On a Cardinality-Constrained Transportation Problem With Market Choice

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Abstract

It is well-known that the intersection of the matching polytope with a cardinality constraint is integral [8]. In this note, we prove a similar result for the polytope corresponding to the transportation problem with market choice (TPMC) (introduced in [4]) when the demands are in the set $\{1,2\}$. This result generalizes the result regarding the matching polytope. The result in this note implies that some special classes of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time.

Keywords: Transportation problem with market choice, cardinality constraint, integral polytope

1. Introduction

The transportation problem with market choice (TPMC), introduced in the paper [4], is a transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. If a market is selected, then its demand must be satisfied fully through shipments from the suppliers. If a market is rejected, then the corresponding potential revenue is lost. The objective is to minimize the total cost of shipping and lost revenues. See [5, 7, 9] for approximation algorithms and heuristics for several other supply chain planning and logistics problems with market choice.

Formally, we are given a set of supply and demand nodes that form a bipartite graph $G = (V_1 \cup V_2, E)$. The nodes in set V_1 represent the supply nodes, where for $i \in V_1$, $s_i \in \mathbb{N}$ represents the capacity of supplier i. The nodes in set V_2 represent the potential markets, where for $j \in V_2$, $d_j \in \mathbb{N}$ represents the demand of market j. The edges between supply and demand nodes have weights that represent shipping costs w_e, where $e \in E$. For each $j \in V_2$, r_j is the revenue lost if the market j is rejected. Let $x_{\{i,j\}}$ be the amount of demand of market j satisfied by supplier i for $\{i,j\} \in E$, and let z_j be an indicator variable taking a value 1 if market j is rejected and 0 otherwise. A mixed-integer programming (MIP) formulation of the problem is given where the objective is to minimize the transportation costs and the lost revenues due to unchosen markets:

$$\min_{x \in \mathbb{R}_+^{|E|},\, z \in \{0,1\}^{|V_2|}} \quad \sum_{e \in E} w_e x_e + \sum_{j \in V_2} r_j z_j \tag{1}$$

$$\text{s.t.} \quad \sum_{i:\{i,j\} \in E} x_{\{i,j\}} = d_j (1 - z_j) \qquad \forall j \in V_2 \tag{2}$$

$$\sum_{j:\{i,j\} \in E} x_{\{i,j\}} \leq s_i \qquad \forall i \in V_1. \tag{3}$$

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We refer to the formulation (1)-(3) as TPMC. The first set of constraints (2) ensures that if market $j \in V_2$ is selected (i.e., $z_j = 0$), then its demand must be fully satisfied. The second set of constraints (3) model the supply restrictions.

TPMC is strongly NP-complete in general [4]. Aardal and Le Bodic [1] give polynomial-time reductions from this problem to the capacitated facility location problem [6], thereby establishing approximation algorithms with constant factors for the metric case and a logarithmic factor for the general case.

When $d_j \in \{1, 2\}$ for each demand node $j \in V_2$, TPMC is polynomially solvable [4]. We call this special class of the problem, the *simple TPMC problem* in the rest of this note.

Observation 1 (Simple TPMC generalizes Matching on General Graphs)**.** *The matching problem can be seen as a special case of the simple TPMC problem. Let $G = (V, E)$ be a graph with n vertices and m edges. We construct a bipartite graph $\hat{G} = (\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows: $\hat{V}^1$ is a set of n vertices corresponding to the n vertices in G, and $\hat{V}^2$ corresponds to the set of edges of G, i.e., $\hat{V}^2$ contains m vertices. We use $\{i, j\}$ to refer to the vertex in $\hat{V}^2$ corresponding to the edge $\{i, j\}$ in E. The set of edges in $\hat{E}$ are of the form $\{i, \{i, j\}\}$ and $\{j, \{i, j\}\}$ for every $i, j \in V$ such that $\{i, j\} \in E$. Now we can construct (the feasible region of) an instance of TPMC with respect to $\hat{G} = (\hat{V}^1 \cup \hat{V}^2, \hat{E})$ as follows:*

$$T = \{(x, z) \in \mathbb{R}^{2m}_+ \times \mathbb{R}^m \mid x_{\{i,e\}} + x_{\{j,e\}} + 2z_e = 2 \ \forall e = \{i, j\} \in \hat{V}^2 \tag{4}$$

$$\sum_{j:\{i,j\} \in E} x_{\{i,\{i,j\}\}} \leq 1 \ \forall i \in \hat{V}^1 \tag{5}$$

$$z_e \in \{0, 1\} \ \forall e \in \hat{V}^2\}. \tag{6}$$

Clearly there is a bijection between the set of matchings in G and the set of solutions in T. Moreover, let

$$H := \{(x, z, y) \in \mathbb{R}^{2m} \times \mathbb{R}^m \times \mathbb{R}^m \mid (x, z) \in T, y = e - z\},$$

where e is the all ones vector in $\mathbb{R}^m$. Then we have that the convex hull of the incidence vectors of all the matchings in $G = (V, E)$ is precisely the set $\mathrm{proj}_y(H)$.

Note that the instances of the form of (4)-(6) are special cases of simple TPMC instances, since in these instances all s_i's are restricted to be exactly 1 and all d_j's are restricted to be exactly 2.

2. Main Result

An important and natural constraint that one may add to the TPMC problem is that of a service level, that is the number of rejected markets is restricted to be at most k. This restriction can be modelled using a *cardinality constraint*, $\sum_{j \in V_2} z_j \leq k$, appended to (1)-(3). We call the resulting problem cardinality-constrained TPMC (CCTPMC). If we are able to solve CCTPMC in polynomial-time, then we can solve TPMC in polynomial time by solving CCTPMC for all $k \in \{0, \ldots, |V_2|\}$. Since TPMC is NP-hard, CCTPMC is NP-hard in general.

In this note, we examine the effect of appending a cardinality constraint to the simple TPMC problem.

Theorem 1. *Given an instance of TPMC with V_2, the set of demand nodes, and E, the set of edges, let $X \subseteq \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|}$ be the set of feasible solutions of the simple TPMC. Let $k \in \mathbb{Z}_+$ and $k \leq |V_2|$. Let $X^k := \mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j \in V_2} z_j \leq k\})$. If $d_j \leq 2$ for all $j \in V_2$, then $X^k = \mathrm{conv}(X) \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times [0, 1]^{|V_2|} \mid \sum_{j \in V_2} z_j \leq k\}$.*

Our proof of Theorem 1 is presented in Section 3. We note that the result of Theorem 1 holds even when X^k is defined as $\mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j \in V_2} z_j \geq k\})$ or $\mathrm{conv}(X \cap \{(x, z) \in \mathbb{R}^{|E|}_+ \times \{0, 1\}^{|V_2|} \mid \sum_{j \in V_2} z_j = k\})$.

In Lemma 1 in Section 3, we give a linear description of $\mathrm{conv}(X)$ by means of a projection of a matching polytope over which we can optimize in polynomial time. Therefore, by invoking the ellipsoid algorithm and the use of Theorem 1 we obtain the following corollary.

Corollary 1. *Cardinality constrained simple TPMC is polynomially solvable.*

We note that, as a consequence of Theorem 1 (but also inherent in our proof), a special class of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time: Simple TPMC can be reduced to a minimum weight perfect matching problem on a general (non-bipartite) graph $G' = (V', E')$ [4]. (Note that Observation 1, in contrast, provides a reduction *from* matching *to* a special case of simple TPMC.) Therefore, it is possible to reduce CCTPMC with $d_j \leq 2$ for all $j \in V_2$ to a *minimum weight perfect matching problem with a cardinality constraint on a subset of edges.* Hence, Corollary 1 implies that a special class of minimum weight perfect matching problems with a cardinality constraint on a subset of edges can be solved in polynomial time.

Note that the intersection of the perfect matching polytope with a cardinality constraint on a strict subset of edges is not always integral.

Example 1. *Consider the cycle C_4 of length 4 with edge set $E = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$, and the cardinality constraint $x_{12} + x_{34} = 1$. The only perfect matchings are $\{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ for which the cardinality constraint has activity 2 and 0, respectively. Thus the perfect matching polytope is a line which is intersected by the hyperplane defined by the cardinality constraint in the (fractional) center.*

To the best of our knowledge, the complexity status of minimum weight perfect matching problem on a general graph with a cardinality constraint on a subset of edges is open. This can be seen by observing that if one can solve minimum weight perfect matching problem with a cardinality constraint on a subset of edges in polynomial time, then one can solve the exact perfect matching problem, in polynomial time. Given a weighted graph, *the exact perfect matching problem* is to find a perfect matching that has a total weight equal to a given number. The complexity status of exact perfect matching is open; see discussion in the last section in [2].

Finally we ask the natural question: Does the statement of Theorem 1 hold when $d_j \leq 2$ does not hold for every j? The next example illustrates that the statement does not hold in such case.

Example 2. *Consider an instance of TPMC where $G = (V_1 \cup V_2, E)$ is a bipartite graph with*

$$\begin{aligned} &V_1 = \{i_1, i_2, i_3, i_4, i_5, i_6\}, \qquad V_2 = \{j_1, j_2, j_3, j_4\},\\ &E = \{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}, \{i_4, j_1\}, \{i_4, j_4\}, \{i_5, j_2\}, \{i_5, j_4\}, \{i_6, j_3\}, \{i_6, j_4\}\},\\ &s_i = 1, i \in V_1, \qquad d_{j_1} = d_{j_2} = d_{j_3} = 2,\, d_{j_4} = 3. \end{aligned}$$

For $k = 2$ it can be verified that we obtain a non-integer extreme point of $\mathrm{conv}(X) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n \mid \sum_{j=1}^n z_j \leq k\}$, given by $x_{\{i_1,j_1\}} = x_{\{i_2,j_2\}} = x_{\{i_3,j_3\}} = x_{\{i_4,j_1\}} = x_{\{i_4,j_4\}} = x_{\{i_5,j_2\}} = x_{\{i_5,j_4\}} = x_{\{i_6,j_3\}} = x_{\{i_6,j_4\}} = z_1 = z_2 = z_3 = z_4 = \frac{1}{2}$. To see this, consider the face defined by the supply constraints of nodes $\{i_4, i_5, i_6\}$ and observe that this face has precisely two solutions having 1 and 3 markets, respectively.

Therefore, $X^k \neq \mathrm{conv}(X) \cap \{(x,z) \in \mathbb{R}^p_+ \times [0,1]^n \mid \sum_{j=1}^n z_j \leq k\}$ in this example.

3. Proof of Theorem 1

To prove Theorem 1 we use an improved reduction to a minimum weight matching problem (compared to the reduction in [4]) and then use the well-known adjacency properties of the vertices of the perfect matching polytope. Since the integrality result does not hold for the perfect matching polytope on a general graph with a cardinality constraint on any subset of edges, as illustrated in Example 1, we need to refine the adjacency criterion.

We begin with some notation. For a graph $G = (V, E)$ with node set V and edge set E, and a node $v \in V$, we denote by $\delta(v) := \delta_G(v) := \{e \in E \mid v \in e\}$ the set of edges incident to v. For a vector $x \in \mathbb{R}^{|E|}$ and a subset $F \subseteq E$ of its ground set, we define $x(F) := \sum_{f \in F} x_f$.

We now describe the improved reduction to a minimum weight matching problem. Consider a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{|V_1|}$, demands $d \in \{1,2\}^{|V_2|}$, edge weights

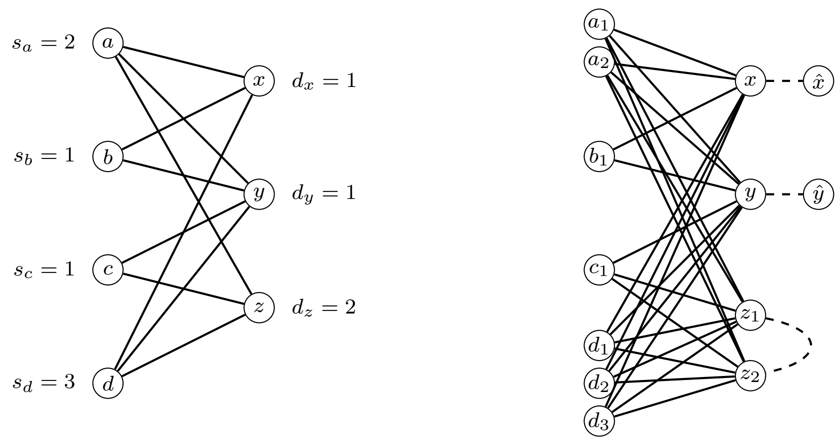

Figure 1: Improved Reduction to a Matching Problem

$w \in \mathbb{R}^{|E|}$, and revenues $r \in \mathbb{R}^{|V_2|}$. Let $D_k = \{j \in V_2 \mid d_j = k\}$ be the partitioning of V_2 into two classes corresponding to the demands.

We create the auxiliary graph G^* (see Figure 1) with nodes $V_1^s \cup D_1 \cup \widehat{D}_1 \cup D_2^1 \cup D_2^2$ and edges $E_1 \cup E_2 \cup F_1 \cup F_2$ with

$$
\begin{aligned}
V_1^s &= \{i_\ell \mid i \in V_1 \text{ and } \ell \in \{1, 2, \ldots, s_i\}\},\\
\widehat{D}_1 &= \{\hat{j} \mid j \in D_1\},\\
D_2^k &= \{j_k \mid j \in D_2\} \text{ for } k = 1, 2,\\
E_1 &= \{\{i_\ell, j\} \mid \{i, j\} \in E,\ \ell \in \{1, 2, \ldots, s_i\} \text{ and } j \in D_1\},\\
E_2 &= \{\{i_\ell, j_k\} \mid \{i, j\} \in E,\ \ell \in \{1, 2, \ldots, s_i\}, j \in D_2 \text{ and } k \in \{1, 2\}\},\\
F_1 &= \{\{j, \hat{j}\} \mid j \in D_1\}, \text{ and}\\
F_2 &= \{\{j_1, j_2\} \mid j \in D_2\}.
\end{aligned}
$$

In the construction every node $i \in V_1$ with supply s_i is split into s_i identical nodes with intended supply value of 1. Furthermore, to every node $j \in V_2$ with demand 1 we attach an edge with a dead end $\hat{j}$, and every node $j \in V_2$ with demand 2 is split into nodes j_1 and j_2 which are connected by an edge. Note that this is a polynomial construction, because the supply, s_i, is at most $2|V_2|$ for any $i \in V_1$.

Lemma 1. *Let $X \subseteq \mathbb{R}_+^{|E|} \times \{0,1\}^{|V_2|}$ be the set of feasible solutions of a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{|V_1|}$ and demands $d \in \{1,2\}^{|V_2|}$. Let the sets D_k and the auxiliary graph G^* be defined as above.*

Then $P := \text{conv}(X)$ is equal to the projection of the face of the matching polytope $P_{match}(G^*)$ of G^*

$$Q := \{y \in P_{match}(G^*) \mid y(\delta(v)) = 1 \text{ for all } v \in D_1 \cup D_2^1 \cup D_2^2\}$$

via the map π defined by $x_{\{i,j\}} = \sum_{\ell=1}^{s_i} y_{\{i_\ell,j\}}$ for $\{i,j\} \in E$ and $j \in D_1$, $x_{\{i,j\}} = \sum_{\ell=1}^{s_i} (y_{\{i_\ell,j_1\}} + y_{\{i_\ell,j_2\}})$ for $\{i,j\} \in E$ and $j \in D_2$, $z_j = y_{\{j,\hat{j}\}}$ for $j \in D_1$, and $z_j = y_{\{j_1,j_2\}}$ for $j \in D_2$.

Proof. We first show $\pi(Q) \subseteq P$. Let y be a vertex of Q and $(x,z) = \pi(y)$ be the projection.

Clearly, for all $i \in V_1$ we have $x(\delta_G(i)) = \sum_{\ell=1}^{s_i} y(\delta_{G^*}(i_\ell)) \le s_i$, i.e., (x,z) satisfies (3). For every node $j \in D_1$ we have $x(\delta_G(j)) + z_j = y(\delta_{G^*}(j) \setminus \{\{j,\hat{j}\}\}) + y_{\{j,\hat{j}\}} = y(\delta_{G^*}(j)) = 1$. Furthermore, for every node $j \in D_2$ we have $x(\delta_G(j)) + 2z_j = y(\delta_{G^*}(j_1) \setminus \{\{j_1,j_2\}\}) + y(\delta_{G^*}(j_1) \setminus \{\{j_1,j_2\}\}) + 2y_{\{j_1,j_2\}} = y(\delta_{G^*}(j_1)) + y(\delta_{G^*}(j_2)) = 2$. Hence, (x,z) satisfies (2) proving $(x,z) \in \text{conv}(X)$ since z is binary.

We now show $P \subseteq \pi(Q)$ for which it suffices to consider only integer points in P since both polytopes are integral. Note that P is integral since for integral z the remaining system is totally unimodular with integral right-hand side. Let $(x, z) \in P \cap (\mathbb{Z}_+^{|E|} \times \{0,1\}^{|V_2|})$ be an integral point in P. For $j \in D_1$ with $z_j = 0$, let $e_j \in E$ be the unique edge with $x_{\{i,j\}} > 0$, and for $j \in D_2$ with $z_j = 0$, let $\{e_j, f_j\}$ be the set of edges incident to j with positive x-value. Observe that if $e_j = f_j$ holds, then $x_{e_j} = 2$, and otherwise $x_{e_j} = x_{f_j} = 1$.

Construct a matching M satisfying

$$M = \{\{j, \hat{j}\} \mid j \in D_1 \text{ with } z_j = 1\} \cup \{\{j_1, j_2\} \mid j \in D_2 \text{ with } z_j = 1\} \tag{7}$$
$$\cup \{\{i_\ell, j\} \mid j \in D_1 \text{ and } i \in e_j \text{ with } z_j = 0\} \tag{8}$$
$$\cup \{\{i_\ell, j_1\} \mid j \in D_2 \text{ and } i \in e_j \text{ with } z_j = 0\} \tag{9}$$
$$\cup \{\{i_\ell, j_2\} \mid j \in D_2 \text{ and } i \in f_j \text{ with } z_j = 0\} \tag{10}$$

choosing ℓ in (8)–(10) such that every node $i_\ell \in V_1^s$ has at most one incident edge in M. This is possible since for each $i \in V_1$, G^* has s_i identical copies $i_1, \ldots, i_{s_i}$ and M has to contain at most $x(\delta_G(i)) \leq s_i$ edges incident to one of the copies because x is integral.

We first prove that M is indeed a matching. A node $j \in D_1$ is matched either to $\hat{j}$ (if $z_j = 1$) or by e_j. Similarly, either j_1 and j_2 are matched by the edge $\{j_1, j_2\}$ (again if $z_j = 1$) or by e_j and f_j, respectively.

The fact that M projects to (x, z) is easy to check by the construction of M according to (7)–(10). This concludes the proof. □

We now turn to the proof of Theorem 1. By definition of the projection map π in Lemma 1, the equation $z(V_2) = k$ corresponds to the equation $y(F_1 \cup F_2) = k$ in Q, that is,

$$P \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times [0,1]^{|V_2|} \mid z(V_2) = k\} = \{\pi(y) \mid y \in Q \text{ with } y(F_1 \cup F_2) = k\}$$

holds. Hence, in order to show that the former is integral (and since π projects integral vectors to integral vectors), it suffices to prove the following claim:

Claim 1. *Let $X \subseteq \mathbb{R}_+^{|E|} \times \{0,1\}^{|V_2|}$ be the set of feasible solutions of a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{|V_1|}$ and demands $d \in \{1,2\}^{|V_2|}$. Let the sets D_k and the auxiliary graph G^* be defined as above and let Q be as in Lemma 1.*

Then $\{y \in Q \mid y(F_1 \cup F_2) = k\}$ is an integral polytope for any integer $k \in \mathbb{Z}_+$.

Proof. Let $H = \{y \mid y(F_1 \cup F_2) = k\}$ denote the intersecting hyperplane and assume, for the sake of contradiction, that $Q \cap H$ is not integral. Then there must exist two adjacent (in Q) matchings M_1 and M_2 defining an edge of Q that is intersected by H in its relative interior, i.e., $|M_1 \cap (F_1 \cup F_2)| < k$ and $|M_2 \cap (F_1 \cup F_2)| > k$.

By the adjacency characterization of the matching polytope [3], the symmetric difference $C := M_1 \Delta M_2$ must be a connected component (a cycle or a path) in G^* containing edges of M_1 and M_2 in an alternating fashion.

We now verify that there must exist a path e-P-f in C of odd length consisting of two edges $e, f \in C \cap (F_1 \cup F_2)$ and a subpath P in $C \setminus (F_1 \cup F_2)$: If for every choice of $e, f \in M_2 \cap C \cap (F_1 \cup F_2)$ there exists an edge belonging to $M_1 \cap (F_1 \cup F_2)$ in all subpath(s) e-P-f of C, then M_2 can have at most one more edge of $F_1 \cup F_2$ than M_1 in C. However since M_2 contains at least two more edges of $F_1 \cup F_2$ than M_1 does, we have that there exists a path e-P-f in C consisting of two edges $e, f \in M_2 \cap C \cap (F_1 \cup F_2)$ and a subpath P in $C \setminus (F_1 \cup F_2)$. Now since e-P-f is subpath of C and $e, f \in M_2$, we have that P is of odd length.

Clearly, since we have $P \cap (F_1 \cup F_2) = \emptyset$, all of P's edges must go between V_1^s and $D_1 \cup (D_2^1 \cup D_2^2)$. Since P also has odd length, one of its endpoints is in V_1^s. But no edge in $F_1 \cup F_2$ is incident to any node in V_1^s which yields a contradiction. □

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