On a Cardinality-Constrained Transportation Problem With Market Choice

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Abstract

It is well-known that the intersection of the matching polytope with a cardinality constraint is integral [8]. In this note, we prove a similar result for the polytope corresponding to the transportation problem with market choice (TPMC) (introduced in [4]) when the demands are in the set \{1, 2\}. This result generalizes the result regarding the matching polytope. The result in this note implies that some special classes of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time.

Keywords: Transportation problem with market choice, cardinality constraint, integral polytope

1. Introduction

The transportation problem with market choice (TPMC), introduced in the paper [4], is a transportation problem in which suppliers with limited capacities have a choice of which demands (markets) to satisfy. If a market is selected, then its demand must be satisfied fully through shipments from the suppliers. If a market is rejected, then the corresponding potential revenue is lost. The objective is to minimize the total cost of shipping and lost revenues. See [5, 7, 9] for approximation algorithms and heuristics for several other supply chain planning and logistics problems with market choice.

Formally, we are given a set of supply and demand nodes that form a bipartite graph \(G = (V_1 \cup V_2, E)\). The nodes in set \(V_1\) represent the supply nodes, where for \(i \in V_1\), \(s_i \in \mathbb{N}\) represents the capacity of supplier \(i\). The nodes in set \(V_2\) represent the potential markets, where for \(j \in V_2\), \(d_j \in \mathbb{N}\) represents the demand of market \(j\). The edges between supply and demand nodes have weights that represent shipping costs \(w_e\), where \(e \in E\). For each \(j \in V_2\), \(r_j\) is the revenue lost if the market \(j\) is rejected. Let \(x_{i,j}\) be the amount of demand of market \(j\) satisfied by supplier \(i\) for \(\{i,j\} \in E\), and let \(z_j\) be an indicator variable taking a value 1 if market \(j\) is rejected and 0 otherwise. A mixed-integer programming (MIP) formulation of the problem is given where the objective is to minimize the transportation costs and the lost revenues due to unchosen markets:

\[
\begin{align*}
\min_{x \in \mathbb{R}^{|E|}, z \in \{0,1\}^{|V_2|}} & \quad \sum_{e \in E} w_e x_e + \sum_{j \in V_2} r_j z_j \\
\text{s.t.} & \quad \sum_{i:\{i,j\} \in E} x_{i,j} = d_j (1 - z_j) \quad \forall j \in V_2 \\
& \quad \sum_{j:\{i,j\} \in E} x_{i,j} \leq s_i \quad \forall i \in V_1.
\end{align*}
\]
We refer to the formulation (1)-(3) as TPMC. The first set of constraints (2) ensures that if market \( j \in V_2 \) is selected (i.e., \( z_j = 0 \)), then its demand must be fully satisfied. The second set of constraints (3) model the supply restrictions.

TPMC is strongly \( \text{NP} \)-complete in general [4]. Aardal and Le Bodic [1] give polynomial-time reductions from this problem to the capacitated facility location problem [6], thereby establishing approximation algorithms with constant factors for the metric case and a logarithmic factor for the general case.

In Lemma 1 in Section 3, we give a linear description of \( \text{conv}(X) \) by means of a projection of a matching polytope over which we can optimize in polynomial time. Therefore, by invoking the ellipsoid algorithm and the use of Theorem 1 we obtain the following corollary.

**Observation 1** (Simple TPMC generalizes Matching on General Graphs). The matching problem can be seen as a special case of the simple TPMC problem. Let \( G = (V,E) \) be a graph with \( n \) vertices and \( m \) edges. We construct a bipartite graph \( G = (V^1 \cup V^2, \hat{E}) \) as follows: \( V^1 \) is a set of \( n \) vertices corresponding to the \( n \) vertices in \( G \), and \( V^2 \) corresponds to the set of edges of \( G \), i.e., \( V^2 \) contains \( m \) vertices. We use \( \{i,j\} \) to refer to the vertex in \( V^2 \) corresponding to the edge \( \{i,j\} \) in \( E \). The set of edges in \( \hat{E} \) are of the form \( \{i,\{i,j\}\} \) and \( \{j,\{i,j\}\} \) for every \( i,j \in V \) such that \( \{i,j\} \in E \). Now we can construct (the feasible region of) an instance of TPMC with respect to \( G = (V^1 \cup V^2, \hat{E}) \) as follows:

\[
T = \{(x,z) \in \mathbb{R}_{+}^n \times \mathbb{R}^m \mid x_{\{i,e\}} + x_{\{j,e\}} + 2z_e = 2 \forall e = \{i,j\} \in \hat{E} \}
\]

(4)

\[
\sum_{j:\{i,j\} \in E} x_{\{i,j\}} \leq 1 \forall i \in \hat{V}^1
\]

(5)

\[
z_e \in \{0,1\} \forall e \in \hat{V}^2.
\]

(6)

Clearly there is a bijection between the set of matchings in \( G \) and the set of solutions in \( T \). Moreover, let

\[
H := \{(x,z,y) \in \mathbb{R}^{2m} \times \mathbb{R}^n \times \mathbb{R}^m \mid (x,z) \in T, y = e - z\},
\]

where \( e \) is the all ones vector in \( \mathbb{R}^m \). Then we have that the convex hull of the incidence vectors of all the matchings in \( G = (V,E) \) is precisely the set \( \text{proj}_y(H) \).

Note that the instances of the form of (4)-(6) are special cases of simple TPMC instances, since in these instances all \( s_i \)'s are restricted to be exactly 1 and all \( d_j \)'s are restricted to be exactly 2.

**2. Main Result**

An important and natural constraint that one may add to the TPMC problem is that of a service level, that is the number of rejected markets is restricted to be at most \( k \). This restriction can be modelled using a cardinality constraint, \( \sum_{j \in V_2} z_j \leq k \), appended to (1)-(3). We call the resulting problem cardinality-constrained TPMC (CCTPMC). If we are able to solve CCTPMC in polynomial-time, then we can solve TPMC in polynomial time by solving CCTPMC for all \( k \in \{0, \ldots, |V_2|\} \). Since TPMC is \( \text{NP} \)-hard, CCTPMC is \( \text{NP} \)-hard in general.

In this note, we examine the effect of appending a cardinality constraint to the simple TPMC problem.

**Theorem 1.** Given an instance of TPMC with \( V_2 \), the set of demand nodes, and \( E \), the set of edges, let \( X \subseteq \mathbb{R}_{+}^{|E|} \times \{0,1\}^{|V_2|} \) be the set of feasible solutions of the simple TPMC. Let \( k \in \mathbb{Z}_+ \) and \( k \leq |V_2| \). Let \( X^k := \text{conv}(X \cap \{(x,z) \in \mathbb{R}_{+}^{|E|} \times \{0,1\}^{|V_2|} \mid \sum_{j \in V_2} z_j \leq k\}) \). If \( d_j \leq 2 \) for all \( j \in V_2 \), then \( X^k \) is a polytope.

Our proof of Theorem 1 is presented in Section 3. We note that the result of Theorem 1 holds even when \( X^k \) is defined as \( \text{conv}(X \cap \{(x,z) \in \mathbb{R}_{+}^{|E|} \times \{0,1\}^{|V_2|} \mid \sum_{j \in V_2} z_j \geq k\}) \) or \( \text{conv}(X \cap \{(x,z) \in \mathbb{R}_{+}^{|E|} \times \{0,1\}^{|V_2|} \mid \sum_{j \in V_2} z_j = k\}) \).

In Lemma 1 in Section 3, we give a linear description of \( \text{conv}(X) \) by means of a projection of a matching polytope over which we can optimize in polynomial time. Therefore, by invoking the ellipsoid algorithm and the use of Theorem 1 we obtain the following corollary.
Corollary 1. Cardinality constrained simple TPMC is polynomially solvable.

We note that, as a consequence of Theorem 1 (but also inherent in our proof), a special class of minimum weight perfect matching problem with a cardinality constraint on a subset of edges can be solved in polynomial time: Simple TPMC can be reduced to a minimum weight perfect matching problem on a general (non-bipartite) graph $G' = (V', E')$ [4]. (Note that Observation 1, in contrast, provides a reduction from matching to a special case of simple TPMC.) Therefore, it is possible to reduce CCTPMC with $d_j \leq 2$ for all $j \in V_2$ to a minimum weight perfect matching problem with a cardinality constraint on a subset of edges. Hence, Corollary 1 implies that a special class of minimum weight perfect matching problems with a cardinality constraint on a subset of edges can be solved in polynomial time.

Note that the intersection of the perfect matching polytope with a cardinality constraint on a strict subset of edges is not always integral.

Example 1. Consider the cycle $C_4$ of length 4 with edge set $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$, and the cardinality constraint $x_{12} + x_{34} = 1$. The only perfect matchings are $\{\{1,2\}, \{3,4\}\}$ and $\{\{1,4\}, \{2,3\}\}$ for which the cardinality constraint has activity 2 and 0, respectively. Thus the perfect matching polytope is a line which is intersected by the hyperplane defined by the cardinality constraint in the (fractional) center.

To the best of our knowledge, the complexity status of minimum weight perfect matching problem on a general graph with a cardinality constraint on a subset of edges is open. This can be seen by observing that if one can solve minimum weight perfect matching problem with a cardinality constraint on a subset of edges in polynomial time, then one can solve the exact perfect matching problem, in polynomial time. Given a weighted graph, the exact perfect matching problem is to find a perfect matching that has a total weight equal to a given number. The complexity status of exact perfect matching is open; see discussion in the last section in [2].

Finally we ask the natural question: Does the statement of Theorem 1 hold when $d_j \leq 2$ does not hold for every $j$? The next example illustrates that the statement does not hold in such case.

Example 2. Consider an instance of TPMC where $G = (V_1 \cup V_2, E)$ is a bipartite graph with

$$V_1 = \{i_1, i_2, i_3, i_4, i_5, i_6\}, \quad V_2 = \{j_1, j_2, j_3, j_4\},$$

$$E = \{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}, \{i_4, j_1\}, \{i_4, j_4\}, \{i_5, j_2\}, \{i_5, j_4\}, \{i_6, j_3\}, \{i_6, j_4\}\},$$

$$s_i = 1, i \in V_1, \quad d_{j_1} = d_{j_2} = d_{j_3} = 2, d_{j_4} = 3.$$

For $k = 2$ it can be verified that we obtain a non-integer extreme point of $\text{conv}(X) \cap \{(x,z) \in \mathbb{R}_{+}^n \times [0,1]^n \mid \sum_{j=1}^n z_j \leq k\}$, given by $x_{\{i_1,j_1\}} = x_{\{i_2,j_2\}} = x_{\{i_3,j_3\}} = x_{\{i_4,j_1\}} = x_{\{i_4,j_4\}} = x_{\{i_5,j_2\}} = x_{\{i_5,j_4\}} = x_{\{i_6,j_3\}} = x_{\{i_6,j_4\}} = z_1 = z_2 = z_3 = z_4 = \frac{1}{2}$. To see this, consider the face defined by the supply constraints of nodes $\{i_4, i_5, i_6\}$ and observe that this face has precisely two solutions having 1 and 3 markets, respectively.

Therefore, $X^k \neq \text{conv}(X) \cap \{(x,z) \in \mathbb{R}_{+}^n \times [0,1]^n \mid \sum_{j=1}^n z_j \leq k\}$ in this example.

3. Proof of Theorem 1

To prove Theorem 1 we use an improved reduction to a minimum weight matching problem (compared to the reduction in [4]) and then use the well-known adjacency properties of the vertices of the perfect matching polytope. Since the integrality result does not hold for the perfect matching polytope on a general graph with a cardinality constraint on any subset of edges, as illustrated in Example 1, we need to refine the adjacency criterion.

We begin with some notation. For a graph $G = (V, E)$ with node set $V$ and edge set $E$, and a node $v \in V$, we denote by $\delta(v) := \delta_G(v) := \{e \in E \mid v \in e\}$ the set of edges incident to $v$. For a vector $x \in \mathbb{R}^{|E|}$ and a subset $F \subseteq E$ of its ground set, we define $x(F) := \sum_{e \in F} x_e$.

We now describe the improved reduction to a minimum weight matching problem. Consider a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{|V_1|}$, demands $d \in \{1,2\}^{|V_2|}$, edge weights
Let $D_k = \{ j \in V_2 \mid d_j = k \}$ be the partitioning of $V_2$ into two classes corresponding to the demands.

We create the auxiliary graph $G^*$ (see Figure 1) with nodes $V_1^* \cup D_1 \cup \hat{D}_1 \cup D_2^* \cup D_3^*$ and edges $E_1 \cup E_2 \cup F_1 \cup F_2$ with

\[
V_1^* = \{ \ell \mid \ell \in \{1, 2, \ldots, s_i\} \},
\]
\[
\hat{D}_1 = \{ j \mid j \in D_1 \},
\]
\[
D_2^* = \{ j_k \mid j \in D_2 \} \text{ for } k = 1, 2,
\]
\[
E_1 = \{ \{i_\ell, j\} \mid \{i, j\} \in E, \ell \in \{1, 2, \ldots, s_i\} \text{ and } j \in D_1 \},
\]
\[
E_2 = \{ \{i_\ell, j_k\} \mid \{i, j\} \in E, \ell \in \{1, 2, \ldots, s_i\} \text{ and } j \in D_2 \text{ and } k \in \{1, 2\} \},
\]
\[
F_1 = \{ \{j, j\} \mid j \in D_1 \}, \text{ and}
\]
\[
F_2 = \{ \{j_1, j_2\} \mid j \in D_2 \}.
\]

In the construction every node $i \in V_1$ with supply $s_i$ is split into $s_i$ identical nodes with intended supply value of 1. Furthermore, to every node $j \in V_2$ with demand 1 we attach an edge with a dead end $\hat{j}$, and every node $j \in V_2$ with demand 2 is split into nodes $j_1$ and $j_2$ which are connected by an edge. Note that this is a polynomial construction, because the supply, $s_i$, is at most $2|V_2|$ for any $i \in V_1$.

**Lemma 1.** Let $X \subseteq \mathbb{R}_{+}^{\{|E| \times \{0, 1\}|V_2|}$ be the set of feasible solutions of a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{|V_1|}$ and demands $d \in \{1, 2\}^{|V_2|}$. Let the sets $D_k$ and the auxiliary graph $G^*$ be defined as above.

Then $P := \text{conv}(X)$ is equal to the projection of the face of the matching polytope $P_{\text{match}}(G^*)$ of $G^*$

\[
Q := \{ y \in P_{\text{match}}(G^*) \mid y(\delta(v)) = 1 \text{ for all } v \in D_1 \cup D_2^* \}
\]

via the map $\pi$ defined by $x_{\{i,j\}} = \sum_{\ell=1}^{s_i} y_{\{i_\ell,j\}}$ for $\{i,j\} \in E$ and $j \in D_1$, $x_{\{i,j\}} = \sum_{\ell=1}^{s_i}(y_{\{i_\ell,j_1\}} + y_{\{i_\ell,j_2\}})$ for $\{i,j\} \in E$ and $j \in D_2$, $z_j = y_{\{j,j\}}$ for $j \in D_1$, and $z_j = y_{\{j_1,j_2\}}$ for $j \in D_2$.

**Proof.** We first show $\pi(Q) \subseteq P$. Let $y$ be a vertex of $Q$ and $(x,z) = \pi(y)$ be the projection.

Clearly, for all $i \in V_1$ we have $x(\delta_G(i)) = \sum_{\ell=1}^{s_i} y(\delta_{G^*}(i_\ell)) \leq s_i$, i.e., $(x,z)$ satisfies (3). For every node $j \in D_1$ we have $x(\delta_G(j)) = \sum_{\ell=1}^{s_i} z_j = y(\delta_{G^*}(j) \setminus \{j, j\}) + y_{\{j,j\}} = y(\delta_G(j)) = 1$. Furthermore, for every node $j \in D_2$ we have $x(\delta_G(j)) + 2z_j = y(\delta_{G^*}(j_1) \setminus \{j_1, j_2\}) + y(\delta_{G^*}(j_1) \setminus \{j_1, j_2\}) + 2y_{\{j_1,j_2\}} = y(\delta_{G^*}(j_1)) + y(\delta_{G^*}(j_2)) = 2$. Hence, $(x,z)$ satisfies (2) proving $(x,z) \in \text{conv}(X)$ since $z$ is binary.
We now show $P \subseteq \pi(Q)$ for which it suffices to consider only integer points in $P$ since both polytopes are integral. Note that $P$ is integral since for integral $z$ the remaining system is totally unimodular with integral right-hand side. Let $(x, z) \in P \cap (\mathbb{Z}_+^{|E|} \times \{0, 1\}^{|V_2|})$ be an integral point in $P$. For $j \in D_1$ with $z_j = 0$, let $e_j \in E$ be the unique edge with $x_{(i,j)} > 0$, and for $j \in D_2$ with $z_j = 0$, let $\{e_j, f_j\}$ be the set of edges incident to $j$ with positive $x$-value. Observe that if $e_j = f_j$ holds, then $x_{e_j} = 2$, and otherwise $x_{e_j} = x_{f_j} = 1$.

Construct a matching $M$ satisfying

$$M = \{(j, \hat{j}) \mid j \in D_1 \text{ with } z_j = 1\} \cup \{(j_1, j_2) \mid j \in D_2 \text{ with } z_j = 1\}$$

$$\cup \{(i, \hat{e}) \mid i \in e_j \text{ with } z_j = 0\}$$

$$\cup \{(i_1, j_1) \mid j \in D_2 \text{ and } i \in e_j \text{ with } z_j = 0\}$$

$$\cup \{(i_2, j_2) \mid j \in D_2 \text{ and } i \in f_j \text{ with } z_j = 0\}$$

choosing $\ell$ in (8)–(10) such that every node $i_\ell \in V_1^s$ has at most one incident edge in $M$. This is possible since for each $i \in V_1$, $G^*$ has $s_i$ identical copies $i_1, \ldots, i_{s_i}$ and $M$ has to contain at most $x(\delta G(i)) \leq s_i$ edges incident to one of the copies because $x$ is integral.

We first prove that $M$ is indeed a matching. A node $j \in D_1$ is matched either to $\hat{j}$ (if $z_j = 1$) or by $e_j$. Similarly, either $j_1$ and $j_2$ are matched by the edge $\{j_1, j_2\}$ (again if $z_j = 1$) or by $e_j$ and $f_j$, respectively.

The fact that $M$ projects to $(x, z)$ is easy to check by the construction of $M$ according to (7)–(10). This concludes the proof.

We now turn to the proof of Theorem 1. By definition of the projection map $\pi$ in Lemma 1, the equation $z(V_2) = k$ corresponds to the equation $y(F_1 \cup F_2) = k$ in $Q$, that is,

$$P \cap \{(x, z) \in \mathbb{R}_+^{|E|} \times \{0, 1\}^{V_2} \mid z(V_2) = k\} = \{y \mid y = \pi(y) \mid y \in Q \text{ with } y(F_1 \cup F_2) = k\}$$

holds. Hence, in order to show that the former is integral (and since $\pi$ projects integral vectors to integral vectors), it suffices to prove the following claim:

**Claim 1.** Let $X \subseteq \mathbb{R}_+^{|E|} \times \{0, 1\}^{V_2}$ be the set of feasible solutions of a simple TPMC instance on a graph $G = (V_1 \cup V_2, E)$ with supplies $s \in \mathbb{N}^{V_1}$ and demands $d \in \{1, 2\}^{V_2}$. Let the sets $D_k$ and the auxiliary graph $G^*$ be defined as above and let $Q$ be as in Lemma 1.

Then $\{y \in Q \mid y(F_1 \cup F_2) = k\}$ is an integral polytope for any integer $k \in \mathbb{Z}_+$.

**Proof.** Let $H = \{y \mid y(F_1 \cup F_2) = k\}$ denote the intersecting hyperplane and assume, for the sake of contradiction, that $Q \cap H$ is not integral. Then there must exist two adjacent (in $Q$) matchings $M_1$ and $M_2$ defining an edge of $Q$ that is intersected by $H$ in its relative interior, i.e., $|M_1 \cap (F_1 \cup F_2)| < k$ and $|M_2 \cap (F_1 \cup F_2)| > k$.

By the adjacency characterization of the matching polytope [3], the symmetric difference $C := M_1 \Delta M_2$ must be a connected component (a cycle or a path) in $G^*$ containing edges of $M_1$ and $M_2$ in an alternating fashion.

We now verify that there must exist a path $e$-$P$-$f$ in $C$ of odd length consisting of two edges $e, f \in C \cap (F_1 \cup F_2)$ and a subpath $P$ in $C \setminus (F_1 \cup F_2)$: If for every choice of $e, f \in M_2 \setminus C \cap (F_1 \cup F_2)$ there exists an edge belonging to $M_1 \cap (F_1 \cup F_2)$ in all subpath(s) $e$-$P$-$f$ of $C$, then $M_2$ can have at most one more edge of $F_1 \cup F_2$ than $M_1$ in $C$. However since $M_2$ contains at least two more edges of $F_1 \cup F_2$ than $M_1$ does, we have that there exists a path $e$-$P$-$f$ in $C$ consisting of two edges $e, f \in M_2 \cap C \cap (F_1 \cup F_2)$ and a subpath $P$ in $C \setminus (F_1 \cup F_2)$.

Now since $e$-$P$-$f$ is subpath of $C$ and $e, f \in M_2$, we have that $P$ is of odd length.

Clearly, since we have $P \cap (F_1 \cup F_2) = \emptyset$, all of $P$’s edges must go between $V_1^s$ and $D_1 \cup (D_2^1 \cup D_2^2)$. Since $P$ also has odd length, one of its endpoints is in $V_1^s$. But no edge in $F_1 \cup F_2$ is incident to any node in $V_1^s$ which yields a contradiction.

\[\square\]
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