necessary and sufficient conditions under which they define facets of $\mathcal{U}$.

Proposition 5. *If $\bar{u} \geq \ell + \delta$, then for $1 \leq t \leq n$ and $1 \leq j \leq \min\left\{n-t, \frac{\bar{u}-\ell}{\delta}\right\}$, the type-I multi-period ramp-up inequality*

$$p_{t+j} - p_t \leq (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\}s_{t+i} - \ell x_t \tag{20}$$

is valid for $\mathcal{U}$.

Proof. There are two cases to consider.

Case 1. Suppose that $x_{t+j} = 0 (= p_{t+j})$. Because $\min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\} \geq 0$ for $1 \leq i \leq j$, $u \geq \bar{u}$ from (A1) and $p_t \geq \ell x_t$, inequality (20) is clearly valid for this case.

Case 2. Suppose that $x_{t+j} = 1$.

i. Suppose that the last period when start-up occurred during periods 1 through $t + j$ is $t + k$, where $1 \leq k \leq j$, i.e., $s_{t+k} = 1$, and $s_{t+i} = 0$ for all $k + 1 \leq i \leq j$. Then, $p_{t+j} \leq \min\{(\bar{u} + (j - k)\delta), u\}$. Also $p_t \geq \ell x_t$. Therefore,

$$\begin{aligned} p_{t+j} - p_t &\leq \min\{(\bar{u} + (j-k)\delta), u\} - \ell x_t \\ &= (\ell + j\delta)x_{t+j} + \min\{(\bar{u} - \ell - k\delta), (u - \ell - j\delta)\}s_{t+k} - \ell x_t \\ &\leq (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\}s_{t+i} - \ell x_t. \end{aligned}$$

ii. Suppose that the last period when start-up occurred during periods 1 through $t + j$ is k, where $k \leq t$, i.e., $s_{t+i} = 0$ for all $1 \leq i \leq j$ and $x_{t+i} = 1$ for all $0 \leq i \leq j$. Therefore,

$$p_{t+j} - p_t \leq j\delta = (\ell + j\delta)x_{t+j} + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - i\delta), (u - \ell - j\delta)\}s_{t+i} - \ell x_t.$$

□

Observe that the two-period ramping inequality (5f) is a special case of type-I multi-period ramp-up inequality (20) with $j = 1$ if $\bar{u} \geq \ell + \delta$.

Proposition 5 describes a polynomial class of inequalities. Hence, its separation is polynomial. Next, we present another class of multi-period ramp-up inequalities. Let $a^+ = \max\{0, a\}$.

Proposition 6. *For $1 \leq t \leq n$ and $1 \leq j \leq \min\left\{n - t, \frac{u-\ell}{\delta}\right\}$, let $S \subseteq [t + 1, t + j]$, $t + j \in S$, $q = \min\{k \in S\}$, and $d_i = \max\{k \in S \cup \{t + 1\} : k < i\}$, $i \in S$. Then the type-II multi-period ramp-up inequality*

$$p_{t+j} - p_t \leq \bar{u}x_{t+j} + \delta \sum_{i \in S \setminus \{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t, \tag{21}$$

where $\phi = (\ell + \delta - \bar{u})^+$, is valid for $\mathcal{U}$.

Proof. There are two cases to consider.

Case 1. Suppose that $x_{t+j} = 0 (= p_{t+j})$. Inequality (21) is clearly valid for this case.

Case 2. Suppose that $x_{t+j} = 1$. Note that $x_i - s_i = 0$, unless the generator is on in period i but started up earlier than i, in which case $x_i - s_i = 1$.

i. Suppose that the last period when start-up occurred on or before $t + j$ is $t + k$, where $1 \leq k \leq j$, i.e., $s_{t+k} = 1$, and $s_{t+i} = 0$ for all $k + 1 \leq i \leq j$. Then, $p_{t+j} \leq \bar{u} + (j - k)\delta$. Also $p_t \geq \ell x_t$. Therefore,

$$\begin{aligned} p_{t+j} - p_t &\leq \bar{u} + (j - k)\delta - \ell x_t \\ &\leq \bar{u}x_{t+j} + \delta \sum_{i \in (S \cap [t+k+1, t+j]) \setminus \{t+1\}} (i - d_i)(x_i - s_i) - \ell x_t \\ &\leq \bar{u}x_{t+j} + \delta \sum_{i \in S \setminus \{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t. \end{aligned}$$

The second inequality is valid because $x_{t+j} = 1$ and $x_i - s_i = 1$ for all $i \in [t+k+1, t+j]$ so $(j-k)\delta \leq \delta \sum_{i \in (S \cap [t+k+1,t+j]) \setminus \{t+1\}} (i - d_i)(x_i - s_i)$. The last inequality is clearly valid because only non-negative terms are added.

ii. Suppose that the last period when start-up occurred on or before $t+j$ is k, where $k \leq t$, i.e., $s_{t+i} = 0$ for all $1 \leq i \leq j$ and $x_{t+i} = 1$ for all $0 \leq i \leq j$. Therefore,

$$\begin{aligned} p_{t+j} - p_t &\leq j\delta \leq j\delta + (\bar{u} - \delta - \ell + \phi) \\ &= \bar{u} + (j-1)\delta + \phi - \ell \\ &= \bar{u}x_{t+j} + \delta \sum_{i \in S \setminus \{t+1\}} (i - d_i)(x_i - s_i) + \phi(x_q - s_q) - \ell x_t. \end{aligned}$$

Note that $\bar{u} - \delta - \ell + \phi \geq 0$ by the definition of ϕ.

□

Observe that the two-period inequality (5f) is the special case of the type-II multi-period ramp-up inequality (21) with $j = 1$ if $\bar{u} \leq \ell + \delta$.

In Proposition 7 we show that although there are exponentially many inequalities (21), their separation can be done efficiently.

Proposition 7. *Given a point* $(\bar{p}, \bar{x}, \bar{s}) \in \mathbb{R}_+^{3n-1}$*, there is an* $O(n^3)$ *algorithm to find a violated inequality* (21)*, if any.*

Proof. Given t, let $j' = \min\left\{n - t, \frac{u-\ell}{\delta}\right\}$. Consider the longest path problem on a directed acyclic graph $G = (N, A)$ where the vertex set N is given by the source node 0, the sink node t', and nodes $i \in [t, t+j']$ and the arc set, A, is given by the arc $(0, t)$ with length $\ell\bar{x}_t - \bar{p}_t$, arc $(t, t+1)$ with length $\phi(\bar{s}_{t+1} - \bar{x}_{t+1})$, arcs (t, q) with length $(\delta(q - t - 1) + \phi)(\bar{s}_q - \bar{x}_q)$ for $t + 1 < q \leq t + j'$, arcs (i, k) with length $\delta(k - i)(\bar{s}_k - \bar{x}_k)$ for $t + 1 \leq i < k \leq t + j'$, and arcs $(t + a, t')$ with length $\bar{p}_{t+a} - \bar{u}\bar{x}_{t+a}$ for $1 \leq a \leq j'$. The length of the longest path from 0 to t' in this graph is equal to the violation of inequality (21), if any. The visited nodes in the longest path determine the set S. Note that there are $O(n^2)$ arcs and $O(n)$ vertices in this directed acyclic graph, so this longest path problem can be solved in $O(n^2)$ time for a given t. Solving this problem for all t we have an $O(n^3)$ time separation algorithm. □

Next, we define a class of multi-period variable upper bound (VUB) inequalities for $\mathcal{U}$.

Proposition 8. *For* $1 \leq t \leq n$*,* $0 \leq j \leq \min\left\{t - 2, \frac{u-\bar{u}}{\delta}\right\}$ *and any* $M \subseteq [t - j + 1, t - 1]$ *the multi-period VUB ramp-up inequality*

$$p_t \leq \bar{u}x_t + \delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}), \tag{22}$$

where for $i \in [t-j+1, t]$ $e_i = \max\{k \in M \cup \{t-j\} : k < i\}$ *and if* $j = 0$*, then* $e_t = t$ *is valid for* $\mathcal{U}$*.*

Proof. There are two cases to consider.

Case 1. Suppose that $x_t = 0 (= p_t)$. Because $\delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}) \geq 0$, inequality (22) is clearly valid for this case.

Case 2. Suppose that $x_t = 1$.

i. Suppose that the last period when start-up occurred on or before period t is $k \in [t-j, t]$. Then,

$$\begin{aligned} p_t &\leq \bar{u} + (t-k)\delta \leq \bar{u} + \delta \sum_{i \in (M \cap [k+1, t-1]) \cup \{t\}} (i - e_i) \\ &\leq \bar{u}x_t + \delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}). \end{aligned}$$

ii. Suppose that the last period when start-up occurred on or before period t is k, where $k \leq t - j - 1$ then $p_t \leq u$. Therefore,

$$\begin{aligned} p_t &\leq \bar{u} + \delta \sum_{i \in M \cup \{t\}} (i - e_i) + (u - \bar{u} - j\delta) \\ &= \bar{u}x_t + \delta \sum_{i \in M \cup \{t\}} (i - e_i)(x_i - s_i) + (u - \bar{u} - j\delta)(x_{t-j} - s_{t-j}) \\ &= \bar{u} + j\delta + u - \bar{u} - j\delta = u. \end{aligned}$$

□

Observe that the two-period variable upper bound inequality (5g) is the special case of the multi-period VUB ramp-up inequality (22) with $j = 0$.

Next we show that although there are exponential many inequalities (22), their separation can be done efficiently.

Proposition 9. *Given a point $(\bar{p}, \bar{x}, \bar{s}) \in \mathbb{R}_+^{3n-1}$, there is an $O(n^3)$ algorithm to find a violated multi-period VUB ramp-up inequality (22), if any.*

Proof. Given t, let $k = \min\{t - 2, \frac{u - \bar{u}}{\delta}\}$. Consider the shortest path problem on a directed acyclic graph $G' = (N', A')$ where the vertex set $N' = \{0, t', j_{t-k}, \ldots, j_t\}$, and the arc set A' is given by the arcs (j_{i_2}, j_{i_1}) with length $\delta(i_2 - i_1)(\bar{x}_{j_{i_2}} - \bar{s}_{j_{i_2}})$ for $i_1, i_2 \in [t-k, t]$, $i_1 < i_2$, arcs (j_i, t') with length $(u - \bar{u} - (t-i)\delta)(\bar{x}_{j_i} - \bar{s}_{j_i})$ for $i \in [t-k, t]$ and arc $(0, j_t)$ with length $\bar{u}\bar{x}_{j_t} - \bar{p}_{j_t}$. The value of the shortest path from 0 to t' in this graph is equal to the violation of inequality (22), if any. The visited nodes in the shortest path determine the set M. Note that there are $O(k^2)$ arcs and $O(k)$ vertices in this directed acyclic graph, so this shortest path problem can be solved in $O(k^2)$ time for a given t, where k is $O(n)$. Solving this problem for all t we have an $O(n^3)$ time separation algorithm. □

Next, we study the strength of the inequalities defined in this section.

3.2 Strength of the Inequalities

First, we consider inequality (20).

Proposition 10. *Type-I multi-period ramp-up inequality (20) defines a facet of conv($\mathcal{U}$) if and only if $\ell + j\delta < u$.*

Proof. See Appendix B. □

Next, we study the strength of inequalities (21).

Proposition 11. *Type-II multi-period ramp-up inequality* (21) *defines a facet of conv*($\mathcal{U}$) *only if the following conditions hold:*

1. *If* $\bar{u} = \ell$ *and* $j \geq 2$, *then* $|S| = 1$.
2. *If* $\bar{u} > \ell + \delta$, *then* $j > 1$.

In addition, if the following conditions hold, then inequality (21) *is a facet of conv*($\mathcal{U}$)

3. $\ell + j\delta < u$.
4. $\bar{u} \leq \ell + \delta$.

Proof. See Damcı-Kurt (2014). □

In Proposition 12 we study the strength of inequalities (22).

Proposition 12. *The multi-period VUB ramp-up inequality* (22) *defines a facet of conv*($\mathcal{U}$).

Proof. See Damcı-Kurt (2014). □

Even though we give large classes of valid inequalities for $\mathcal{U}$, the next example shows that they are not sufficient to completely describe *conv*($\mathcal{U}$) for $n > 2$.

Example 1. *Consider* $\mathcal{U}$, *where* $n = 4, u = 7, \ell = 1, \bar{u} = 4$, *and* $\delta = 1$. *The following inequalities are facets of conv*($\mathcal{U}$):

$$
\begin{aligned}
p_3 + 2p_4 - p_1 &\leq 10x_4 - x_1 + 7x_3 - 3s_3 - 2s_4, \\
9p_4 - 2p_1 - 3p_2 &\leq 45x_4 - 2x_1 - 3x_2 - 9s_4.
\end{aligned}
$$

Note that these inequalities cannot be expressed as one of the inequalities (20), (21), (22), *because some of the production variables have integer coefficients greater than one in absolute value.*

3.3 Ramp-Down Polytope

In this section, we present the ramp-down analogs of the results given in Section 3.1. We use the symmetry between ramping up and ramping down constraints. Specifically, reversing time $(n, \ldots, 1)$ is sufficient to obtain this symmetry. Nevertheless, it is important to note that there is a difference in the information start-up and shut-down variables provide. For example in any given solution if $s_t = 1$, then $x_{t-1} = 0$, $x_t = 1$ however, if $z_t = 1$, then $x_{t-1} = 1$ and $x_t = 0$. Figure 1 illustrates this difference. We omit the proofs of our results for the ramp-down polytope, because they follow from their ramp-up counterparts with minor adjustments.

Next, we describe the ramp-down analog of type-I multi-period ramp-up inequality (20).

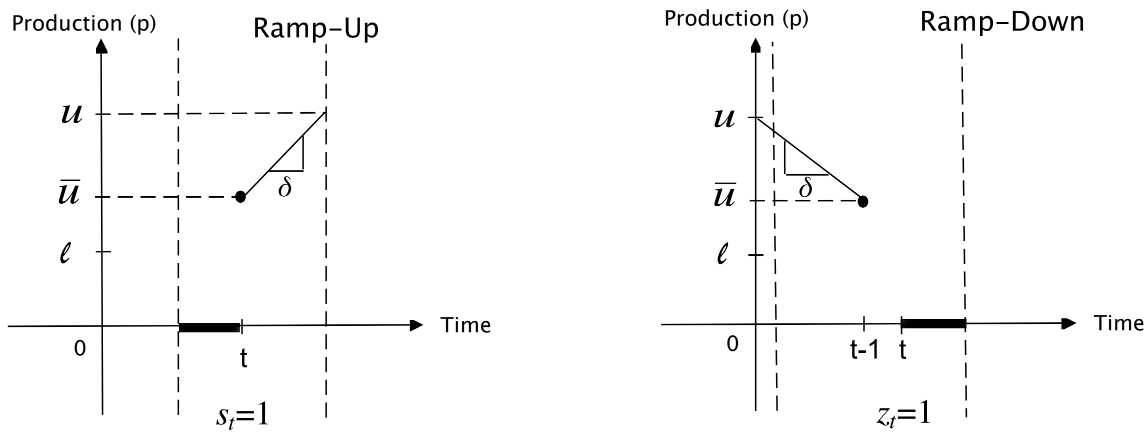

Figure 1: Effect of start-up and shut-down variables on production.

Proposition 13. *If $\bar{u} \geq \ell + \delta$, then for $1 \leq t \leq n$ and $1 \leq j \leq \min\left\{n - t, \frac{\bar{u}-\ell}{\delta}\right\}$, the type-I multi-period ramp-down inequality*

$$p_t - p_{t+j} \leq (\ell + j\delta)x_t + \sum_{i=1}^{j} \min\{(\bar{u} - \ell - (j - i + 1)\delta), (u - \ell - j\delta)\} z_{t+i} - \ell x_{t+j} \tag{23}$$

is valid for $\mathcal{D}$. It defines a facet of conv($\mathcal{D}$) if and only if $\ell + j\delta < u$.

In the next proposition we give the ramp-down analog of type-II multi-period ramp-up inequality (21). This result immediately follows from Proposition 6 by reversing the time.

Proposition 14. *For $1 \leq t \leq n$ and $1 \leq j \leq \min\left\{n - t, \frac{u-\ell}{\delta}\right\}$, let $S' \subseteq [t + 1, t + j]$, $t + 1 \in S'$, $q' = \max\{k \in S'\}$, and $d'_i = \min\{k \in S' \cup \{t + j\} : k > i\}$, $i \in S'$. Then the type-II multi-period ramp-down inequality*

$$p_t - p_{t+j} \leq \bar{u}x_t + \delta \sum_{i \in S' \setminus \{t+j\}} (d'_i - i)(x_{i-1} - z_i) + \phi(x_{q'-1} - z_{q'}) - \ell x_{t+j}, \tag{24}$$

where $\phi = (\ell + \delta - \bar{u})^+$ is valid for $\mathcal{D}$.

As is the case for its ramp-up counterpart, the separation of inequalities (24) can be done efficiently.

Proposition 15. *Given a point $(\bar{p}, \bar{x}, \bar{z}) \in \mathbb{R}_+^{3n-1}$, there is an $O(n^3)$ algorithm to find the most violated inequality* (24)*, if any.*

Next, we give conditions under which inequality (24) defines a facet.

Proposition 16. *Type-II multi-period ramp-down inequality* (24) *defines a facet of conv($\mathcal{D}$) only if the following conditions hold:*

1. *If $\bar{u} = \ell$ and $j \geq 2$ then $|S'| = 1$.*

2. *If $\bar{u} \geq \ell + \delta$ then $j > 1$.*

In addition, if the following conditions hold, then inequality (24) is a facet of conv($\mathcal{D}$)

3. $\ell + j\delta < u$.

4. $\bar{u} \leq \ell + \delta$.

Observe that when $j = 1$ and $\bar{u} \leq \ell + \delta$ inequality (24) reduces to $p_t - p_{t+1} \leq (\ell + \delta)x_t - (\ell + \delta - \bar{u})z_{t+1} - \ell x_{t+1}$. Inequality (24) can be rewritten as $p_t - p_{t+1} \leq \bar{u}z_{t+1} + \delta x_{t+1} - (\ell + \delta)s_{t+1}$, and because $(\ell + \delta)s_{t+1} \geq 0$ it dominates inequality (2f). Similarly, when $j = 1$ and $\bar{u} > \ell + \delta$ inequality (23) dominates inequality (2f) because, in this case, inequality (23) reduces to $p_t - p_{t+1} \leq \bar{u}z_{t+1} + \delta x_{t+1} - (\ell + \delta)s_{t+1}$.

Finally, we give the ramp-down analog of the multi-period VUB ramp-up inequality (22).

Proposition 17. *For $2 \leq t \leq n$, $0 \leq j \leq \min\left\{t-2, \frac{u-\bar{u}}{\delta}\right\}$ and any $M \subseteq [t-j+1, t-1]$ the multi-period VUB ramp-down inequality*

$$p_{t-j-1} \leq \bar{u}x_{t-j-1} + \delta \sum_{i \in M \cup \{t-j\}} (e'_i - i)(x_{i-1} - z_i) + (u - \bar{u} - j\delta)(x_{t-1} - z_t), \tag{25}$$

where for $i \in [t-j, t-1]$ $e'_i = \min\{k \in M \cup \{t\} : k > i\}$, and if $j = 0$, then $e'_t = t$ is valid for $\mathcal{D}$ and it defines a facet of conv($\mathcal{D}$).

As is the case for its ramp-up counterpart, the separation of inequalities (25) can be done efficiently.

Proposition 18. *Given a point $(\bar{p}, \bar{x}, \bar{z}) \in \mathbb{R}_+^{3n-1}$, there is an $O(n^3)$ algorithm to find the most violated inequality (25), if any.*

4 Computational Results

In this section, we report our computational experiments on a unit commitment problem. First we present an MIP formulation of the UC problem. Recall that m is the number of generators, and n is the length of the planning horizon. In period t, the fixed cost of starting up generator g is $f_{t,g}$, the fixed cost of running generator g is $h_{t,g}$, and the unit production cost is $c_{t,g}$, $t \in [1, n]$, $g \in [1, m]$. For any period $t \in [1, n]$ the load is given by $\tilde{d}_t$ and the spinning reserve constant is given by $r_t \geq 1$. In the UC problem the physical constraints of generators include limits on production levels. Repeating the notation from Section 1, but now using subscript g for each generator, we have the following. The maximum production level when a generator $g \in [1, m]$ is started up (and before shut-down) is given by $\bar{u}_g$. The maximum change in production from one operating period to the next, in absolute value, is limited by $\delta_g > 0$, and the minimum and maximum production levels in any period are limited by ℓ_g and u_g, respectively. Finally, in addition to the ramping constraints, it is assumed that when a generator g is turned on, it needs to remain on for at least $\overline{v}_g$ periods. This is known as the minimum up time of a generator. Similarly, if a generator g is turned off, then it needs to remain off for at least $\underline{v}_g$ periods. This is referred to as the minimum down time of a generator.

Next we describe the decision variables. For $g \in [1, m]$, let $p_{t,g}$ be the production level in period t for generator g, and $x_{t,g}$ be 1 if generator g is operating in period t and 0 otherwise, for $t \in [1, n]$. Let $s_{t,g}$ be 1 if generator g is started in period t and 0 otherwise, and $z_{t,g}$ be 1 if generator g is stopped in period t and 0 otherwise, for $t \in [2, n]$, $g \in [1, m]$. An MIP formulation of the UC problem where the objective is to minimize the operational cost of generators over n periods is

$$\min \sum_{g=1}^{m} \Big(\sum_{t=1}^{n} c_{t,g} p_{t,g} + \sum_{t=1}^{n} h_{t,g} x_{t,g} + \sum_{t=2}^{n} f_{t,g} s_{t,g}\Big) \tag{26a}$$

$$\text{s.t.} \quad \sum_{g=1}^{m} p_{t,g} \geq \tilde{d}_t, \qquad t \in [1, n] \tag{26b}$$

$$\sum_{g=1}^{m} u_g x_{t,g} \geq r_t \tilde{d}_t, \qquad t \in [1, n] \tag{26c}$$

$$\ell_g x_{t,g} \leq p_{t,g}, \qquad t \in [1, n], g \in [1, m] \tag{26d}$$

$$p_{t,g} \leq u_g x_{t,g}, \qquad t \in [1, n], g \in [1, m] \tag{26e}$$

$$x_{t+1,g} - x_{t,g} \leq s_{t+1,g}, \qquad t \in [1, n-1], g \in [1, m] \tag{26f}$$

$$\sum_{i=t-\underline{v}_g+1}^{t} s_{i,g} \leq 1 - x_{t-\underline{v}_g,g}, \qquad t \in [\underline{v}_g + 1, n], g \in [1, m] \tag{26g}$$

$$\sum_{i=t-\overline{v}_g+1}^{t} s_{i,g} \leq x_{t,g}, \qquad t \in [\overline{v}_g + 1, n], g \in [1, m] \tag{26h}$$

$$s_{t,g} - z_{t,g} = x_{t,g} - x_{t-1,g}, \qquad t \in [2, n], g \in [1, m] \tag{26i}$$

$$p_{t+1,g} - p_{t,g} \leq \bar{u}_g s_{t+1,g} + \delta_g x_{t,g}, \qquad t \in [1, n-1], g \in [1, m] \tag{26j}$$

$$p_{t,g} - p_{t+1,g} \leq \bar{u}_g z_{t+1,g} + \delta_g x_{t+1,g}, \qquad t \in [1, n-1], g \in [1, m] \tag{26k}$$

$$s \in \{0,1\}^{(n-1)m}, x \in \{0,1\}^{nm}, z \in \mathbb{R}_+^{(n-1)m}, p \in \mathbb{R}_+^{nm}. \tag{26l}$$

The objective (26a) minimizes the total operating cost, including the power generation, setup and start-up costs. We refer to the feasible set defined by (26b)-(26l) as $\mathcal{UC}$. Constraints (26b) ensure that the load is met in every period $t \in [1, n]$. Constraints (26c) are the so-called spinning reserve constraints, which require that the total maximum capacity of all online generators is enough to satisfy a constant factor of the load in every period $t \in [1, n]$. Constraints (26d) and (26e) are the minimum and maximum production constraints for any period t and generator g, $t \in [1, n]$, $g \in [1, m]$. Constraints (26f) describe the relationship between the start-up variables s and generator on/off status variables x. The minimum up- and down-time restrictions are modeled by constraints (26g) and (26h), respectively, for any period t and generator g, $t \in [1, n]$, $g \in [1, m]$. They ensure that if a generator g is turned on (off), then it stays on (off) for at least $\overline{v}_g$ ($\underline{v}_g$) time periods. Equations (26i) expresses the shut-down variable $z_{t,g}$ in terms of variables $s_{t,g}$ and $x_{t,g}$, $t \in [2, n]$, $g \in [1, m]$. Ramp-up and ramp-down are described by constraints (26j) and (26k), respectively. Note that constraints (26i) and (26l) ensure that the turn-off variables z are binary, and hence we do not add this restriction to the model.

We report our computational experiments with valid inequalities (20), (21), (22), (23), (24) and (25) in a branch-and-cut algorithm. We test the strength of these inequalities using formulation $\mathcal{UC}$. For test purposes we create ten combinations of numbers of generators (m). The number of time periods (n) for each instance is 24 hours. For each combination we generate 3 instances and report the averages. We add a user cut if it is violated by 0.01 units and we do not limit the number of cuts that are added at each branch and bound node. We categorize the types of generators as slow-start and fast-start. Each instance has exactly the same number of each type of generator. Table 2 presents generator specific parameters that are created using a uniform distribution. Similarly, for each time period $t \in [1, n]$ a uniform distribution is utilized for parameters demand $\tilde{d}_t$ and spinning reserve $r_t \in [1, 1.1]$. For demand values we divide the time horizon into four. For time periods 1 to 6, $\tilde{d}_t \in \left[0.5 \sum_{g=1}^{m} u, 0.6 \sum_{g=1}^{m} u\right]$, for time periods 7 to 12, $\tilde{d}_t \in \left[0.6 \sum_{g=1}^{m} u, 0.8 \sum_{g=1}^{m} u\right]$, for time periods 13 to 18, $\tilde{d}_t \in \left[0.8 \sum_{g=1}^{m} u, 0.9 \sum_{g=1}^{m} u\right]$ and for time periods 19 to 24, $\tilde{d}_t \in \left[0.7 \sum_{g=1}^{m} u, 0.8 \sum_{g=1}^{m} u\right]$. We conduct the experiments on an Intel Xeon x5650 Processor at 2.67GHz with 4GB RAM. We use IBM ILOG CPLEX 12.4 as the MIP solver. We turn the dynamic search option off, use a single thread, permit only linear reductions in the presolve phase of CPLEX, set the optimality tolerance to 0.05%, and impose a time limit of two hours in all our experiments. Our test instances are available at `http://ise.osu.edu/ISEFaculty/kucukyavuz/data/UCDataSet.zip`.

Table 2: Parameters for Two Types of Generators

Parameter	Slow-start	Fast-start
u	$[3000, 10000]$	$[800, 1000]$
ℓ	$\frac{u}{6}$	$\frac{u}{4}$
$\bar{u}$	ℓ	$[\ell, 1.2\ell]$
δ	$[\ell, 2\ell]$	$[\ell, 1.6\ell]$
$\underline{v}$, $\overline{v}$	$[4, 8]$	$[1, 3]$
c	0.1	0.3
h	$[50, 100]$	$[5, 10]$
f	$[5000, 10000]$	$[50, 100]$

Our study complements the polyhedral study of Ostrowski et al. (2012), which also takes minimum up and minimum down time into consideration. Ostrowski et al. (2012) present two polynomial classes of three-period ramping inequalities, but both of them are valid only when $\overline{v} \geq 2$, and are therefore not valid for $\mathcal{U}$. The authors also present a polynomial class of VUB inequalities, but these are the same as our two-period VUB inequality (4) when $\overline{v} = 1$, and are not valid for $\mathcal{U}$ when $\overline{v} \geq 2$. As a result, our inequalities cannot be directly compared to the inequalities in Ostrowski et al. (2012) analytically. We note that Ostrowski et al. (2012) also assume that $\bar{u} - \delta < \ell$ in all but one class of their inequalities. In contrast, among the six classes of inequalities we propose, four of them are exponential and are valid without any restriction on the data. Only the type-I multi-period ramping inequalities (20) and (23) are polynomial, and they are valid under the assumption $\bar{u} \geq \ell + \delta$.

In Table 3, we test the strength of our inequalities empirically at the root node, by comparing the performance of three algorithms:

UCE-N (User Cuts with Exact Separation): $\mathcal{UC}$ formulation with only user cuts. Note that user cuts refers to inequalities (20), (21), (22), (23), (24) and (25). We use variants of the exact separation algorithms we described in Section 3 for inequalities (21), (22), (24) and (25). For example, for the separation of inequalities (21) given in Proposition 7, instead of searching for all j to find the most violated cut for each t, we generate *a violated cut for* the smallest j for each t, if there are any. Note that this is still an exact separation, because we continue our search until we find a violated inequality.

OC-N (Ostrowski et al. (2012) cuts): $\mathcal{UC}$ formulation with ramping and VUB inequalities of Ostrowski et al. (2012) added to the user cuts table for CPLEX (no separation algorithm is implemented), and all CPLEX cuts are turned off.

CD (CPLEX Default Cut Settings): $\mathcal{UC}$ formulation with default CPLEX cut settings.

In Table 3, we report **LPGap**, the initial gap percentage at the root node, which is $100 \times (zub - zlp)/zub$, where zub is the objective function value of the best integer solution obtained within time limit (among all the compared algorithms if the optimal solution is unknown) and zlp is the objective function value of the LP relaxation to formulation $\mathcal{UC}$. Column **RGap** gives the integrality gap percentage at the root node just before branching which is $100 \times (zub - zrb)/zub$, where zrb is the best lower bound obtained at the root node. We report the number of cuts added at the root node by column **RCuts**. We denote the user cuts added by prefix u, whereas we do not use a prefix for the cuts added by CPLEX. For the gap values in the tables we report the numbers rounded to the second decimal place. For all the tables we report the overall averages in the last row **Avg**.

Table 3: Comparison of Algorithms UCE-N, OC-N, and CD at the root node.

	LPGap	**RGap**			**RCuts**		
m		UCE-N	OC-N	CD	UCE-N	OC-N	CD
30	1.87%	0.25%	0.86%	0.80%	u1468	u180	288
60	1.39%	0.18%	0.78%	0.58%	u1783	u212	701
90	1.02%	0.20%	0.53%	0.29%	u2970	u305	776
120	2.52%	0.51%	1.14%	1.04%	u5508	u760	1587
150	1.64%	0.63%	0.99%	0.70%	u5574	u854	2150
180	1.98%	0.43%	1.08%	1.01%	u7407	u975	1981
210	1.84%	0.40%	1.01%	0.83%	u8629	u1043	2366
240	1.56%	0.34%	0.88%	0.63%	u7631	u1046	2888
270	0.86%	0.31%	0.51%	0.24%	u7144	u758	2880
300	2.06%	0.53%	1.78%	1.02%	u12228	u1674	3702
Avg	1.67%	0.38%	0.96%	0.71%	u6034	u781	1932

Because CPLEX cuts are turned off for algorithms UCE-N and OC-N we can observe the benefits of adding the inequalities defined in this paper compared to inequalities proposed by Ostrowski et al. (2012) without the interference of CPLEX cuts. In all the rows, except for $m = 270$ we observe that the smallest root gap is found by algorithm UCE-N. Algorithm UCE-N adds the largest number of cuts which also supports this conclusion. On average default CPLEX cut settings (CD) performs better in terms of root gap compared to algorithm OC-N. This result is likely due to the small number of inequalities defined by Ostrowski et al. (2012) compared to our inequalities. Solution times are not provided in Table 3 because on average the longest solution time takes less than 180 seconds. The run time for algorithm UCE-N is longest because of the large number of cuts added.

Next, in Table 4, we compare the following algorithms within a branch-and-cut framework on the same set of instances as those in Table 3:

UCH (User Cuts with a Heuristic Separation): $\mathcal{UC}$ formulation with the cuts defined in this paper and default CPLEX cuts. For the heuristic separation of inequalities (20), (21), (23) and (24) we use a modification of the exact separation algorithm described in UCE-N, where for multi-period VUB inequalities (22) and (25), we set the largest possible value of j to $\lceil n/4 \rceil$. Note that this is a heuristic because there may be violated inequalities for $j > \lceil n/4 \rceil$. User cuts are generated and added for the first fifty branch-and-cut nodes.

OC (Ostrowski et al. (2012) Cuts): Same as algorithm OC-N, but CPLEX default cuts are also enabled.

CD (CPLEX Default Cut Settings): $\mathcal{UC}$ formulation with default CPLEX cut settings, as before.

In our experiments with UCH, we restricted the multi-period VUB inequalities (22) and (25) to those with $j \leq \lceil n/4 \rceil$, because this provided a slight improvement over UCE-N. A similar restriction for the ramping inequalities did not provide any advantage over the exact separation. In Table 4, column **EGap** reports the end gap percentage at termination output by CPLEX, which is $100 \times (zub - zbest)/zub$, where $zbest$ is the best lower bound available within time limit. Column **ECuts** reports the number of cuts added after the problem is solved to optimality (with a tolerance of 0.05%) within the time limit. Column **Time(uslvd)** reports the solution time in seconds and the total number of unsolved instances. The time for an unsolved instance is not included in the average calculations. In column **B&C Nodes** we report the number of branch-and-cut tree nodes explored.

The root gap trend obtained in Table 3 changes for algorithm OC and CD. Now that CPLEX cuts are turned on for all the algorithms, the worst performance in terms of root gap, end gap and time is given by algorithm CD. For all the instances, the smallest root gap, end gap, and time values are found by UCH. In addition, the average number of branch-and-cut nodes explored is significantly smaller for UCH than for OC and CD. Overall, algorithm UCH solves more instances to optimality (with a tolerance of 0.05%) compared to OC and CD within the time limit and solves them several times faster. As a result, algorithm UCH outperforms both OC and CD for all instances.

We also tested the performance of our inequalities on a set of unit commitment instances described in Frangioni et al. (2008). However, this data set also considers quadratic production costs. Therefore, we use the MIQP solver of CPLEX 12.4 instead of the MIP solver. To handle the quadratic terms in the objective function we use the formulation described in Aktürk et al. (2009), whereas Frangioni et al. (2008) replace the quadratic terms in the objective function by a (convex) piecewise-linear approximation. Specifically, we remove $c^2_{t,g} p^2_{t,g}$ from the objective function, where

Table 4: Comparison of Algorithms UCH, OC and CD.

m	Alg	**RGap**	**EGap**	**ECuts**	**Time(uslvd)**	**B&C Nodes**
30	UCH	0.19%	$\leq 0.05\%$	190, u749	6	287
	OC	0.49%	$\leq 0.05\%$	382, u214	88	21711
	CD	0.80%	$\leq 0.05\%$	519	495	137897
60	UCH	0.09%	$\leq 0.05\%$	368, u636	310	35812
	OC	0.48%	0.29%	762, u217	26(1)	376375
	CD	0.58%	0.41%	955	121(1)	454744
90	UCH	0.10%	$\leq 0.05\%$	543, u981	30	838
	OC	0.23%	0.08%	903, u340	173(1)	189762
	CD	0.29%	0.12%	1124	95(2)	403616
120	UCH	0.23%	$\leq 0.05\%$	1192, u2022	545(1)	106042
	OC	0.70%	0.40%	1550, u808	(3)	505840
	CD	1.04%	0.75%	2032	(3)	541800
150	UCH	0.27%	$\leq 0.05\%$	973, u2631	1543(2)	234691
	OC	0.57%	0.39%	1816, u883	(3)	421925
	CD	0.70%	0.51%	2532	(3)	430834
180	UCH	0.27%	$\leq 0.05\%$	1343, u3064	48(2)	180640
	OC	0.70%	0.48%	1983, u936	(3)	403088
	CD	1.01%	0.82%	2457	(3)	426553
210	UCH	0.27%	0.08%	1867, u2545	6142(2)	233458
	OC	0.62%	0.46%	2211, u1062	(3)	328170
	CD	0.83%	0.71%	2799	(3)	390562
240	UCH	0.14%	$\leq 0.05\%$	2363, u3135	16(2)	113433
	OC	0.45%	0.36%	3032, u978	33(2)	189649
	CD	0.63%	0.59%	3281	40(2)	227782
270	UCH	0.12%	$\leq 0.05\%$	1826, u2796	1260	31620
	OC	0.25%	0.13%	2402, u746	40(2)	150198
	CD	0.24%	0.15%	3261	70(2)	140047
300	UCH	0.28%	0.08%	2388, u4582	(3)	187456
	OC	0.74%	0.61%	3257, u1581	(3)	200227
	CD	1.02%	0.89%	4040	(3)	214116
Avg	UCH	0.20%	0.06%	1305, u2314	759(12)	112428
	OC	0.52%	0.31%	1830, u777	82(21)	278694
	CD	0.71%	0.50%	2300	242(22)	336795

$c_{t,g}^2$ is the quadratic cost coefficient at period t for generator g, and add the following constraints,

$$c_{t,g}^2 p_{t,g}^2 \leq x_{t,g} w_{t,g}, \quad t \in [1,n], g \in [1,m],$$
$$w_{t,g} \geq 0, \quad t \in [1,n], g \in [1,m],$$

to the $\mathcal{UC}$ formulation. The newly introduced variables $w_{t,g}$, $t \in [1,n], g \in [1,m]$ in the reformulation are added to the objective function.

Table 5 summarizes our experiments with this data set. Note that the data set is such that one can fully ramp at most two or three periods consecutively. Therefore, for such data we suggest that our two-period inequalities be added to the formulation a priori without the use of separation routines. We refer to the $\mathcal{UC}$ formulation with our two-period ramping inequalities in place of the original ramping inequalities as **UC-2P**. As can be seen from our experiments in Table 5, these instances are much too easy for current optimization software. We do not report end gaps, because all algorithms solve all instances well within the time limit. (Note that Frangioni et al. (2008) uses CPLEX 9.1 in their experiments with this data.) We also observe that having our two-period ramping inequalities in the formulation is helpful when we compare the results of UC-2P and CD.

Table 5: Performance of UC-2P, OC and CD on Frangioni et al. (2008) instances.

m	**Alg**	**RGap**	**ECuts**	**Time**	**B&C Nodes**
10	UC-2P	0.02%	79	1	0
	OC	0.02%	137, u0	1	0
	CD	0.02%	137	1	0
20	UC-2P	0.03%	67	1	0
	OC	0.03%	147, u0	1	0
	CD	0.03%	147	1	0
50	UC-2P	0.05%	110	23	437
	OC	0.06%	364, u2	35	791
	CD	0.06%	372	35	858
75	UC-2P	0.06%	257	206	3218
	OC	0.06%	661, u3	260	5452
	CD	0.06%	651	269	5888
100	UC-2P	0.05%	103	135	1508
	OC	0.05%	565, u4	358	5768
	CD	0.06%	542	320	5052
150	UC-2P	0.05%	145	299	3386
	OC	0.05%	606, u4	293	5698
	CD	0.05%	652	350	6265
200	UC-2P	0.04%	40	46	85
	OC	0.04%	699, u9	108	1377
	CD	0.04%	668	39	165
Avg	UC-2P	0.04%	115	101	1234
	OC	0.04%	454, u3	151	2727
	CD	0.05%	453	145	2605

5 Conclusion

In this paper we study the ramp-up and ramp-down polytopes arising for modeling production ramping. For the two-period problem, we give a complete linear description of the convex hull of feasible solutions. For the multi-period case, we give exponential classes of facet-defining inequalities and efficient separation algorithms. However, the convex hull of the feasible set for the case of more than two periods is an open problem that merits further research. Our computational results show the effectiveness of the proposed inequalities when used as cuts in a branch-and-cut algorithm to solve the UC problem with ramping constraints.

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A Proof of Convex Hull of Two-Period Ramping Polytope without Start Variables

Corollary 1. *For* $\bar{u} \leq \ell + \delta$, *conv*$(\mathcal{UNS}_t^2) = \{(p, x) \in \mathbb{R}^{2n} : (7) - (15)\}$, *and for* $\bar{u} > \ell + \delta$, *conv*$(\mathcal{UNS}_t^2) = \{(p, x) \in \mathbb{R}^{2n} : (7) - (13), (16), (17)\}$.

Proof. We use Fourier-Motzkin elimination (see Schrijver (1998) and Nemhauser and Wolsey (1988)) of variable s_{t+1} from the convex hull of the feasible solutions to the formulation with start-up variables given by inequalities (5a)-(5i). Inequalities (5d), (5e) and (5h) continue to be facets (given by inequalities (7)-(9)) because these inequalities do not include variable s_{t+1} in their description. Similarly, if $\bar{u} = \ell + \delta$, then inequality (5f) is also a facet, because the coefficient of s_{t+1} is equal to 0. In this case, inequality (5f) is equivalent to inequalities (14) and (15). If $u = \bar{u}$, then inequality (5g) is also a facet, because the coefficient of s_{t+1} is equal to 0. In this case, inequality (5g) reduces to (12).

We need to consider all possible cross products of inequalities (5b), (5c), and (5f) (if $\bar{u} > \ell + \delta$) that provide a lower bound on s_{t+1} with inequalities (5a), (5g) (if $u > \bar{u}$), (5i), and (5f) (if $\bar{u} < \ell + \delta$) that provide an upper bound on s_{t+1}.

We first consider the cross product of lower-bounding inequalities (5b) and (5c) with upper-bounding inequalities defined by (5a), (5g) (if $u > \bar{u}$) and (5i).

- The pair of inequalities (5b) and (5a) gives $x_t \geq 0$ which is dominated by inequalities (5e) and (5h).
- The pair of inequalities (5b) and (5g) gives inequality (10) (if $u > \bar{u}$).
- The pair of inequalities (5b) and (5i) gives inequality (11).
- The pair of inequalities (5c) and (5a) gives $x_{t+1} \geq 0$ which is dominated by inequalities (5d) and (12).
- The pair of inequalities (5c) and (5g) gives inequality (12) (if $u > \bar{u}$).
- The pair of inequalities (5c) and (5i) gives inequality (13).

Note that depending on whether the coefficient of s_{t+1} in inequality (5f) is positive or negative, we get a lower-bounding or upper-bounding inequality for s_{t+1}, respectively. Therefore, we consider the following two cases:

Case 1. If $\bar{u} < \ell + \delta$, then inequality (5f) is an upper-bounding inequality given by

$$s_{t+1} \leq \frac{(\ell + \delta)x_{t+1} - \ell x_t - p_{t+1} + p_t}{(\ell + \delta - \bar{u})}. \tag{27}$$

- The pair of inequalities (5b) and (27) gives inequality (14).
- The pair of inequalities (5c) and (27) gives inequality (15).

Case 2. If $\bar{u} > \ell + \delta$, then inequality (5f) is a lower-bounding inequality given by

$$s_{t+1} \geq \frac{-(\ell + \delta)x_{t+1} + \ell x_t + p_{t+1} - p_t}{(\bar{u} - \ell - \delta)}. \tag{28}$$

- The pair of inequalities (5a) and (28) gives

$$p_{t+1} - p_t \leq \bar{u}x_{t+1} - \ell x_t, \tag{29}$$

which is dominated by inequalities (17) and (8). To see this, note that multiplying inequality (8) by $-(\bar{u}-\ell-\delta)$ and adding to inequality (17) gives inequality (29) multiplied by $(u - \ell - \delta)$.

- The pair of inequalities (5g) (when $u > \bar{u}$) and (28) gives

$$\frac{ux_{t+1} - p_{t+1}}{(u - \bar{u})} \geq \frac{-(\ell + \delta)x_{t+1} + \ell x_t + p_{t+1} - p_t}{(\bar{u} - \ell - \delta)}.$$

Rearranging terms we get inequality (17).

- The pair of inequalities (5i) and (28) gives inequality (16).

□

B Facet-definition Proof of Type-I Multi-Period Ramp-Up Inequality

Proposition 10. *Type-I multi-period ramp-up inequality* (20) *defines a facet of conv*$(\mathcal{U})$ *if and only if* $\ell + j\delta < u$.

Proof. Necessity. For contradiction assume that $\ell + j\delta \geq u$. From validity of inequality (20) we have $\ell + j\delta \leq u$, so $\ell + j\delta = u$. Then inequality (20) can be written as $p_{t+j} - p_t \leq (\ell + j\delta)x_{t+j} - \ell x_t = ux_{t+j} - \ell x_t$, and it is dominated by inequalities (1a) and (1b).
Sufficiency. We use the technique in Theorem 3.6 of §I.4.3 in Nemhauser and Wolsey (1988). We show that inequality (20) is the only inequality that is satisfied at equality by all points $(p, x, s) \in \mathcal{U}$ that are tight at (20), i.e., we show that if all points of $\mathcal{U}$ at which inequality (20) is tight satisfy

$$\sum_{k=1}^{n} \alpha_k p_k + \sum_{k=1}^{n} \beta_k x_k + \sum_{k=2}^{n} \gamma_k s_k = \alpha_0, \tag{30}$$

then

1. $\alpha_0 = 0$,
2. $\alpha_k = 0$, $k \in [1, n] \setminus \{t, t + j\}$,
3. $\alpha_t = -\bar{\alpha}$, $\alpha_{t+j} = \bar{\alpha}$,
4. $\beta_k = 0$, $k \in [1, n] \setminus \{t, t + j\}$,
5. $\beta_t = \bar{\alpha}\ell$,
6. $\beta_{t+j} = -\bar{\alpha}(\ell + j\delta)$,
7. $\gamma_k = 0$, $k \in [2, t] \cup [t + j + 1, n]$,

8. $\gamma_{t+i'} = -\bar{\alpha}\min\{(\bar{u}-\ell-i'\delta),(u-\ell-j\delta)\}$, $i' \in [1,j]$.

In order to establish the values of the coefficients α_k, β_k, γ_k and α_0, we construct a feasible solution to $\mathcal{U}$ on the face defined by (20). Then a small change in the solution is made to obtain another feasible solution which is on the face defined by inequality (20). Comparing the resulting expressions, the possible values of a set of coefficients are obtained. Also note that from the validity assumption and (A2), $\bar{u} \geq \ell+\delta > \ell$. We start by describing several points feasible to $\mathcal{U}$ that will be used throughout the facet proofs. We assume that $k \geq 2$ if we set the value of s_k. In the following feasible solutions (except for the zero vector ($S1$)) if the value of a variable is not given, then its value is equal to zero. Let $k_1, k_2 \in [2,n]$ be two periods such that $k_1 < k_2$. Let ϵ be a very small number greater than zero.

$$x_r = p_r = 0, r \in [1,n], s_r = 0, r \in [2,n], \tag{$S1$}$$
$$x_{k_1} = 1, p_{k_1} = \ell, s_{k_1} = 1, \tag{$S2$}$$
$$x_{k_1} = 1, p_{k_1} = \ell+\epsilon, s_{k_1} = 1, \tag{$S3$}$$
$$x_1 = 1, p_1 = \ell, \tag{$S4$}$$
$$x_r = 1, p_r = \ell, r \in [k_1,k_2], s_{k_1} = 1, \tag{$S5$}$$
$$x_r = 1, p_r = \ell, r \in [k_1,k_2-1], x_{k_2} = 1, p_{k_2} = \ell+\epsilon, s_{k_1} = 1, \tag{$S6$}$$
$$x_r = 1, p_r = \ell+(r-t)\delta, r \in [t,t+j], s_t = 1, \tag{$S7$}$$
$$x_r = 1, p_r = \ell+\epsilon+(r-t)\delta, r \in [t,t+j], s_t = 1, \tag{$S8$}$$
$$x_r = 1, p_r = \ell, r \in [1,t], x_r = 1, p_r = \ell+(r-t)\delta, r \in [t+1,t+j], \tag{$S9$}$$
$$x_{k_1-1} = x_{k_1} = 1, p_{k_1-1} = p_{k_1} = \ell, s_{k_1-1} = 1 \text{ (assuming } k_1-1 \geq 2), \tag{$S10$}$$

Note that points ($S3$), ($S6$), ($S7$), ($S8$), and ($S9$) are feasible because $\bar{u} > \ell$ and $\ell + j\delta < u$.

Next we show the values of the coefficients α_k, β_k, $k \in [1,n]$, γ_k, $k \in [2,n]$ and α_0.

1. $\alpha_0 = 0$.

   Consider solution ($S1$). Clearly, this point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Hence, $\alpha_0 = 0$.

2. $\alpha_k = 0$, $k \in [1,n] \setminus \{t, t+j\}$.

   Consider the following two cases:

   (a) $k \in [1,t-1] \cup [t+j+1,n]$.
   Consider solution ($S2$) with $k_1 = k$. Clearly, this point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Now, consider solution ($S3$) with $k_1 = k$. This point also satisfies inequality (20) at equality and is a valid point because $\bar{u} > \ell$ by assumption. Then evaluating (30) at both solutions we get $\alpha_k \ell = \alpha_k(\ell+\epsilon)$. Hence, $\alpha_k = 0$.

   (b) $k \in [t+1,t+j-1]$.
   Consider solution ($S5$) with $k_1 = t$ and $k_2 = k$. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $-\ell$. Now, consider solution ($S6$) with $k_1 = t$ and $k_2 = k$ (this point also satisfies inequality (20) at equality). Then evaluating (30) at both solutions we get $\alpha_k \ell = \alpha_k(\ell+\epsilon)$. Hence, $\alpha_k = 0$.

3. $\alpha_t = -\bar{\alpha}$, $\alpha_{t+j} = \bar{\alpha}$.

   Consider solution ($S7$). This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. Now, consider solution ($S8$). This point also satisfies inequality (20) at equality. Because we showed that $\alpha_k = 0$, $k \in [1, n] \setminus \{t, t+j\}$ in part 2, evaluating (30) at both solutions we get $\alpha_t \epsilon = -\alpha_{t+j}\epsilon$. Let $\bar{\alpha} := -\alpha_t = \alpha_{t+j}$.

4. $\beta_k = 0$, $k \in [1, n] \setminus \{t, t+j\}$.

   Consider the following two cases:

   (a) $k > t + j$.
   Consider a solution to $\mathcal{U}$ with $x_r = 1$, $r \in [t, k]$, $s_t = 1$, $p_{t+i} = \ell + i\delta$, $i \in [0, j]$, $p_r = \ell + j\delta$, $r \in [t + j + 1, k]$ and all other variables are equal to zero. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. Now, consider the same solution except we set $x_k = 0 = p_k$ (this solution is on the face defined by inequality (20)). Evaluating (30) at both solutions we get $\alpha_k(\ell + j\delta) + \beta_k = 0$. Because we showed that $\alpha_k = 0$ in part 2 we get $\beta_k = 0$.

   (b) $t < k < t + j$.
   Consider solution ($S5$) with $k_1 = t$ and $k_2 = k$. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $-\ell$. Now, consider solution ($S5$) with $k_1 = t$ and $k_2 = k - 1$ if $t < k - 1$, and solution ($S2$) with $k_1 = t$ if $t = k - 1$. Both of the points satisfy inequality (20) at equality. Note that if $t = k - 1$ we use solution ($S2$) because both $k_1 = t$ and $k_2 = k - 1 = t$ and we define $k_1 < k_2$ in solution ($S5$). Evaluating (30) at the described solutions we get $\alpha_k \ell + \beta_k = 0$. Because we showed that $\alpha_k = 0$ in part 2 we get $\beta_k = 0$.

   (c) $k \leq t - 1$ for $t \geq 2$.
   Consider the following two cases:

      i. $k = 1$.
      Consider solution ($S4$). This point is on the face defined by inequality (20) because both the left- and the right-hand sides of the inequality are equal to zero. Evaluating (30) at this solution we get $\alpha_k \ell + \beta_k = \alpha_0 = 0$ and since $\alpha_k = 0$ (from part 2) we get $\beta_k = 0$.

      ii. $k \geq 2$.
      Consider solution ($S10$) with $k_1 = k$. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to zero. Now, consider solution ($S2$) with $k_1 = k - 1$. Then evaluating (30) at both solutions we get $\alpha_k \ell + \beta_k = 0$, and because $\alpha_k = 0$ (from part 2) we get $\beta_k = 0$.

5. $\beta_t = \bar{\alpha}\ell$.

   If $t = 1$, then we use solution ($S4$). Because $\alpha_t = \alpha_1 = -\bar{\alpha}$ (from part 3) evaluating this solution at equality (30) we get $\alpha_1 \ell + \beta_1 = 0$ so $\beta_1 = \bar{\alpha}\ell$. For $t \geq 2$ consider solution ($S5$) with $k_1 = 1$ and $k_2 = t$. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $-\ell$. Now consider solution ($S5$) with

$k_1 = 1$ and $k_2 = t - 1$. This point is on the face defined by inequality (20) because both the left- and the right-hand sides of the inequality are equal to zero. Then evaluating (30) at both solutions we obtain $\alpha_t \ell + \beta_t = 0$. Because $\alpha_t = -\bar{\alpha}$ (from part 3) we get $\beta_t = \bar{\alpha}\ell$.

6. $\beta_{t+j} = -\bar{\alpha}(\ell + j\delta)$.

   Consider solution $(S9)$. This point satisfies inequality (20) at equality because both the left- and the right-hand sides of the inequality are equal to $j\delta$. Consider the same solution except now we set $x_{t+j} = p_{t+j} = 0$. This point is on the face defined by inequality (20) because both the left- and the right-hand sides of the inequality are equal to $-\ell$. Then evaluating (30) at both solutions we obtain $\alpha_{t+j}(\ell + j\delta) + \beta_{t+j} = 0$. Because $\alpha_{t+j} = \bar{\alpha}$ (from part 3) we get $\beta_{t+j} = -\bar{\alpha}(\ell + j\delta)$.

7. $\gamma_k = 0$, $k \in [2, t] \cup [t + j + 1, n]$.

   Consider solution $(S2)$ with $k_1 = k$. This point satisfies inequality (20) at equality because both the left-and the right-hand sides of the inequality are equal to zero unless $k = t$. If $k = t$, then both the left- and the right-hand sides of the inequality are equal to $-\ell$. Evaluating (30) at this solution we obtain $\alpha_k \ell + \beta_k + \gamma_k = 0$. If $k \neq t$, then we have $\alpha_k = \beta_k = 0$ (from parts 2 and 4) so we get $\gamma_k = 0$. If $k = t$, then because $\alpha_t \ell = -\bar{\alpha}\ell$ and $\beta_t = \bar{\alpha}\ell$ (from parts 3 and 5), we get $\gamma_t = 0$.

8. $\gamma_{t+i'} = -\bar{\alpha} \min\{(\bar{u} - \ell - i'\delta), (u - \ell - j\delta)\}$, $i' \in [1, j]$.

   Consider a solution to $\mathcal{U}$ with $x_{t+i'} = 1$, $p_{t+i'} = \bar{u}$, $s_{t+i'} = 1$, $x_{t+i} = 1$, $p_{t+i} = \min\{\bar{u} + (i - i')\delta, u\}$, $i \in [i' + 1, j]$ and all other variables are equal to zero. This point satisfies inequality (20) at equality because either both the left- and the right-hand sides of the inequality are equal to $\bar{u} + (j - i')\delta$ or u depending on the value of p_{t+j}. Evaluating (30) at this solution we obtain,

$$\alpha_{t+i'}\bar{u} + \beta_{t+i'} + \gamma_{t+i'} + \sum_{i=i'+1}^{j-1} (\alpha_{t+i}p_{t+i} + \beta_{t+i}) + \alpha_{t+j}p_{t+j} + \beta_{t+j} = 0. \tag{32}$$

   From parts 1-4 and 6 we obtain $\gamma_{t+i'} = -\bar{\alpha}(p_{t+j} - \ell - j\delta)$. Furthermore, because p_{t+j} is either $\bar{u} + (j - i')\delta$ or u, proof is complete.

□