A POLYHEDRAL STUDY OF MULTI-ECHELON LOT SIZING WITH INTERMEDIATE DEMANDS

MINJIAO ZHANG, SİMGE KÜÇÜKYAVUZ

Department of Integrated Systems Engineering, The Ohio State University

HANDE YAMAN

Department of Industrial Engineering, Bilkent University

Abstract. In this paper, we study a multi-echelon uncapacitated lot-sizing problem in series (m-ULS), where the output of the intermediate echelons has its own external demand, and is also an input to the next echelon. We propose a polynomial-time dynamic programming algorithm, which gives a tight, compact extended formulation for the two echelon case (2-ULS). Next, we present a family of valid inequalities for m-ULS, show its strength and give a polynomial-time separation algorithm. We establish a hierarchy between the alternative formulations for 2-ULS. In particular, we show that our valid inequalities can be obtained from the projection of the multi-commodity formulation. Our computational results show that this extended formulation is very effective in solving our uncapacitated multi-item 2-echelon test problems. In addition, for capacitated multi-item multi-echelon problems, we demonstrate the effectiveness of a branch-and-cut algorithm using the proposed inequalities.

Keywords: Lot-sizing, multi-echelon, facets, extended formulation, fixed-charge networks

1. Introduction

Managing inventory can be a challenging task for many enterprises. In particular, this task becomes significantly more complex for firms with multi-echelon supply chains, where replenishments of inventory located in multiple tiers must be synchronized. In this paper, we study a multi-echelon

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E-mail addresses: {zhang.769, kucukyavuz.2}@osu.edu, hyaman@bilkent.edu.tr.

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lot-sizing problem in series and with intermediate demands, which arises frequently for many wholesalers, retail chains and manufacturers. For example, consider a two-echelon distribution system for a wholesaler, which consists of regional and forward distribution centers (DCs). The regional DCs (first echelon) place orders to receive products directly from suppliers and then ship these products to forward DCs (second echelon). The forward DCs fulfill demand for most end-customers. However, the regional DCs may also ship directly to some end-customers in close proximity. Similarly, consider a two-echelon distribution system for a multi-channel retailer which consists of DCs and customer-facing stores. The DCs ship to all stores but may also ship directly to end-customers who order online. Finally, consider a two-echelon production system for a vertically-integrated manufacturer. The firm produces a part at the first echelon, which is used at the second echelon to assemble the final product. In addition, the same part may also be used to fulfill external demand from the repair or field service business.

In all of these examples, demand is dynamic and time-varying, and there are economies-of-scale in production/shipping of orders. The goal is to determine the production/order plan over a finite horizon to meet the demand at both echelons in each period with the minimum total cost, which includes fixed and variable production/order costs, and variable holding costs at each echelon. This problem can be seen as a fixed-charge network flow problem on a grid (see Figure 1).

In a seminal paper on the single-echelon uncapacitated lot-sizing problem (ULS), Wagner and Whitin (1958) analyze the properties of optimal solutions to ULS, and propose a polynomial-time algorithm. The running time was later improved by Aggarwal and Park (1993), Federgruen and Tzur (1991), Wagelmans et al. (1992). Krarup and Bilde (1977) give an uncapacitated facility location extended formulation for ULS and show that the linear programming (LP) relaxation of this formulation always has an optimal solution with integer setup variables. Barany et al. (1984) give a complete linear description of the ULS polyhedron using the so-called (ℓ, S) inequalities. Since then, several extensions of the single-echelon ULS polyhedron have been considered, to incorporate backlogging (Pochet and Wolsey, 1988, Küçükyavuz and Pochet, 2009), uncertainty in demands (Guan et al., 2006a,b), production or inventory capacities (Pochet and Wolsey, 1993, Atamtürk and Muñoz, 2004, Atamtürk and Küçükyavuz, 2005), among others (see Pochet and Wolsey (2006) for a review). Belvaux and Wolsey (2000, 2001) and Wolsey (2002) illustrate the utility of valid inequalities and reformulations for fundamental lot-sizing problems in solving more complex practical problems.

Multi-echelon lot-sizing problems have been considered primarily under the assumption that there is demand only at the final echelon. We refer to these problems as m-ULS-F, where m is the number of echelons. Zangwill (1969) proposes an $O(mn^4)$ dynamic programming algorithm for m-ULS-F and van Hoesel et al. (2005) show that for $m = 2$, this algorithm runs in $O(n^3)$ time. Love (1972) shows that if the production costs are non-increasing over time and the holding costs are non-decreasing over echelons, then there exists an optimal nested schedule. Exploiting this nested structure, an $O(mn^3)$ algorithm is proposed. Lee et al. (2003) give an $O(n^6)$ algorithm for 2-ULS-F when backlogging is allowed and there is a stepwise shipment cost between the two echelons. Melo and Wolsey (2010) propose a dynamic programming algorithm with an improved running time, $O(n^2 \log n)$, and a compact tight extended reformulation for 2-ULS-F. For a review of valid inequalities and extended formulations for m-ULS-F, we refer the reader to Pochet and Wolsey (2006). An effective heuristic for capacitated m-ULS-F using strong formulations for each echelon is proposed in Akartunalı and Miller (2009).

Various heuristic algorithms are proposed for the more complicated multi-echelon lot-sizing problems with demands in intermediate echelons (see, for example, Stadtler (2003) and the references therein). However, to the best of our knowledge, the polyhedral study of serial multi-echelon lot-sizing problems with demands in intermediate echelons (m-ULS) has received little attention in the literature. A notable exception is due to Gaglioppa et al. (2008), who study a multi-echelon production planning problem with complex assembly structures (not necessarily serial), where intermediate products (sub-assemblies) have external demand. They give a polynomial class of echelon inequalities valid for this problem. In contrast, we give an exponential class of inequalities (with polynomial separation) for the multi-echelon lot-sizing problem in series.

In this paper, we are interested in exact methods for m-ULS based on its polyhedral characterizations. In Section 2, we give an $O(n^4)$ dynamic program for 2-ULS, where n is the length of the finite planning horizon. In Section 3, we propose valid inequalities for m-ULS and study their strength. We also give a polynomial-time separation algorithm. In Section 4, we establish a hierarchy of alternative extended formulations for 2-ULS, and show that our inequalities can be obtained from the projection of the so-called multi-commodity formulation. Our computational results, summarized in Section 5, illustrate that the multi-commodity formulation is very effective in solving a difficult class of uncapacitated multi-item two-echelon lot-sizing problems. In addition, for capacitated multi-item multi-echelon problems, we demonstrate the effectiveness of a branch-and-cut algorithm using the proposed inequalities.

1.1. **Mathematical Model.** Let $d_t^i \geq 0$ denote the demand in period t at the ith echelon, and $d_{tk}^i = \sum_{j=t}^k d_j^i$, with $d_{tk}^i = 0$ if $t > k$. If we order in period t at echelon i, we incur a fixed cost f_t^i and a variable cost $\tilde{c}_t^i$. Let h_t^i denote the unit holding cost at echelon i at the end of period t. Let x_t^i be the order quantity at the ith echelon in period t, s_t^i be the inventory at echelon i at the end of period t, y_t^i be the order setup variable at the ith echelon in period t, where $y_t^i = 1$ if $x_t^i > 0$; $y_t^i = 0$ otherwise. Throughout the paper, we let $[i, j]$ denote the interval $\{i, i+1, \dots, j\}$ for $i \leq j$, and $[i, j] = \emptyset$ for $i > j$.

Figure 1 depicts a two-echelon 4-period uncapacitated lot-sizing network with demand in both echelons, where node (i, j) represents echelon j and period i. A *natural* formulation of 2-ULS is:

$$\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + \tilde{c}_t^i x_t^i + h_t^i s_t^i) \tag{1}$$

$$\text{s.t. } s_{t-1}^1 + x_t^1 = d_t^1 + x_t^2 + s_t^1 \qquad t \in [1, n], \tag{2}$$

$$s_{t-1}^2 + x_t^2 = d_t^2 + s_t^2 \qquad t \in [1, n], \tag{3}$$

$$s_0^i = s_n^i = 0 \qquad i \in [1, 2], \tag{4}$$

$$x_t^1 \leq (d_{tn}^1 + d_{tn}^2) y_t^1 \qquad t \in [1, n], \tag{5}$$

$$x_t^2 \leq d_{tn}^2 y_t^2 \qquad t \in [1, n], \tag{6}$$

$$y_t^i \in \{0, 1\} \qquad t \in [1, n], i \in [1, 2], \tag{7}$$

$$x_t^i \geq 0 \qquad t \in [1, n], i \in [1, 2], \tag{8}$$

$$s_t^i \geq 0 \qquad t \in [1, n], i \in [1, 2]. \tag{9}$$

The objective function (1) is to minimize the sum of fixed and variable ordering costs and the inventory holding costs. Constraints (2) and (3) are flow balance equations for the first and second echelon, respectively. We assume that the initial and ending inventories at both echelons are 0 as stated in constraints (4). Note that the assumption that $s_0^2 = 0$ is without loss of generality similar to the single echelon case (Pochet and Wolsey, 2006). However, for the first echelon, the assumption that $s_0^1 = 0$ is not without loss of generality. Constraints (5) and (6) are variable upper bound constraints that force the binary variables y_t^1 and y_t^2 to be 1 if there is a positive order in period t

at the first and second echelon, respectively. Finally, constraints (7)–(9) are variable restrictions. The formulation of m-ULS for $m \geq 3$ follows similarly.

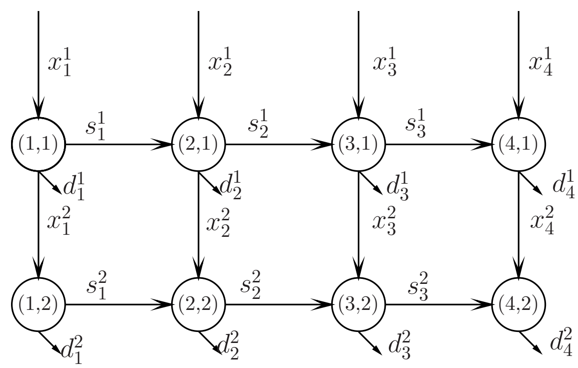

FIGURE 1. 2-echelon 4-period uncapacitated lot-sizing network

Note that from (2)-(4), the stock variables can be projected out by letting $s_t^1 = \sum_{j=1}^t (x_j^1 - x_j^2) - d_{1t}^1, s_t^2 = \sum_{j=1}^t x_j^2 - d_{1t}^2$ for $t \in [1, n]$, and we get an alternative formulation:

$$\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + c_t^i x_t^i) - B$$

s.t. (5) – (8)

$$\sum_{t=1}^{n} x_t^1 = d_{1n}^1 + d_{1n}^2, \tag{10}$$

$$\sum_{t=1}^{n} x_t^2 = d_{1n}^2, \tag{11}$$

$$\sum_{j=1}^{t} x_j^2 \geq d_{1t}^2 \qquad t \in [1, n], \tag{12}$$

$$\sum_{j=1}^{t} x_j^1 \geq \sum_{j=1}^{t} x_j^2 + d_{1t}^1 \qquad t \in [1, n], \tag{13}$$

where the unit order costs are updated as $c_t^1 = \tilde{c}_t^1 + \sum_{i=t}^n h_i^1, c_t^2 = \tilde{c}_t^2 + \sum_{i=t}^n (h_i^2 - h_i^1)$, for $t \in [1, n]$ and $B = \sum_{t=1}^n (h_t^1 d_{1t}^1 + h_t^2 d_{1t}^2)$ is a constant. In the sequel, we drop the constant term B from the objective function. We also make a realistic assumption that $\tilde{c}^1$ and $\tilde{c}^2$ are non-negative, and $h_i^2 \geq h_i^1$ for all $i \in [1, n]$. Thus, c^1 and c^2 are non-negative. In addition, we let $\mathcal{S}$ denote the set of feasible solutions to (5)–(8), (10)–(13).

2. Dynamic Programming Recursion and Reformulation

In this section, we give a dynamic programming (DP) recursion for 2-ULS that generalizes the algorithm of Zangwill (1969) by allowing positive demands at the first echelon. As 2-ULS is a single-source uncapacitated fixed-charge network (SSFCN) flow problem, we can apply the well-known result that the extreme points of SSFCN correspond to a spanning tree (Zangwill, 1968, Veinott, 1969) to conclude that there exists an optimal basic feasible solution to 2-ULS with $s^i_{t-1}x^i_t = 0$ for all $t \in [1, n]$ and $i \in [1, 2]$.

For $1 \leq i_2 \leq j_2 \leq n$, we define $(1, i_2, 1, j_2)$ as a *regeneration interval* if $s^1_{i_2} = s^2_{j_2} = 0$, $x^1_1 = d^1_{1i_2} + d^2_{1j_2}$, and $s^1_i > 0$ or $d^1_{i+1,i_2} = 0$ for $i \in [1, i_2 - 1]$. Similarly, for $2 \leq i_1 \leq i_2 \leq j_2 \leq n$, we define (i_1, i_2, j_1, j_2) as a *regeneration interval*, if for $i_1 \leq j_1 \leq j_2$, we have $s^1_{i_1-1} = s^1_{i_2} = s^2_{j_1-1} = s^2_{j_2} = 0$, $x^1_{i_1} = d^1_{i_1i_2} + d^2_{j_1j_2}$, and $s^1_i > 0$ or $d^1_{i+1,i_2} = 0$ for $i \in [i_1, i_2 - 1]$ or for $j_1 = j_2 + 1$, we have $s^1_{i_1-1} = s^1_{i_2} = 0$, $x^1_{i_1} = d^1_{i_1i_2}$, and $s^1_i > 0$ or $d^1_{i+1,i_2} = 0$ for $i \in [i_1, i_2 - 1]$. In addition, we define an interval (j_1, j_2) with $1 \leq j_1 \leq j_2 \leq n$, $s^2_{j_1-1} = s^2_{j_2} = 0$, $x^2_{j_1} = d^2_{j_1j_2}$, and $s^2_j > 0$ or $d^2_{j+1,j_2} = 0$ for $j \in [j_1, j_2 - 1]$ as a *regeneration subinterval* for the second echelon. A regeneration interval can contain several regeneration subintervals or no regeneration subinterval (when $j_1 = j_2 + 1$). In the latter case, the value of j_2 is equal to that of the preceding regeneration interval. For example, in Figure 2, $(1, 3, 1, 5)$, $(4, 4, 6, 5)$ and $(5, 6, 6, 6)$ are regeneration intervals, $(1, 2)$, $(3, 5)$ and $(6, 6)$ are regeneration subintervals. The regeneration interval $(1, 3, 1, 5)$ contains the regeneration subintervals $(1, 2)$ and $(3, 5)$. However, the regeneration interval $(4, 4, 6, 5)$ contains no regeneration subinterval. The spanning tree property of SSFCN implies that there exists an optimal basic feasible solution that is a concatenation of regeneration intervals.

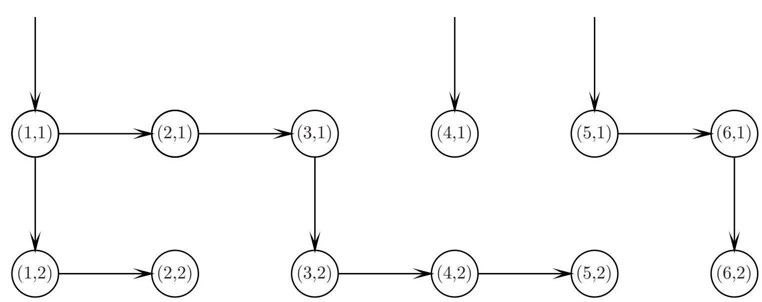

Figure 2. An optimal solution of a two-echelon 6-period uncapacitated lot-sizing problem

Let $G(i_2, j_2)$, $1 \leq i_2 \leq j_2 \leq n$, denote the minimum cost of satisfying the demand in periods 1 to i_2 at the first echelon and the demand in periods 1 to j_2 at the second echelon. In addition, let

$H(j_1, j_2), 1 \leq j_1 \leq n+1, 0 \leq j_2 \leq n$, be the minimum cost to satisfy the demand in periods j_1 to j_2 at the second echelon, where $H(j_1, j_2) = 0$ if $j_1 > j_2$. For $1 \leq i_2 \leq j_2 \leq n$, consider the forward recursions:

$$G(i_2, j_2) = \min \begin{cases} \min\limits_{\substack{2 \leq i_1 \leq i_2 \\ i_1 \leq j_1 \leq j_2+1}} \{G(i_1 - 1, j_1 - 1) + f^1_{i_1} + c^1_{i_1} d^1_{i_1 i_2} + c^1_{i_1} d^2_{j_1 j_2} + H(j_1, j_2)\}, \\ f^1_1 + c^1_1 d^1_{1 i_2} + c^1_1 d^2_{1 j_2} + H(1, j_2), \end{cases} \tag{14}$$

where for $1 \leq j_1 \leq j_2 \leq n$,

$$H(j_1, j_2) = \min_{j_1 \leq j_3 \leq j_2} \{H(j_1, j_3 - 1) + f^2_{j_3} + c^2_{j_3} d^2_{j_3 j_2}\}. \tag{15}$$

The minimum total cost over the entire planning horizon for the original problem is given by $G(n, n) - B$.

Proposition 1. *The dynamic program given by the recursions* (14) *and* (15) *solves 2-ULS in* $O(n^4)$ *time.*

Proof. Note that the recursion (14) evaluates the minimum cost to satisfy the demand in periods 1 to i_2 at the first echelon and the demand in periods 1 to j_2 at the second echelon such that the last regeneration interval is (i_1, i_2, j_1, j_2). Similarly, the recursion (15) calculates the minimum cost to satisfy the demand in periods j_1 to j_2 at the second echelon such that the last regeneration subinterval is (j_3, j_2). As a result, $G(n, n) - B$ gives the optimal objective function value to 2-ULS and is calculated in $O(n^4)$ time. □

In the special case that the intermediate demands at the first echelon are zero, we can drop the index i_2 in the recursion (14). Then the resulting recursions for $G(j_2)$ and $H(j_1, j_2)$ are identical to the dynamic programming recursions in Melo and Wolsey (2010).

We note that, using the approach proposed by Eppen and Martin (1987), Martin (1987), we can obtain a tight extended formulation for 2-ULS based on the proposed DP. This formulation has $O(n^4)$ variables and $O(n^4)$ constraints, including nonnegativities.

3. Valid Inequalities

In this section, we give valid inequalities for 2-ULS.

3.1. **2-Echelon Inequalities.** We define $\beta(T,k)$ as the set of consecutive elements in set T starting from k, where if $k \notin T, \beta(T,k) = \emptyset$. In other words, if $k \in T$, then $\beta(T,k) = [k,k'] \subseteq T$, for some k' such that $k'+1 \notin T$.

Theorem 2. *For $0 \leq k \leq l \leq n$, let $T_1 \subseteq [1,k]$, $[k+1,l] \subseteq T_2 \subseteq [1,l]$ and $T_3 \subseteq T_2$. Then the 2-echelon inequality*

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \sum_{j\in T_1} \phi_j y_j^1 + \sum_{j\in T_2\setminus T_3} x_j^2 + \sum_{j\in T_3} \psi_j y_j^2 \geq d_{1k}^1 + d_{1l}^2 \tag{16}$$

is valid for $\mathcal{S}$, where $\psi_j = \sum_{i\in\beta(T_2,j)} d_i^2$ and $\phi_j = d_{jk}^1 + d_{jl}^2 - \psi_j$.

Proof. We prove the validity of inequality (16) considering two cases.

(1) If $y_j^1 = 0$ for all $j \in T_1$, then $x_j^1 = 0$ for all $j \in T_1$. Let $i_1 := \min\{i \in T_2 \setminus T_3 : x_i^2 > 0, i \geq k+1\}$; if $\{i \in T_2 \setminus T_3 : x_i^2 > 0, i \geq k+1\} = \emptyset$, then let $i_1 := l+1$. Let $i_2 := \min\{i \in T_3 : x_i^2 > 0, i \geq k+1\}$; if $\{i \in T_3 : x_i^2 > 0, i \geq k+1\} = \emptyset$, then let $i_2 := l+1$. Note that $i_1 \neq i_2$ unless $i_1 = i_2 = l+1$.

- If $i_1 > i_2$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 \geq d_{1k}^1 + d_{1,i_2-1}^2$ and $\psi_{i_2} y_{i_2}^2 = \psi_{i_2} = d_{i_2 l}^2$. Summing these two inequalities up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \psi_{i_2} y_{i_2}^2 \geq d_{1k}^1 + d_{1l}^2.$$

- If $i_1 < i_2$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 + \sum_{j\in[i_1,i_2-1]\setminus T_3} x_j^2 \geq d_{1k}^1 + d_{1,i_2-1}^2$ and $\psi_{i_2} y_{i_2}^2 = d_{i_2 l}^2 y_{i_2}^2$. Summing these two inequalities up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \sum_{j\in[i_1,i_2-1]\setminus T_3} x_j^2 + \psi_{i_2} y_{i_2}^2 \geq d_{1k}^1 + d_{1l}^2.$$

  Note that $([i_1, i_2-1] \setminus T_3) \subseteq (T_2 \setminus T_3)$.

- If $i_1 = i_2 = l+1$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 \geq d_{1k}^1 + d_{1l}^2$.

Since all terms on the left hand side of inequality (16) are non-negative, inequality (16) is valid if $y_j^1 = 0$ for all $j \in T_1$.

(2) If there exists $j \in T_1$ such that $y_j^1 = 1$, then let $j_1 := \min\{j \in T_1 : y_j^1 = 1\}$.

(a) If $j_1 \notin T_2$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 \geq d_{1,j_1-1}^1 + d_{1,j_1-1}^2$ and $\phi_{j_1} y_{j_1}^1 = \phi_{j_1} = d_{j_1 k}^1 + d_{j_1 l}^2$. Summing them up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \phi_{j_1} y_{j_1}^1 \geq d_{1k}^1 + d_{1l}^2.$$

(b) If $j_1 \in T_2$, then let $v := \max\{j \in \beta(T_2, j_1)\}$.

(i) If $x_j^2 = 0$ for all $j \in \beta(T_2, j_1)$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 \geq d_{1,j_1-1}^1 + d_{1v}^2$ and $\phi_{j_1} y_{j_1}^1 = \phi_{j_1} = d_{j_1 k}^1 + d_{v+1,l}^2$. Summing these two inequalities up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \phi_{j_1} y_{j_1}^1 \geq d_{1k}^1 + d_{1l}^2.$$

(ii) If there exists $j \in \beta(T_2, j_1)$ such that $x_j^2 > 0$, then let $j_2 := \min\{j \in \beta(T_2, j_1) : x_j^2 > 0\}$.

- If $j_2 \in T_3$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 \geq d_{1,j_1-1}^1 + d_{1,j_2-1}^2$, $\phi_{j_1} y_{j_1}^1 = \phi_{j_1} = d_{j_1 k}^1 + d_{v+1,l}^2$ and $\psi_{j_2} y_{j_2}^2 = \psi_{j_2} = d_{j_2 v}^2$. Summing them up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \phi_{j_1} y_{j_1}^1 + \psi_{j_2} y_{j_2}^2 \geq d_{1k}^1 + d_{1l}^2.$$

- If $j_2 \in T_2 \setminus T_3$, then consider the following two cases:
  - If $\{j \in [j_2+1, v] \cap T_3 : x_j^2 > 0\} \neq \emptyset$, then let $j_3 := \min\{j \in [j_2+1, v] \cap T_3 : x_j^2 > 0\}$. Then $\sum_{j\in[1,k]\setminus T_1} x_j^1 + \sum_{j\in[j_2,j_3-1]\setminus T_3} x_j^2 \geq d_{1,j_1-1}^1 + d_{1,j_3-1}^2$, $\phi_{j_1} y_{j_1}^1 = \phi_{j_1} = d_{j_1 k}^1 + d_{v+1,l}^2$ and $\psi_{j_3} y_{j_3}^2 = d_{j_3 v}^2$. Summing them up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \phi_{j_1} y_{j_1}^1 + \sum_{j\in[j_2,j_3-1]\setminus T_3} x_j^2 + \psi_{j_3} y_{j_3}^2 \geq d_{1k}^1 + d_{1l}^2.$$

    Note that $([j_2, j_3-1] \setminus T_3) \subseteq (T_2 \setminus T_3)$.
  - If $\{j \in [j_2+1, v] \cap T_3 : x_j^2 > 0\} = \emptyset$, then $\sum_{j\in[1,k]\setminus T_1} x_j^1 + \sum_{j\in[j_2,v]\setminus T_3} x_j^2 \geq d_{1,j_1-1}^1 + d_{1v}^2$ and $\phi_{j_1} y_{j_1}^1 = \phi_{j_1} = d_{j_1 k}^1 + d_{v+1,l}^2$. Summing them up, we get

$$\sum_{j\in[1,k]\setminus T_1} x_j^1 + \phi_{j_1} y_{j_1}^1 + \sum_{j\in[j_2,v]\setminus T_3} x_j^2 \geq d_{1k}^1 + d_{1l}^2.$$

    Note that $([j_2, v] \setminus T_3) \subseteq (T_2 \setminus T_3)$.

Since all terms on the left hand side of inequality (16) are non-negative, inequality (16) is valid if there exists $j \in T_1$ such that $y_j^1 > 0$.

Hence, the inequality (16) is valid.

□

An alternative proof can be obtained by using the dicut collection inequalities of Rardin and Wolsey (1993). We provide the precise correspondence between the simple dicut collection inequalities and the 2-echelon inequalities in Corollary 9.

Example 1. To illustrate the 2-echelon inequalities, consider a four-period problem as shown in Figure 1 with $d_i^1 = d_i^2 = 1$ for $i \in [1, 4]$. For $k = 2$ and $l = 3$, we have $x_1^1 + 3y_2^1 + x_3^2 \geq 5$ where $T_1 = \{2\}$, $T_2 = \{3\}$, $T_3 = \emptyset$. For $k = l = 3$, we have $x_1^1 + 4y_2^1 + y_3^1 + x_3^2 \geq 6$ where $T_1 = \{2, 3\}$, $T_2 = \{3\}$, $T_3 = \emptyset$, and $x_1^1 + 4y_2^1 + y_3^1 + y_3^2 \geq 6$ where $T_1 = \{2, 3\}$, $T_2 = \{3\}$, $T_3 = \{3\}$. For $k = 3$ and $l = 4$, we have $x_1^1 + 4y_2^1 + 3y_3^1 + x_2^2 + x_4^2 \geq 7$ where $T_1 = \{2, 3\}$, $T_2 = \{2, 4\}$, $T_3 = \emptyset$, and $x_1^1 + 4y_2^1 + 3y_3^1 + y_2^2 + x_4^2 \geq 7$ where $T_1 = \{2, 3\}$, $T_2 = \{2, 4\}$, $T_3 = \{2\}$.

Note that for $k = 0$, we have $T_1 = \emptyset$, $T_2 = [1, l]$ and $T_3 \subseteq T_2$, so inequality (16) is equivalent to the (ℓ, S) inequality of Barany et al. (1984) for the second echelon only, where $\ell = l$ and $T_3 = S$. For example,

$$x_1^2 + x_2^2 + y_3^2 \geq 3 \tag{17}$$

is the (ℓ, S) inequality for the second echelon only, with $\ell = 3$ and $S = \{3\}$. In addition, for $l = n$, $T_2 = [1, n], T_3 = \emptyset$, inequality (16) is equivalent to the (ℓ, S) inequality of Barany et al. (1984) for the first echelon only, where $\ell = k$ and $T_1 = S$. For example,

$$x_1^1 + x_2^1 + y_3^1 \geq 3 \tag{18}$$

is the (ℓ, S) inequality for the first echelon only, with $\ell = 3$, $S = \{3\}$. As a result, single echelon (ℓ, S) inequalities are valid for 2-ULS, and they are subsumed by the 2-echelon inequalities.

Also, for $k = l$ and $T_2 = \emptyset$, inequality (16) is equivalent to the (ℓ, S) inequality for the aggregation of the two echelons. For example,

$$x_1^1 + x_2^1 + 2y_3^1 \geq 6 \tag{19}$$

is the (ℓ, S) inequality for the aggregation of the two echelons with $\ell = 3$, $S = \{3\}$.

Using a similar argument, we can show that the 2-echelon inequalities obtained by aggregating the demands in echelons $[m_1, m_2]$ (echelon 1) and those in $[m_2 + 1, m_3]$ (echelon 2) for $1 \leq m_1 \leq m_2 < m_3 \leq m$, are valid for m-ULS for any $m \geq 2$. For example, for 4-period 5-echelon lot-sizing problem with unit demands in all echelons, letting $m_1 = 1, m_2 = 2, m_3 = 4$:

$$x_1^1 + 8y_2^1 + 6y_3^1 + x_2^3 + x_4^3 \geq 14 \tag{20}$$

is a valid 2-echelon inequality where $k = 3$, $l = 4$, $T_1 = \{2, 3\}$, $T_2 = \{2, 4\}$ and $T_3 = \emptyset$.

3.2. **Facet Conditions.** Next we give necessary and sufficient conditions for 2-echelon inequalities (16) to be facet-defining for conv($\mathcal{S}$). We assume that $\mathbf{d^1}$ and $\mathbf{d^2}$ are positive for ease of exposition. Note that under this assumption, $y_1^1 = y_1^2 = 1$. Denote a feasible point in conv($\mathcal{S}$) as $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2})$.

The dimension of conv($\mathcal{S}$) is $4n - 4$ for $\mathbf{d^1} > \mathbf{0}$ and $\mathbf{d^2} > \mathbf{0}$ (see Appendix A).

Proposition 3. *For $\mathbf{d^1} > \mathbf{0}$ and $\mathbf{d^2} > \mathbf{0}$, inequality (16) is facet-defining for conv($\mathcal{S}$) if and only if*

(1) $1 \notin T_1$;
(2) $1 \notin T_2$ *if* $k \neq 0$;
(3) $1 \notin T_3$ *if* $k = 0$;
(4) $k \neq 1$;
(5) *if* $k = 0$, $l = n$, *then* $|T_3| = 1$;
(6) *for every* $j \in T_2 \cap [2, k]$, *there exists* $i \in T_1$ *such that* $j \in \beta(T_2, i)$;
(7) *if* $2 \leq k \leq l = n$ *with* $T_3 \neq \emptyset$, *then* $T_3 \cap [k + 1, n] = \emptyset$ *and for each* $j \in T_3 \cap [2, k]$, *there exists* $j^* \in [j + 1, k]$ *such that* $j^* \notin T_2$;
(8) *if* $2 \leq k \leq l < n$, *then there exists* $j \in [p^1, k]$ *such that* $j \notin T_2$;
(9) *if* $k = l = n$, *then either* $T_2 = \emptyset$ *with* $|T_1| = 1$, *or* $T_2 \neq \emptyset$ *is a consecutive set with* $p^2 = p^1$ *and* $[p^1, w^1] \subseteq T_2 = [p^1, w^2] \subseteq [p^1, n]$;
(10) *if* $k \neq 0$, *then* $T_1 \neq \emptyset$; *if* $k = 0$, *then* $T_3 \neq \emptyset$;

where $p^1 := \min\{j \in T_1\}$, $w^1 := \max\{j \in T_1\}$, $p^2 := \min\{j \in T_2\}$ *and* $w^2 := \max\{j \in T_2\}$.

Proof. See Appendix B. □

Using the facet conditions, we see that (ℓ, S) inequalities for the second echelon only and for the aggregation of two echelons are facet-defining for 2-ULS problem, such as inequalities (17) and (19). But (ℓ, S) inequality for the first echelon only, such as inequality (18), is not facet-defining because it violates facet condition 2.

Based on our experiments with PORTA (Christof and Löbel, 2008), in a three-period two-echelon lot-sizing problem with unit demands in both echelons, all facets of the convex hull of 2-ULS solutions are defined by the 2-echelon inequalities. However, in a four-period problem with unit demands in both echelons, 65 out of the 81 facets are defined by the 2-echelon inequalities. 4 out of these 65 facets are (ℓ, S) inequalities for the aggregation of the first and second echelons, and 4 out of these 65 facets are (ℓ, S) inequalities for the second echelon only.

3.3. **Separation.**

Proposition 4. *Given a fractional point* $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2}) \in \mathbb{R}^{4n}$*, there is an* $O(n^4)$ *algorithm to find the most violated inequality* (16)*, if any.*

Proof. As stated earlier, when $k = 0$, 2-echelon inequalities are (ℓ, S) inequalities of Barany et al. (1984) for the second echelon, which have an $O(n \log n)$ separation algorithm (c.f., Pochet and Wolsey (2006)). When $k = 1$, the 2-echelon inequalities are not facet-defining due to facet condition 4. Next, for given k and l such that $2 \leq k \leq l \leq n$, we give an $O(n^2)$ algorithm that minimizes the left-hand-side of inequality (16). Note that for a given k and l, the right-hand-side of inequality (16) is fixed, so this algorithm maximizes the violation, if any.

Note that by definition, $[k+1, l] \subseteq T_2$. To minimize $\sum_{j \in T_2 \cap [k+1,l] \setminus T_3} x_j^2 + \sum_{j \in T_3 \cap [k+1,l]} \psi_j y_j^2$, let $T_3 \cap [k+1, l] := \{j \in [k+1, l] : x_j^2 \geq d_{jl}^2 y_j^2\}$. This takes $O(n)$ time. Now we need to determine the sets $T_1, T_2 \cap [1, k]$ and $T_3 \cap [1, k]$. Note that the coefficients of the variables in T_1 depend on the choice of T_2, because they contain the term $\psi_j = \sum_{i \in \beta(T_2, j)} d_i^2$.

Consider a shortest path network $G = (V, A)$. For example, Figure 3 is the shortest path network for separating a 2-echelon inequality (16) with $k = 4$. The node set is $V = \{1'\} \cup \{i : i \in [2, k+1]\} \cup \{i' : i \in [2, k]\}$, where $(k+1)$ is the sink node. Node i' represents $i \notin T_2$ and node i represents $i \in T_2$. By definition, we know that if $k \neq l$, then $(k+1) \in T_2$. From the facet conditions, we know that $1 \notin T_2$. The arc set is $A = \{(i', i+1) : i \in [1, k]\} \cup \{(i', (i+1)') : i \in [1, k-1]\} \cup \{(i, (v+1)') : i \in [2, k-1], v \in [i, k-1]\} \cup \{(i, (k+1)) : i \in [2, k]\}$.

(1) A shortest path visiting the arc $(i', i+1)$ for $i \in [1, k]$ implies that to minimize the left-hand-side of inequality (16), we let $i \notin T_2$ and $(i+1) \in T_2$. The cost on this arc is $\bar{c}_{i', i+1} = \min\{x_i^1, (d_{ik}^1 + d_{il}^2) y_i^1\}$. Note that when $i \notin T_2$, $\phi_i = d_{ik}^1 + d_{il}^2$. Therefore, if $x_i^1 \leq (d_{ik}^1 + d_{il}^2) y_i^1$, then we let $i \notin T_1$, else we let $i \in T_1$.

(2) A shortest path visiting the arc $(i', (i+1)')$ for $i \in [1, k-1]$ implies that to minimize the left-hand-side of inequality (16), we let $i \notin T_2$ and $(i+1) \notin T_2$. The cost on this arc is $\bar{c}_{i',(i+1)'} = \min\{x_i^1, (d_{ik}^1 + d_{il}^2)y_i^1\}$. If $x_i^1 \leq (d_{ik}^1 + d_{il}^2)y_i^1$, then we let $i \notin T_1$, else we let $i \in T_1$.

(3) A shortest path visiting the arc $(i, (v+1)')$ for $i \in [2, k-1]$ and $v \in [i, k-1]$ represents $[i, v] \subseteq T_2$ and $(i-1) \notin T_2$ and $(v+1) \notin T_2$. As a result, $\beta(T_2, j) = [j, v]$ for all $j \in [i, v]$ and the decision on which elements to include in $T_1 \cap [i, v]$ can be made easily as the coefficients ϕ_j depend on $\beta(T_2, j)$. The cost on this arc is $\bar{c}_{i,(v+1)'} = \sum_{t=i}^{v} \min\{x_t^1, (d_{tk}^1 + d_{(v+1),l}^2)y_t^1\} + \sum_{t=i}^{v} \min\{x_t^2, d_{tv}^2 y_t^2\}$. As before, if $x_i^1 \leq (d_{ik}^1 + d_{(v+1),l}^2)y_i^1$, then we let $i \notin T_1$, else we let $i \in T_1$. Similarly, if $x_i^2 \leq d_{iv}^2 y_i^2$, then we let $i \in T_2 \setminus T_3$, else we let $i \in T_3$.

(4) A shortest path visiting the arc $(i, (k+1))$ for $i \in [2, k]$ represents $[i, l] \subseteq T_2$, $(i-1) \notin T_2$, and $(k+1) \in T_2$ if $k < l$. As a result, $\beta(T_2, j) = [j, l]$ for all $j \in [i, k]$. Hence, the cost on this arc is $\bar{c}_{i,(k+1)} = \sum_{t=i}^{k} \min\{x_t^1, d_{tk}^1 y_t^1\} + \sum_{t=i}^{l} \min\{x_t^2, d_{tl}^2 y_t^2\}$. As before, if $x_i^1 \leq d_{ik}^1 y_i^1$, then we let $i \notin T_1$, else we let $i \in T_1$. Similarly, if $x_i^2 \leq d_{il}^2 y_i^2$, then we let $i \in T_2 \setminus T_3$, else we let $i \in T_3$.

Note that there are $O(n)$ nodes and $O(n^2)$ arcs in this network. In addition, G is directed acyclic. Hence, the shortest path problem for a given k and l can be solved in $O(n^2)$ time. Overall, this separation algorithm takes $O(n^4)$ time considering all k, l such that $0 \leq k \leq l \leq n$.

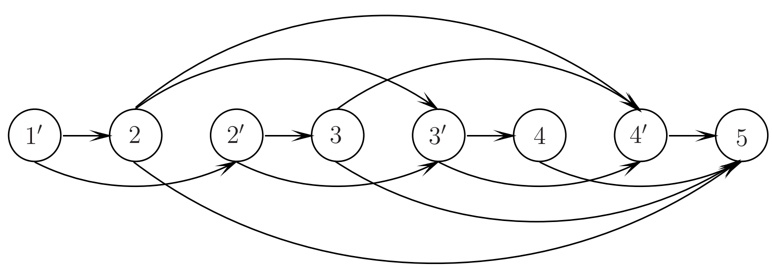

FIGURE 3. Separation network for 2-echelon inequality (16) with $k = 4$

□

4. Alternative Extended Formulations for 2-ULS

A tight and compact extended formulation for 2-ULS can be obtained from the dynamic program given in Section 2. However, the size of this formulation is large and its projection is non-trivial. In this section, we consider alternative extended formulations obtained by adapting those for m-ULS-F from the literature, such as the multi-commodity formulation (Krarup and Bilde, 1977, Rardin and Wolsey, 1993) and the echelon stock formulation (Wolsey, 2002, Belvaux and Wolsey, 2001) (see also Pochet and Wolsey (2006)). We establish a hierarchy of formulations by studying their relative strength.

4.1. **Multi-commodity Formulation.** In this section, we propose a multi-commodity extended formulation similar to that of Pochet and Wolsey (2006) for m-ULS-F. Let z^{11}_{ut} be the order quantity in period u at the first echelon to satisfy the intermediate demand in period t, z^{12}_{ut} be the order quantity in period u at the first echelon to satisfy the demand at the second echelon in period t, and z^{22}_{ut} be the order quantity in period u at the second echelon to satisfy the demand at the second echelon in period t for $1 \leq u \leq t \leq n$. Using these additional variables, we can model 2-ULS as follows:

$$\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f^i_t y^i_t + c^i_t x^i_t)$$

$$\text{s.t.} \sum_{u=1}^{t} z^{11}_{ut} = d^1_t \qquad t \in [1, n], \tag{21}$$

$$\sum_{u=1}^{t} z^{12}_{ut} = d^2_t \qquad t \in [1, n], \tag{22}$$

$$\sum_{u=1}^{t} z^{22}_{ut} = d^2_t \qquad t \in [1, n], \tag{23}$$

$$\sum_{u=1}^{j} z^{12}_{ut} \geq \sum_{u=1}^{j} z^{22}_{ut} \qquad t \in [1, n], j \in [1, t], \tag{24}$$

$$d^1_t y^1_u \geq z^{11}_{ut} \qquad t \in [1, n], u \in [1, t], \tag{25}$$

$$d^2_t y^1_u \geq z^{12}_{ut} \qquad t \in [1, n], u \in [1, t], \tag{26}$$

$$d^2_t y^2_u \geq z^{22}_{ut} \qquad t \in [1, n], u \in [1, t], \tag{27}$$

$$x^1_t = \sum_{v=t}^{n} (z^{11}_{tv} + z^{12}_{tv}) \qquad t \in [1, n], \tag{28}$$

$$x^2_t = \sum_{v=t}^{n} z^{22}_{tv} \qquad t \in [1, n], \tag{29}$$

$$z^{11}_{ut}, z^{12}_{ut}, z^{22}_{ut} \geq 0 \qquad t \in [1, n], u \in [1, t], \tag{30}$$

$$y^i_t \in \{0, 1\} \qquad t \in [1, n], i \in [1, 2]. \tag{31}$$

Here constraints (21)–(24) ensure that the demand is satisfied on time. In particular, constraints (24) enforce that the order quantity at the second echelon until period j to satisfy the second echelon demand in period t cannot be larger than the order quantity at the first echelon until period j to satisfy the second echelon demand in period t. Constraints (25)–(27) ensure that there are no orders in periods with no order setup. Constraints (28) and (29) relate the values of the order variables in the natural formulation with the additional variables in the extended formulation. We refer to the formulation (21)–(31) as the multi-commodity (MC) formulation.

4.1.1. *Comparison of MC formulation with the natural formulation strengthened with 2-echelon inequalities.* Here we prove that the LP relaxation of MC formulation is at least as strong as the natural formulation strengthened with 2-echelon inequalities. It is easy to see that the constraints of the natural formulation (5)–(8), (10)–(13) are implied by MC formulation. Next, we show that the 2-echelon inequalities are implied by MC formulation. To do this, we study the projection of the feasible set of MC formulation onto the space of order and setup variables.

Note that because c^1 and c^2 are non-negative, equality (22) for a given t can be relaxed as $\sum_{u=1}^{t} z_{ut}^{12} \geq d_t^2$, which is implied by equality (23) for that t and inequality (24) for $j = t$. We associate dual variables α_t^1, α_t^2, ρ_{jt}, γ_{ut}^{11}, γ_{ut}^{12}, γ_{ut}^{22}, σ_t^1 and σ_t^2 to constraints (21), (23)–(29), respectively. From Farkas' Lemma, for a given $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2})$ satisfying these constraints, the LP relaxation of MC formulation has a solution if and only if

$$\sum_{t=1}^{n} \sigma_t^1 x_t^1 + \sum_{t=1}^{n} \sigma_t^2 x_t^2 + \sum_{u=1}^{n} \sum_{t=u}^{n} (\gamma_{ut}^{11} d_t^1 + \gamma_{ut}^{12} d_t^2) y_u^1 + \sum_{u=1}^{n} \sum_{t=u}^{n} \gamma_{ut}^{22} d_t^2 y_u^2 \geq \sum_{t=1}^{n} (d_t^1 \alpha_t^1 + d_t^2 \alpha_t^2) \qquad (32)$$

for all $(\sigma^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)$ satisfying

$$\gamma_{ut}^{11} + \sigma_u^1 \geq \alpha_t^1 \qquad 1 \leq u \leq t \leq n, \qquad (33)$$

$$\gamma_{ut}^{12} + \sigma_u^1 \geq \sum_{j=u}^{t} \rho_{jt} \qquad 1 \leq u \leq t \leq n, \qquad (34)$$

$$\gamma_{ut}^{22} + \sigma_u^2 \geq \alpha_t^2 - \sum_{j=u}^{t} \rho_{jt} \qquad 1 \leq u \leq t \leq n. \qquad (35)$$

$$\gamma_{ut}^{11}, \gamma_{ut}^{12}, \gamma_{ut}^{22}, \rho_{ut} \geq 0 \qquad 1 \leq u \leq t \leq n,$$

Proposition 5. *If a projection inequality* (32) *defined by a non-negative extreme ray* (σ^1, σ^2, γ^{11}, γ^{12}, γ^{22}, α^1, α^2, ρ) *of the projection cone with equal positive entries is not dominated, then it has the following form:*

$$\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\in A_1\setminus S_1} \hat{\phi}_u y_u^1 + \sum_{u\in A_2\setminus S_2} \hat{\psi}_u y_u^2 \geq d^1_{1t^1} + d^2_{1t^2}, \tag{36}$$

where $0 \leq t^1 \leq t^2 \leq n$, $A_1 = [1,t^1]$, $A_2 = [1,t^2]$, $S_1 \subseteq A_1$, $S_2 \subseteq A_2$, $j(1) \in [0,1]$, $j(t+1) \in \{j(t), t+1\}$ *for all* $t \in A_2$, $t \leq n-1$, $j(t) \leq t^1$ *for* $t \in A_2$, $\hat{\phi}_u = d^1_{ut^1} + \sum_{t\in A_2:u\leq j(t)} d^2_t$ *for* $u \in A_1 \setminus S_1$ *and* $\hat{\psi}_u = \sum_{t\in A_2:j(t)<u\leq t} d^2_t$ *for* $u \in A_2 \setminus S_2$, *where* $j(t)$ *is the largest index* $j \in [1,t]$ *with positive* ρ_{jt} *(if none exists, then* $j(t) = 0$*).*

Proof. See Appendix C. □

Proposition 6. *If a projection inequality* (32) *defined by a non-negative extreme ray of the projection cone with equal positive entries is not dominated, then it is a 2-echelon inequality* (16)*.*

Proof. Let $0 \leq t^1 \leq t^2 \leq n$, $A_1 = [1,t^1]$, $A_2 = [1,t^2]$, $S_1 \subseteq A_1$, $S_2 \subseteq A_2$, $j(1) \in [0,1]$, $j(t+1) \in \{j(t), t+1\}$ for all $t \in A_2$, $t \leq n-1$, $j(t) \leq t^1$ for $t \in A_2$, $\hat{\phi}_u = d^1_{ut^1} + \sum_{t\in A_2:u\leq j(t)} d^2_t$ for $u \in A_1 \setminus S_1$ and $\hat{\psi}_u = \sum_{t\in A_2:j(t)<u\leq t} d^2_t$ for $u \in A_2 \setminus S_2$.

Define $k = t^1$, $l = t^2$, and $C = \{t \in [1,k] : j(t) \neq t\}$. Let $T_2 = C \cup [k+1,l]$. As $j(t) \leq t^1$ for $t \in A_2$, $T_2 = \{t \in A_2 : j(t) \neq t\}$. Let $T_1 = A_1 \setminus S_1$ and $T_3 \subseteq A_2 \setminus S_2$.

Let $u \in A_2 \setminus S_2$. If $u \notin T_2$, then $\psi_u = 0 = \hat{\psi}_u$ and we let $u \in T_3$. If $u \in T_2$, then $j(u) < u$. Now $\psi_u = \sum_{t\in\beta(T_2,u)} d^2_t = \sum_{t\in A_2:u\leq t, j(t)=j(u)} d^2_t = \sum_{t\in A_2:j(t)<u\leq t} d^2_t = \hat{\psi}_u$ and we let $u \in T_3$.

Let $u \in T_1 = A_1 \setminus S_1$. Then $\phi_u = d^1_{uk} + d^2_{ul} - \sum_{t\in\beta(T_2,u)} d^2_t$. If $u \notin T_2$, then $j(u) = u$ and for all $t \in A_2$ with $t \geq u$, we have $j(t) \geq j(u)$. Hence $\sum_{t\in A_2:u\leq j(t)} d^2_t = d^2_{ul}$ and $\phi_u = d^1_{uk} + d^2_{ul} = \hat{\phi}_u$. If $u \in T_2$, then $j(u) \neq u$. Let u' be the smallest index greater than u with $j(u') = u'$. We have $\sum_{t\in A_2:u\leq j(t)} d^2_t = d^2_{u',l}$. This is the same as $d^2_{ul} - d^2_{u,u'-1} = \sum_{t\in\beta(T_2,u)} d^2_t$. Hence $\phi_u = \hat{\phi}_u$.

The resulting 2-echelon inequality is $\sum_{u\in S_1} x^1_u + \sum_{u\in S_2:j(u)\neq u} x^2_u + \sum_{u\in T_1} \hat{\phi}_u y^1_u + \sum_{u\in T_3} \hat{\psi}_u y^2_u \geq d^1_{1t^1} + d^2_{1t^2}$ and dominates the projection inequality if there exists $u \in S_2$ with $j(u) = u$.

□

Proposition 7. *Inequalities* (16) *can be obtained by projecting the MC formulation onto the* $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2})$ *space.*

Proof. Consider the 2-echelon inequality (16) defined by $0 \leq k \leq l \leq n$, $T_1 \subseteq [1,k]$, $[k+1,l] \subseteq T_2 \subseteq [1,l]$, $C = T_2 \cap [1,k]$, and $T_3 \subseteq T_2$. Let $T_2 = \cup_{s=1}^r T_2^s$ where T_2^s is a maximal consecutive component, i.e., $T_2^s = [a(s), b(s)] \subseteq T_2$ with $a(s)-1 \notin T_2$ and $b(s)+1 \notin T_2$ for each $s = 1,\ldots,r$ and r is the number of maximal consecutive components comprising T_2.

Now define $t^1 = k$, $t^2 = l$, $A_1 = [1,k]$, $A_2 = [1,l]$, $S_1 = [1,k] \setminus T_1$, $S_2 = T_2 \setminus T_3$ and $j(t) = t$ for $t \in [1,k] \setminus C$ and $j(t) = a(s) - 1$ if $t \in T_2^s$ for $s = 1,\ldots,r$.

For $u \in A_1 \setminus S_1$, $\hat{\phi}_u = d^1_{ut^1} + \sum_{t \in A_2 : u \leq j(t)} d^2_t = d^1_{uk} + \sum_{t \in [1,k]\setminus C : u \leq t} d^2_t + \sum_{s=1}^r \sum_{t \in T_2^s : u \leq a(s)-1} d^2_t$. If $u \notin T_2$, then $\sum_{t \in [1,k]\setminus C : u \leq t} d^2_t + \sum_{s=1}^r \sum_{t \in T_2^s : u \leq a(s)-1} d^2_t = d^2_{ul}$. If $u \in T_2$, let $\bar{s}$ be the interval that u falls into, i.e., $u \in T_2^{\bar{s}}$. Then $\sum_{t \in [1,k]\setminus C : u \leq t} d^2_t + \sum_{s=1}^r \sum_{t \in T_2^s : u \leq a(s)-1} d^2_t = d^2_{b(\bar{s})+1,l}$. In both cases, $\hat{\phi}_u = \phi_u$.

Let $u \in A_2 \setminus S_2$. Then $\hat{\psi}_u = \sum_{t \in A_2 : j(t) < u \leq t} d^2_t = \sum_{t \in [1,k]\setminus C : j(t) < u \leq t} d^2_t + \sum_{s=1}^r \sum_{t \in T_2^s : j(t) < u \leq t} d^2_t = \sum_{t \in [1,k]\setminus C : t < u \leq t} d^2_t + \sum_{s=1}^r \sum_{t \in T_2^s : a(s)-1 < u \leq t} d^2_t$. Observe that $\sum_{t \in [1,k]\setminus C : t < u \leq t} d^2_t = 0$. If $u \notin T_2$, then $\sum_{s=1}^r \sum_{t \in T_2^s : a(s)-1 < u \leq t} d^2_t = 0$. If $u \in T_2$, then $\sum_{s=1}^r \sum_{t \in T_2^s : a(s)-1 < u \leq t} d^2_t = d^2_{b(\bar{s})+1,l}$ where $\bar{s}$ is the interval that u falls into. Hence, $\hat{\psi}_u = 0$ if $u \notin T_2$ and $\hat{\psi}_u = \psi_u$ if $u \in T_2$.

As a result, the projection inequality for these choices is the same as the 2-echelon inequality (16). □

Using the Propositions (6) and (7), we have the following theorem.

Theorem 8. *The formulation obtained by adding the projection inequalities* (32) *corresponding to the non-negative extreme rays with equal positive entries has the same strength as the formulation obtained by adding all 2-echelon inequalities* (16).

Rardin and Wolsey (1993) give a class of dicut collection inequalities for single-source uncapacitated fixed-charge networks, which are obtained by projecting the multi-commodity extended formulation to the original space. Dicut collection inequalities are written implicitly as a function of a collection of dicuts in a graph. Therefore, there are no known explicit conditions for dicut collection inequalities to be facet-defining, and as a result, many of these inequalities are dominated. In addition, there are no known combinatorial separation algorithms for these inequalities.

Corollary 9. *2-echelon inequalities are special cases of dicut collection inequalities.*

Proof. This follows from Theorem 8. Here we give the dicut collection that corresponds to the 2-echelon inequalities. For $t \in [1,n]$ and $i \in [1,2]$, Γ^i_t is a collection of variables such that removing the arcs corresponding to these variables will disconnect the flows from source node to nodes (t,i) in the single-source network depicted in Figure 1. To yield the 2-echelon inequality (T_1,T_2,T_3,k,l), the required dicut collection $\Gamma=\{\Gamma_t^1\}_{t\in[1,n]}\cup\{\Gamma_t^2\}_{t\in[1,n]}$ has each Γ_t^j as a singleton $\{Q_t^j\}$ for $t\in[1,n]$ and $j\in[1,2]$. We define $\beta^{-1}(T,\cdot)$ as the inverse function of $\beta(T,\cdot)$, i.e., $t\in\beta(T,i)$ if and only if $i\in\beta^{-1}(T,t)$. Then the dicut collection that gives the 2-echelon inequality is:

- For $t\in[1,k]$, $\Gamma_t^1=\{Q_t^1\}=\{x_i^1: i\in[1,t]\setminus T_1\}\cup\{y_i^1: i\in[1,t]\cap T_1\}$.
- For $t\in[1,l]$, $\Gamma_t^2=\{Q_t^2\}=\{x_i^1: i\in[1,t]\setminus T_1\}\cup\{x_i^2: i\in[1,t]\cap(T_2\setminus T_3)\}\cup\{y_i^1: i\notin\beta^{-1}(T_2,t), i\in[1,t]\cap T_1\}\cup\{y_i^2: i\in\beta^{-1}(T_2,t)\cap T_3\}$.
- For $t\in[k+1,n]$, $\Gamma_t^1=\emptyset$.
- For $t\in[l+1,n]$, $\Gamma_t^2=\emptyset$.

We refer the reader to Rardin and Wolsey (1993) for further details on the dicut collection inequalities.

□

Nevertheless, as 2-echelon inequalities are in closed-form, we are able to show that they are facet-defining under certain conditions (Proposition 3) and give a combinatorial separation algorithm for them (Proposition 4).

Example 1 (continued). Based on our experiments with PORTA (Christof and Löbel, 2008), the LP relaxation of MC formulation is not tight for 2-ULS with more than 3 periods. Consider the four-period 2-ULS problem with $d^1=d^2=(1,1,1,1)$. As stated before, 65 out of 81 facets are defined by 2-echelon inequalities. Besides these 65 facets, 3 out of the 16 remaining facets are defined by the projection of MC formulation. For example, $x_1^1+x_2^1+2y_3^1-x_2^2-2y_2^2\geq 6$ is a projection inequality, but it is clearly not a 2-echelon inequality because of the negative coefficients of x_2^2 and y_2^2. Thus, the MC formulation is strictly contained in the natural formulation with 2-echelon inequalities.

Let $h^1=h^2=(0,0,0,0)$, $f^1=(0,2,2,2)$, $f^2=(0,2,0,0)$, $c^1=(8,7,6,5)$, $c^2=(0,0,2,2)$. The solution to the linear relaxation of the MC formulation is $x^1=(3,2.5,1.5,1)$, $x^2=(1.5,1.5,0.5,0.5)$, $y^1=(1,0.5,0.5,0.5)$, $y^2=(1,0.5,1,1)$. Because binary variables y^1 and y^2 are fractional at the optimal solution, the MC formulation is not tight in this example. So we conclude that the exact DP-based formulation is stronger than the MC formulation.

4.2. **Echelon Stock Reformulation.** Pochet and Wolsey (2006) derive an alternative formulation for m-ULS-F using the so-called "echelon stock variables". Here we adapt this formulation to our problem. The first echelon stock variable $e_t^1 = s_t^1 + s_t^2$ is the total inventory at the first echelon at the end of period t and the second echelon stock variable $e_t^2 = s_t^2$ is the total inventory at the second echelon at the end of period t. Using these variables, we obtain the following model.

$$
\begin{aligned}
\min \sum_{i=1}^{2} \sum_{t=1}^{n} & (f_t^i y_t^i + c_t^i x_t^i) \\
\text{s.t. } & (5) - (8), \\
& e_{t-1}^1 + x_t^1 = d_t^1 + d_t^2 + e_t^1 && t \in [1,n], \\
& e_{t-1}^2 + x_t^2 = d_t^2 + e_t^2 && t \in [1,n], \\
& e_0^i = e_n^i = 0 && i \in [1,2], \\
& e_t^1 \geq e_t^2 && t \in [1,n], \\
& e_t^i \geq 0 && t \in [1,n], i \in [1,2].
\end{aligned}
$$

4.2.1. *Comparison of the natural formulation strengthened with 2-echelon inequalities and the echelon stock reformulation with (ℓ, S) inequalities.* The echelon stock reformulation has the same linear programming relaxation bound as the natural formulation. However, if we consider the variables and the constraints associated with a given echelon, then we have the same structure as that of ULS. Now, we can generate (ℓ, S)-inequalities for each echelon. Let $\ell \in [1,n]$, $L = [1,\ell]$ and $S \subseteq L$. The first echelon (ℓ, S)-inequality is

$$\sum_{j \in S} x_j^1 \leq \sum_{j \in S} (d_{j\ell}^1 + d_{j\ell}^2) y_j^1 + e_\ell^1$$

which is the same as

$$d_{1\ell}^1 + d_{1\ell}^2 \leq \sum_{j \in S} (d_{j\ell}^1 + d_{j\ell}^2) y_j^1 + \sum_{j \in L \setminus S} x_j^1 \tag{37}$$

after substituting $e_\ell^1 = \sum_{j=1}^{\ell} x_j^1 - d_{1\ell}^1 - d_{1\ell}^2$. Similarly, the second echelon (ℓ, S)-inequality is

$$d_{1\ell}^2 \leq \sum_{j \in S} d_{j\ell}^2 y_j^2 + \sum_{j \in L \setminus S} x_j^2. \tag{38}$$

We refer to inequalities (37) and (38) as echelon stock inequalities.

Proposition 10. *The natural formulation with 2-echelon inequalities is stronger than the echelon stock reformulation with echelon stock inequalities.*

Proof. Let $\ell \in [1, n]$, $L = [1, \ell]$ and $S \subseteq L$. If we let $k = l = \ell$, $T_1 = S$, $T_2 = T_3 = \emptyset$, then the 2-echelon inequality (16) simplifies to

$$\sum_{j \in L \setminus S} x_j^1 + \sum_{j \in S} (d_{j\ell}^1 + d_{j\ell}^2) y_j^1 \geq d_{1\ell}^1 + d_{1\ell}^2,$$

which is the same as the echelon stock inequality (37).

Also, if we let $k = 0$, $l = \ell$, $T_1 = \emptyset$, $T_2 = [1, l]$, $T_3 = S$, inequality (16) is the same as inequality (38). Thus, the natural formulation with 2-echelon inequalities is stronger than the echelon stock reformulation with the echelon stock inequalities. □

4.3. **Hierarchy of formulations.** A *formulation* of a mixed-integer program is formally defined as the polyhedron given by the linear programming relaxation of its constraints (Definition 1.2 of Wolsey (1998)). From Sections 2, 3, 4.1 and 4.2, we establish a hierarchy of formulations for 2-ULS, in its natural space, from stronger to weaker as: projection of the DP-based exact extended formulation; projection of the MC formulation; natural formulation with 2-echelon inequalities (16); echelon stock formulation with echelon stock inequalities; natural formulation. Also, the inclusion in each case is strict. For example, we know that not all projection inequalities of MC formulation are 2-echelon inequalities (16).

5. Computations

In this section, we report our computational experiments with a class of multi-item multi-echelon lot-sizing problems with mode constraints. In these problems, we have n time periods, m echelons, and r items. The mode constraints allow at most κ orders to be placed in each period and each echelon. Let M_{at}^i be the order capacity of item a at echelon i in period t, $1 \leq i \leq m$, $1 \leq a \leq r$ and $1 \leq t \leq n$. Let $\hat{d}_{at}^i$ be the demand of item a in period t at echelon i, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$. Define $\hat{d}_{aut}^i := \sum_{j=u}^{t} \hat{d}_{aj}^i$.

Let x_{at}^i denote the total order quantity of item a in period t at echelon i, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$. The mixed-integer programming formulation of capacitated multi-item lot-sizing problem with mode constraint is as follows:

$$\begin{aligned}
\min \quad & \sum_{a=1}^{r}\sum_{i=1}^{m}\sum_{t=1}^{n}(f_{at}^{i}y_{at}^{i}+c_{at}^{i}x_{at}^{i}) & \\
\text{s.t.} \quad & \sum_{t=1}^{n} x_{at}^{i} = \sum_{j=i}^{m} \hat{d}_{a1n}^{j} & 1 \le i \le m, 1 \le a \le r, \\
& \sum_{j=1}^{t} x_{aj}^{i} \ge \sum_{j=1}^{t} x_{aj}^{i+1} + \hat{d}_{a1t}^{i} & 1 \le i \le m-1, 1 \le a \le r, 1 \le t \le n, \\
& \sum_{j=1}^{t} x_{aj}^{m} \ge \hat{d}_{a1t}^{m} & 1 \le t \le n, 1 \le a \le r, \\
& x_{at}^{i} \le M_{at}^{i} y_{at}^{i} & 1 \le i \le m, 1 \le a \le r, 1 \le t \le n, \\
& \sum_{a=1}^{r} y_{at}^{i} \le \kappa & 1 \le t \le n, 1 \le i \le m, \\
& x_{at}^{i} \ge 0 & 1 \le i \le m, 1 \le t \le n, 1 \le a \le r, \\
& y_{at}^{i} \in \{0,1\} & 1 \le i \le m, 1 \le t \le n, 1 \le a \le r.
\end{aligned}$$

Let z_{aut}^{ij} denote the order quantity of item a in period u at echelon i to satisfy the demand in period t at echelon j, $1 \le i \le j \le m$, $1 \le u \le t \le n$, $1 \le a \le r$. The multi-commodity formulation of capacitated multi-item lot-sizing problem with mode constraint is as follows:

$$\begin{aligned}
\min \quad & \sum_{a=1}^{r}\sum_{i=1}^{m}\sum_{t=1}^{n}(f_{at}^{i}y_{at}^{i}+c_{at}^{i}x_{at}^{i}) & \\
\text{s.t} \quad & \sum_{u=1}^{t} z_{aut}^{ij} = \hat{d}_{at}^{j} & 1 \le i \le j \le m, 1 \le a \le r, 1 \le t \le n, \\
& \sum_{u=1}^{k} z_{aut}^{ij} \ge \sum_{u=1}^{k} z_{aut}^{(i+1)j} & 1 \le i < j \le m, 1 \le a \le r, 1 \le k \le t \le n, \\
& x_{au}^{i} = \sum_{j=i}^{m}\sum_{t=u}^{n} z_{aut}^{ij} & 1 \le i \le m, 1 \le a \le r, 1 \le u \le n, \\
& z_{aut}^{ij} \le \hat{d}_{at}^{j} y_{au}^{i} & 1 \le i \le j \le m, 1 \le a \le r, 1 \le u \le t \le n, \\
& x_{at}^{i} \le M_{at}^{i} y_{at}^{i} & 1 \le i \le m, 1 \le a \le r, 1 \le t \le n, \\
& \sum_{a=1}^{r} y_{at}^{i} \le \kappa & 1 \le i \le m, 1 \le t \le n, \\
& z_{aut}^{i} \ge 0 & 1 \le i \le m, 1 \le a \le r, 1 \le u \le t \le n, \\
& x_{at}^{i} \ge 0 & 1 \le i \le m, 1 \le a \le r, 1 \le t \le n, \\
& y_{at}^{i} \in \{0,1\} & 1 \le i \le m, 1 \le a \le r, 1 \le t \le n.
\end{aligned}$$

We conduct all the experiments on a 1 GHz Dual-Core AMD Opteron(tm) Processor 1218 with 2GB RAM. We use IBM ILOG CPLEX 12.0 as the MIP solver.

5.1. **Strength of alternative formulations for uncapacitated multi-item two-echelon instances.** In this subsection, we investigate the strength of alternative formulations and cuts. We limit ourselves to *uncapacitated* instances with 30 periods and two echelons, where $M_{at}^{i} = \sum_{j=i}^{m} \hat{d}_{atn}^{j}$ for $1 \le i \le m$, $1 \le t \le n$, $1 \le a \le r$. The variable costs of the first and second echelons are generated using a discrete uniform distribution in the interval [0, 50] and [0, 100], respectively. Unit inventory costs of the both echelons are generated using a discrete uniform distribution in the interval [0, 6]. Let δ be the ratio of fixed and unit order costs. For various values of r, κ, and δ, we generate five instances and report the averages in Table 1.

For each formulation, we report the average percentage duality gap (rounded to two significant digits) and the average number of cuts added (if applicable). First, we solve the LP relaxations of the natural and multi-commodity formulations, which we refer to as NF and MCF, respectively.

The gap reported for NF and MCF is calculated as $100 \times (\texttt{zub} - \texttt{zlb})/\texttt{zub}$, where zub is objective function value of the optimal solution and zlb is the optimal value of the initial LP relaxation. The MCF is very strong and has zero gap for all the instances considered, whereas the initial gap of NF can be as high as 25%. Next, we solve NF by letting CPLEX generate its cuts and report the root gap and the average number of cuts generated before branching. The root gap is calculated similarly by letting zlb be the optimal value of the LP relaxation strengthened by cutting planes. We refer to the natural formulation with CPLEX cuts as CPX. We observe that CPLEX can close a big portion of the gap. Finally, using cutting plane algorithms, we solve the LP relaxations of the natural formulation strengthened with the 2-echelon inequalities (referred to as 2ULS) and the echelon stock formulation with echelon stock inequalities (referred to as ES). We can see that the echelon stock inequalities reduce the duality gap significantly but the remaining gaps are slightly higher than those with CPLEX cuts. The 2-echelon inequalities, however, close almost all the gap, with the average gap being below 0.5%. This comparison shows that using 2-echelon inequalities, we obtain a formulation that is almost as strong as the multi-commodity formulation and significantly stronger than the formulation obtained by adding only the echelon stock inequalities. Because our goal in this experiment is to test the strength of 2ULS empirically, we do not report the solution times. The exact separation of the 2-echelon inequalities can be quite time consuming in practice due to its $O(n^4)$ time complexity. In the next subsection, we employ a heuristic separation to make 2ULS practicable.

Table 1. Gaps for different formulations and valid inequalities for uncapacitated 2-echelon multi-item lot-sizing problems

$n.m.r.\kappa.\delta$	NF	CPX		2ULS		ES		MCF
	Gap	Gap	Cuts	Gap	Cuts	Gap	Cuts	Gap
30.2.5.2.500	25.40%	3.66%	111.8	0.42%	5990.2	4.41%	1637.0	0%
30.2.5.3.500	27.52%	4.31%	115.6	0.62%	5498.8	4.34%	1408.4	0%
30.2.10.3.500	25.26%	4.63%	208.8	0.42%	13367.8	4.96%	4188.2	0%
30.2.10.5.500	25.61%	2.94%	223.6	0.31%	11563.4	4.69%	2894.6	0%
30.2.5.2.1000	18.71%	4.71%	62.6	0.16%	3608.4	5.30%	1279.2	0%
30.2.5.3.1000	22.21%	4.22%	75.2	0.33%	2868.8	5.84%	941.8	0%
30.2.10.3.1000	17.93%	5.39%	127.4	0.11%	7810.8	5.34%	3631.6	0%
30.2.10.5.1000	18.80%	3.83%	127.8	0%	6497.8	5.55%	2246.4	0%
30.2.5.2.2500	4.46%	0.48%	34.0	0%	1740.8	0.23%	685.8	0%
30.2.5.3.2500	7.08%	0.07%	37.8	0%	1213.8	0.45%	475.6	0%
30.2.10.3.2500	3.90%	1.38%	75.8	0%	4910.2	0.27%	2195.0	0%
30.2.10.5.2500	4.50%	0.03%	67.8	0%	3791.6	0.02%	1328.4	0%

In our computational experience, MCF is highly effective in solving uncapacitated multi-item lot-sizing instances also for more echelons with $2 \leq m \leq 5$. However, in the next subsection, we show that for capacitated instances a branch-and-cut algorithm using our proposed inequalities is more effective than the MCF formulation.

5.2. **Effectiveness of 2-echelon inequalities for capacitated multi-item multi-echelon instances.** In this subsection, we test the multi-commodity formulation and three alternative branch-and-cut methods on *capacitated* multi-item multi-echelon lot-sizing problem with mode constraints:

(1) Algorithm 1: multi-commodity formulation with all CPLEX cuts (denoted by MCF),
(2) Algorithm 2: echelon stock formulation with echelon stock inequalities (37)–(38) and all CPLEX cuts (denoted by ES),
(3) Algorithm 3: natural formulation with a subset of 2-echelon inequalities and all CPLEX cuts (denoted by 2ULS),
(4) Algorithm 4: natural formulation with all CPLEX cuts (denoted by CPX).

Note that echelon stock inequalities are special cases of 2-echelon inequalities. We impose an hour time limit for all algorithms.

In 2ULS, we generate a subset of the violated 2-echelon inequalities at the root node only. We add all violated echelon stock inequalities for a single echelon obtained by aggregating the echelons $[m_1, m]$ for $m_1 \in [1, m]$. To apply the 2-echelon inequalities in the multi-echelon setting, we aggregate echelons $[m_1, m_2]$ and treat as echelon one, and we aggregate echelons $[m_2+1, m_3]$ and treat as echelon 2, for certain choices of m_1, m_2, m_3, where $1 \leq m_1 \leq m_2 < m_3 \leq m$. In particular, we only consider the facet-defining 2-echelon inequalities for the following cases:

(a) echelons $[m_1, m-1]$ aggregated as echelon 1 and $[m, m]$ aggregated as echelon 2 (i.e., $m_2 = m-1, m_3 = m$) for all k, l with $2 \leq k < l = n$,
(b) echelon m_1 used as echelon 1 and $[m_1+1, m]$ aggregated as echelon 2 (i.e., $m_2 = m_1, m_3 = m$) for all k, l with $k = l = n$.

We add all the cuts aggressively and we force CPLEX to start branching if the improvement of lower bound at the root node is less than 0.01% after adding all cuts generated in one iteration.

In our experimental setup, the demands, fixed costs, variable costs and holding cost of each item in each echelon and each period are generated using a discrete uniform distribution in the intervals [0,50], [1000,2000], [0, 20] and [0, 6], respectively. The capacity M^i_{at} is set to be $3\lceil \frac{\hat{d}^i_{a1n}}{n} \rceil$ for $i \in [1,m]$, $a \in [1,r]$ and $t \in [1,n]$.

We report our results in Table 2 for various settings $n.m.r.\kappa$. For each setting, we generate five instances and report the averages. In column **RGap(noint)**, we report the average percentage integrality gap at the root node just before branching, which is $100 \times (\texttt{zub} - \texttt{zrb})/\texttt{zub}$, where `zub` is objective function value of the best integer solution obtained within time limit and `zrb` is the best lower bound obtained at the root node. The number of instances without integer solutions obtained within time limit is given in parentheses in cases where not all five instances are solved with integer solutions. In column **GClos(noint)**, we report the average percentage closure of the integrality gap at the root node before branching, which is $100 \times (\texttt{zrb} - \texttt{zlb})/(\texttt{zub} - \texttt{zlb})$, and in parantheses, we give the number of instances with no feasible integer solutions obtained within time limit. In columns **EGap(noint)**, we report the average percentage end gap at termination output by CPLEX, which is $100 \times (\texttt{zub} - \texttt{zbest})/\texttt{zub}$, where `zbest` is the best lower bound available within time limit, and the number of instances without integer solutions obtained within the time limit in parentheses. Columns **Time(unslvd)** report the average solution time in seconds and the number of unsolved instances in parentheses in cases where not all five instances are solved to optimality within time limit. Columns **Nodes(nobr)** report the average number of branch-and-cut tree nodes explored and the number of instances without branching in parentheses in cases where not all five instances start branching. In columns **Cuts**, we report the average number of CPLEX cuts and user inequalities (echelon stock inequalities for ES and 2-echelon inequalities for 2-ULS) added separately.

The branch-and-cut method with the MC formulation was not able to obtain any integer feasible solutions for any of the five instances from 30.5.5.3 setting within an hour. Therefore, the gap closure and the end gap for the MC formulation is not calculated. Also, for all five instances from 20.5.5.3 and 30.5.3.2 settings, the MC formulation was not able to start branching, although it was able to solve the initial LP relaxation, add CPLEX default cuts at the root node and even obtained integer feasible solutions in all but one instance of the 30.5.3.2 setting. These experiments demonstrate that the MC formulation may not scale up for capacitated problems as the number of echelons, items or periods increase. Overall, 2-echelon inequalities are the most effective method in obtaining optimal solutions in shortest time, or solutions with the smallest end gaps within an hour.

Table 2. Comparison of MCF and alternative branch-and-cut methods for capacitated multi-item multi-echelon lot-sizing problems

			GClos	Time	Nodes	Cuts		EGap
$n.m.r.\kappa$	Alg.	RGap	(noint)	(unslvd)	(nobr)	CPLEX	User	(noint)
20.2.5.3	MCF	1.19%	47.11%	≥3600	36344.4	11830.2	0	0.64%
	ES	0.37%	91.85%	63.75	8098.4	898.0	384.6	0
	2ULS	0.37%	92.01%	46.80	4155.6	1148.8	201.2	0
	CPX	0.44%	90.38%	49.29	4994.4	1295.2	0	0
20.3.3.2	MCF	1.18%	45.80%	530.52(4)	15270.2	4678.4	0	0.76%
	ES	0.64%	86.20%	111.26	10007.0	1004.4	128.6	0
	2ULS	0.60%	87.19%	102.37	7496.6	1172.2	167.8	0
	CPX	0.71%	84.72%	146.92	11106.4	1097.6	0	0
20.5.3.2	MCF	1.72%	29.78%	≥3600	156.4	85.2	0	1.68%
	ES	0.85%	82.13%	3243.24(4)	56425.6	1970.8	173.6	0.10%
	2ULS	0.73%	84.43%	2205.35(3)	38654.6	1896.2	248.8	0.06%
	CPX	0.95%	80.00%	≥3600	67661.0	2145.8	0	0.18%
20.5.5.3	MCF	3.78%	10.78%	≥3600	-(5)	68.4	0	3.78%
	ES	1.16%	78.07%	≥3600	23126.4	2949.2	345.8	0.74%
	2ULS	1.14%	78.79%	≥3600	28193.2	2886.8	488.2	0.74%
	CPX	1.43%	73.71%	≥3600	30513.6	3389.8	0	1.02%
20.3.10.5	MCF	4.21%(1)	14.04%(1)	≥3600	2234(4)	227.8	0	3.35%(1)
	ES	0.69%	85.06%	≥3600	24695.4	3432.8	437.2	0.43%
	2ULS	0.61%	86.68%	≥3600	24014.0	3308.4	561.6	0.37%
	CPX	0.79%	82.95%	≥3600	25431.4	3881.2	0	0.54%
30.2.5.3	MCF	1.43%	29.58%	≥3600	27301.0	5529.8	0	1.21%
	ES	0.61%	81.65%	562.37(4)	126946.4	1769.2	240.2	0.14%
	2ULS	0.54%	83.61%	468.12(4)	127468.4	1754.6	267.4	0.12%
	CPX	0.68%	79.29%	903.38(4)	155664.6	2027.2	0	0.17%
30.3.3.2	MCF	1.77%	19.78%	≥3600	11853.2	2100.2	0	1.53%
	ES	0.86%	76.61%	1925.4(3)	86462.4	1439.2	191.0	0.30%
	2ULS	0.82%	77.64%	1353.1(3)	72181.2	1692.6	224.4	0.30%
	CPX	0.96%	73.94%	1795.25(4)	122084.8	1923.4	0	0.35%
30.5.3.2	MCF	2.96%(1)	9.01%(1)	≥3600	-(5)	56.8	0	2.96%(1)
	ES	1.21%	69.11%	≥3600	31201.0	2945.8	249.2	0.88%
	2ULS	1.12%	71.04%	≥3600	21739.0	2866.8	328.2	0.84%
	CPX	1.31%	66.25%	≥3600	25162.6	4377.8	0	0.95%
30.5.5.3	MCF	-(5)	-(5)	≥3600	87.0	131.4	0	-(5)
	ES	1.73%(2)	60.21%(2)	≥3600	19904.6	4372.4	432.4	1.54%(2)
	2ULS	1.19%(2)	69.03%(2)	≥3600	17803.8	4011.8	566.0	1.04%(2)
	CPX	2.45%	51.38%	≥3600	30277.2	4418.2	0	2.27%
30.3.10.5	MCF	3.76%(2)	6.15%(2)	≥3600	-(5)	148.0	0	3.76%(2)
	ES	1.77%(2)	58.98%(2)	≥3600	16600.2	4703.0	634.4	1.63%(2)
	2ULS	1.93%(2)	57.06%(2)	≥3600	11799.4	4362.8	735.8	1.83%(2)
	CPX	2.87%(3)	46.50%(3)	≥3600	20846.6	5052.4	0	2.75%(3)

6. Conclusions

In this paper, we studied an m-echelon lot-sizing problem with intermediate demands (m-ULS). We gave a polynomial-time dynamic program, which implies a tight and compact extended formulation to solve 2-ULS. In addition, we presented a class of valid inequalities for m-ULS, which are separable in polynomial time. Our computational experience with these inequalities demonstrate the effectiveness of these inequalities for multi-item multi-echelon instances. We conjecture that these inequalities are enough to give the convex hull of solutions to 2-ULS for $n = 3$. However, they are not enough to give the convex hull for $n > 3$. In addition, we compared the theoretical strength of alternative formulations such as the multi-commodity and echelon stock reformulations, and established a hierarchy between them. Finally, we presented our computational experiments with the multi-commodity formulation and our valid inequalities. The multi-commodity formulation performs extremely well for uncapacitated problems and the branch and cut algorithm outperforms the multi-commodity formulation when capacity constraints are introduced.

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Appendix A. Dimension of conv($\mathcal{S}$)

Let $\eta_j^i \in \mathbb{B}^{4n}$ and $e_j^i \in \mathbb{B}^{4n}$, $j \in [1,n]$, $i \in \{1,2\}$, be the unit vectors corresponding to the variables x_j^i and y_j^i. The component of η_j^i, which has the same position with x_j^i in the feasible solution, is 1; all other components of η_j^i are 0. The component of e_j^i, which has the same position with y_j^i in the feasible solution, is 1; all other components of e_j^i are 0.

Proposition 11. *The dimension of conv($\mathcal{S}$) is $4n-4$ if $\mathbf{d^1} > \mathbf{0}$ and $\mathbf{d^2} > \mathbf{0}$.*

Proof. Since there are $4n$ variables and 4 linearly independent equalities (10), (11), $y_1^1 = 1$, $y_1^2 = 1$, the dimension of conv($\mathcal{S}$) is at most $4n-4$. Then, consider the following $4n-3$ points: $\hat{u}_0 = (d_{1n}^1 + d_{1n}^2)\eta_1^1 + e_1^1 + d_{1n}^2\eta_1^2 + e_1^2$, and for $j \in [2,n]$, $\hat{u}_j^1 = \hat{u}_0 + e_j^1$, $\hat{u}_j^2 = \hat{u}_0 + e_j^2$, $\tilde{u}_j^1 = \hat{u}_j^1 - \epsilon\eta_1^1 + \epsilon\eta_j^1$, $\tilde{u}_j^2 = \hat{u}_j^2 - \epsilon\eta_1^2 + \epsilon\eta_j^2$, where $0 < \epsilon < \min\{d_j^i : j \in [1,k], i \in \{1,2\}\}$. It is easy to see that these $4n-3$ points are affinely independent and the dimension of conv($\mathcal{S}$) is at least $4n-4$. Hence, the dimension of conv($\mathcal{S}$) is $4n-4$.

□

Appendix B. Proof of Proposition 3

Proposition 3. *For $\mathbf{d^1} > \mathbf{0}$ and $\mathbf{d^2} > \mathbf{0}$, inequality* (16) *is facet-defining for conv($\mathcal{S}$) if and only if*

(1) $1 \notin T_1$;

(2) $1 \notin T_2$ *if* $k \neq 0$;

(3) $1 \notin T_3$ *if* $k = 0$;

(4) $k \neq 1$;

(5) *if* $k = 0$, $l = n$, *then* $|T_3| = 1$;

(6) *for every* $j \in T_2 \cap [2, k]$, *there exists* $i \in T_1$ *such that* $j \in \beta(T_2, i)$;

(7) *if* $2 \leq k \leq l = n$ *with* $T_3 \neq \emptyset$, *then* $T_3 \cap [k+1, n] = \emptyset$ *and for each* $j \in T_3 \cap [2, k]$, *there exists* $j^* \in [j+1, k]$ *such that* $j^* \notin T_2$;

(8) *if* $2 \leq k \leq l < n$, *then there exists* $j \in [p^1, k]$ *such that* $j \notin T_2$;

(9) *if* $k = l = n$, *then* $T_1 \neq \emptyset$ *and either* $T_2 = \emptyset$ *with* $|T_1| = 1$, *or* $T_2 \neq \emptyset$ *is a consecutive set with* $p^2 = p^1$ *and* $[p^1, w^1] \subseteq T_2 = [p^1, w^2] \subseteq [p^1, n]$;

(10) *if* $k \neq 0$, *then* $T_1 \neq \emptyset$; *if* $k = 0$, *then* $T_3 \neq \emptyset$.

Proof. **Necessity.** For simplicity, we denote the 2-echelon inequality (16) with the particular choice of T_1, T_2, T_3, k, l, by (T_1, T_2, T_3, k, l). Note that $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2}) \geq \mathbf{0}$.

(1) Suppose that $1 \in T_1$. Since $y_1^1 = 1$, $x_j^i \geq 0$ and $y_j^i \geq 0$ for $j \in [1, n]$, $i \in \{1, 2\}$, then the 2-echelon inequality (T_1, T_2, T_3, k, l) is dominated by the inequality $y_1^1 \geq 1$ and 2-echelon inequality ($\emptyset$, $\beta(T_2, 1)$, $\beta(T_2, 1) \cap T_3$, 0, $\max\{j : j \in \beta(T_2, 1)\}$).

(2) Suppose that $1 \notin T_1$ and $1 \in T_2$ with $k \neq 0$. Since $x_1^2 > 0$ and $y_1^2 = 1$, the 2-echelon inequality (T_1, T_2, T_3, k, l) is dominated by the 2-echelon inequality (T_1, $T_2 \setminus \{1\}$, $T_3 \setminus \{1\}$, k, l).

(3) Note that if $k = 0$, then $T_1 = \emptyset$ and $T_2 = [1, l]$. Suppose $1 \in T_3$. Then the 2-echelon inequality ($\emptyset$, T_2, T_3, 0, l) is dominated by the inequality $y_1^2 \geq 1$.

(4) By facet conditions (1)-(2) and the fact that $x_1^1 \geq d_1^1$, if $k = 1$, then the 2-echelon inequality ($\emptyset$, T_2, T_3, 1, l) is dominated by the 2-echelon inequality ($\emptyset$, T_2, T_3, 0, l).

(5) Suppose that $k = 0$, $l = n$. In this case, $T_2 = [1, n]$. If $T_3 = \emptyset$, then the face defined by 2-echelon inequality ($\emptyset$, T_2, $\emptyset$, 0, n) is equivalent to the flow balance equation (11), so it is not proper. If $|T_3| > 1$, then the 2-echelon inequality ($\emptyset$, T_2, T_3, 0, n) is dominated by

the 2-echelon inequalities ($\emptyset$, T_2, $\{j\}$, 0, n), $j \in T_3$. Note that when $T_3 = \{j\}$ for some $j \in [1, n]$, the 2-echelon inequality ($\emptyset$, T_2, T_3, 0, n) is equivalent to the variable upper bound constraint $x_j^2 \leq d_{jn}^2 y_j^2$ given by (6).

(6) Suppose that there exists $j \in T_2$ such that $j \notin \beta(T_2, i)$ for all $i \in T_1$, then the 2-echelon inequality (T_1, T_2, T_3, k, l) is dominated by the 2-echelon inequality (T_1, $T_2 \setminus \{j\}$, $T_3 \setminus \{j\}$, k, l).

(7) Suppose that $2 \leq k \leq l = n$ and $T_3 \neq \emptyset$. If there exists $j \in T_3 \cap [2, k]$ such that $[j+1, k] \subseteq T_2$, or there exists $j \in T_3 \cap [k+1, n]$, then the 2-echelon inequality (T_1, T_2, T_3, k, n) is dominated by the 2-echelon inequality (T_1, T_2, $T_3 \setminus \{j\}$, k, n) and inequality $x_j^2 \leq d_{jn}^2 y_j^2$.

(8) Suppose that $k \leq l < n$ and $[p^1, k] \subseteq T_2$. Note that in this case, the coefficients ϕ_j, $j \in T_1$ of the 2-echelon inequality (T_1, T_2, T_3, k, l) are the same with the coefficients ϕ_j, $j \in T_1$ of the 2-echelon inequality (T_1, $T_2 \cup [l+1, n]$, $\emptyset$, k, n). Then the 2-echelon inequality (T_1, T_2, T_3, k, l) is dominated by the 2-echelon inequalities (T_1, $T_2 \cup [l+1, n]$, $\emptyset$, k, n) and ($\emptyset$, $[1, l]$, T_3, 0, l), because the sum of inequalities (T_1, $T_2 \cup [l+1, n]$, $\emptyset$, k, n) and ($\emptyset$, $[1, l]$, T_3, 0, l) is equal to the sum of 2-echelon inequality (T_1, T_2, T_3, k, l) and flow balance equation (11).

(9) It is easy to see that for $k = l = n$, we cannot have $T_1 = \emptyset$ in a facet-defining inequality. Suppose that $k = l = n$ and $T_2 = \emptyset$. If $|T_1| > 1$, then the 2-echelon inequality $(T_1, \emptyset, \emptyset, n, n)$ is dominated by the 2-echelon inequalities ($\{j\}$, $\emptyset$, $\emptyset$, n, n), $j \in T_1$. Next, suppose that $k = l = n$, $T_2 \neq \emptyset$, $w^1 \leq w^2$ and there exists $j \in [p^1, w^2]$ such that $j \notin T_2$. Let $j' = \min\{j \in [p^1, w^2], j \notin T_2\}$.

- If $j' \in T_1$, then the 2-echelon inequality (T_1, T_2, T_3, n, n) is dominated by the 2-echelon inequalities ($T_1 \cap [1, j'-1]$, $T_2 \cap [1, j'-1]$, $T_3 \cap [1, j'-1]$, n, n), ($T_1 \cap [j'+1, n]$, $T_2 \cap [j'+1, n]$, $T_3 \cap [j'+1, n]$, n, n), and ($\{j'\}$, $\emptyset$, $\emptyset$, n, n).
- If $j' \notin T_1$, then the 2-echelon inequality (T_1, T_2, T_3, n, n) is dominated by the 2-echelon inequalities ($T_1 \cap [1, j'-1]$, $T_2 \cap [1, j'-1]$, $T_3 \cap [1, j'-1]$, n, n) and ($T_1 \cap [j'+1, n]$, $T_2 \cap [j'+1, n]$, $T_3 \cap [j'+1, n]$, n, n).
- If $j' > w^1$, then the 2-echelon inequality (T_1, T_2, T_3, n, n) is dominated by the 2-echelon inequality (T_1, $T_2 \cap [1, j'-1]$, $T_3 \cap [1, j'-1]$, n, n).

Lastly, suppose that $k = l = n$, $T_2 \neq \emptyset$ and $w^1 > w^2$. Let $j'' := \min\{j \in T_1 : j > w^2\}$. Then the 2-echelon inequality (T_1, T_2, T_3, n, n) is dominated by the 2-echelon inequality ($T_1 \cap [1, j''-1]$, T_2, T_3, n, n). Note that if $T_3 \neq \emptyset$, then $w^2 < n$ by facet condition (7).

(10) Suppose that $k \neq 0$ and $T_1 = \emptyset$. It is easy to see that if $k = l = n$, then we cannot have $T_1 = \emptyset$ in a facet-defining inequality. Therefore, we assume that $k < n$. Then the 2-echelon inequality ($\emptyset$, T_2, T_3, k, l) is dominated by 2-echelon inequality ($\{k+1\}$, T_2, T_3, $k+1$, $\max\{l, k+1\}$) and inequality $y^1_{k+1} \leq 1$. Suppose that $k = 0$ and $T_3 = \emptyset$. From facet condition (5), we must have $l < n$ in this case. Note that for $k = 0$, T_2 is a consecutive set $[1, l]$ by its definition in Theorem 2. Then the 2-echelon inequality ($\emptyset$, T_2, $\emptyset$, 0, l) is dominated by 2-echelon inequality $(\emptyset, [1, n], [1, n] \setminus T_2, 0, n)$ and inequalities $y^2_j \leq 1$ for $j \in [1, n] \setminus T_2$.

Sufficiency. To prove sufficiency, we exhibit $4n - 4$ affinely independent points on the face defined by inequality (16). First, note that if $k = 0$, the 2-echelon inequalities are equivalent to (ℓ, S) inequalities for the second echelon, which have been proved to be facet-defining for the convex hull of solutions to ULS by Barany et al. (1984), when $1 \notin T_3$ (facet condition (3)). The dimension of the convex hull of ULS with positive demand is $2n-2$. Then there exist $2n-2$ affinely independent points $(\mathbf{x^2}, \mathbf{y^2}) = a_j \in \mathbb{R}^{2n}_+$, $j = 1, \ldots, 2n-2$ on the face defined by the (ℓ, S) inequality. We can expand these $2n-2$ points to $4n-4$ affinely independent points $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2}) \in \mathbb{R}^{4n}_+$ for 2-ULS, by letting $\hat{a}_{2j-1} = (d^1_{1n} + d^2_{1n})\eta^1_1 + e^1_j + \tilde{a}_j$ and $\hat{a}_{2j} = (d^1_{1n} + d^2_{1n} - d^1_j)\eta^1_1 + d^1_j\eta^1_j + e^1_j + \tilde{a}_j$, where $\tilde{a}_j = (0, \ldots, 0, a_j) \in \mathbb{R}^{4n}_+$. It is easy to see that for $j \in [2, n]$, the points $\{\hat{a}_i\}_{i=1}^{4n-4}$ are in $\mathrm{conv}(\mathcal{S})$ and affinely independent. Thus, the inequalities (16) are facet-defining for 2-ULS when $k = 0$.

From facet condition (4), we have $k \neq 1$ for the 2-echelon inequality to be facet-defining. So we assume $k \geq 2$ in the rest of the proof. Note, from facet condition (10), that $T_1 \neq \emptyset$ in this case. By facet condition (6), we define $g(j) := \max\{i \in T_1 : j \in \beta(T_2, i)\}$ for $j \in T_2 \cap [2, k]$. In addition, let $r(j) = \max\{i \in \beta(T_2, j)\}$ if $\beta(T_2, j) \neq \emptyset$, and $r(j) = j - 1$, otherwise.

Consider the point

$$u_0 = (d^1_{1k} + d^2_{1l})\eta^1_1 + e^1_1 + (d^1_{k+1,n} + d^2_{l+1,n})\eta^1_{k+1} + e^1_{k+1} + d^2_{1l}\eta^2_1 + e^2_1 + d^2_{l+1,n}\eta^2_{l+1} + e^2_{l+1}$$

on the face defined by the 2-echelon inequality (16). Based on u_0, we can generate $4n-4$ points as follows.

For $j \in [k+2, n]$, consider the points

$$u^1_j = \begin{cases} u_0 + (d^1_{jn} + d^2_{l+1,n})\eta^1_j - (d^1_{jn} + d^2_{l+1,n})\eta^1_{k+1} + e^1_j & \text{if } j \in [k+2, \min\{l+1, n\}], \\ u_0 + d^1_{jn}\eta^1_j - d^1_{jn}\eta^1_{k+1} + e^1_j & \text{if } j \in [l+2, n], \end{cases}$$

and $\bar{u}_j^1 = u_0 + e_j^1$.

For $j \in [l+2, n]$, consider the points $u_j^2 = u_0 + d_{jn}^2 \eta_j^2 - d_{jn}^2 \eta_{l+1}^2 + e_j^2$ and $\bar{u}_j^2 = u_0 + e_j^2$.

For $j \in [2, k] \setminus T_1$, consider the points $u_j^1 = u_0 + d_{jk}^1 \eta_j^1 - d_{jk}^1 \eta_1^1 + e_j^1$ and $\bar{u}_j^1 = u_0 + e_j^1$.

For $j \in T_1$, note that either $r(j) < k$ or $r(j) = l$. Also note that $j \neq 1$ from facet condition (1). Consider the points

$$u_j^1 = \begin{cases} u_0 + \phi_j \eta_j^1 - \phi_j \eta_1^1 - d_{r(j)+1,l}^2 \eta_1^2 + d_{r(j)+1,l}^2 \eta_{r(j)+1}^2 + e_j^1 + e_{r(j)+1}^2 & \text{if } r(j) < k, \\ u_0 + \phi_j \eta_j^1 - \phi_j \eta_1^1 + e_j^1 & \text{if } r(j) = l, \end{cases}$$

and

$$\bar{u}_j^1 = \begin{cases} u_j^1 + \epsilon \eta_j^1 - \epsilon \eta_{k+1}^1 & \text{if } k < l = n \text{ or } k \leq l < n, \\ u_j^1 + d_{w^2}^2 \eta_j^1 - d_{w^2}^2 \eta_1^1 - d_{w^2}^2 \eta_1^2 + d_{w^2}^2 \eta_{w^2}^2 + e_{w^2}^2 & \text{if } k = l = n, T_2 \neq \emptyset, w^2 \in T_3, \\ u_j^1 + \epsilon \eta_j^1 - \epsilon \eta_1^1 - \epsilon \eta_1^2 + \epsilon \eta_{w^2}^2 + e_{w^2}^2 & \text{if } k = l = n, T_2 \neq \emptyset, w^2 \in T_2 \setminus T_3, \end{cases}$$

where $0 < \epsilon < d_{k+1}^1$ if $k \leq l < n$; $0 < \epsilon < d_{w^2}^2$ if $k = l = n$ and $T_2 \neq \emptyset$. Note that for $k = l = n$, if $T_3 \neq \emptyset$, then $j \leq w^2 < k = n$ from facet condition (9).

For $j \in [2, l] \setminus T_2$, consider the points $u_j^1 = u_0 + d_{jl}^2 \eta_j^2 - d_{jl}^2 \eta_1^2 + e_j^2$ and $\bar{u}_j^1 = u_0 + e_j^2$.

For $j \in T_3$, by facet conditions (7) and (8), for $j \in T_3 \cap [2, k]$, either $r(j) < k$ or $r(j) = l$. Consider the following points:

$$u_j^2 = \begin{cases} u_0 + (\phi_{g(j)} + d_{j,r(j)}^2) \eta_{g(j)}^1 - (\phi_{g(j)} + d_{j,r(j)}^2) \eta_1^1 - d_{jl}^2 \eta_1^2 & \\ + d_{j,r(j)}^2 \eta_j^2 + d_{r(j)+1,l}^2 \eta_{r(j)+1}^2 + e_{g(j)}^1 + e_j^2 + e_{r(j)+1}^2 & \text{if } j \in T_3 \cap [2, k], r(j) < k, \\ u_0 + (\phi_{g(j)} + d_{jl}^2) \eta_{g(j)}^1 - (\phi_{g(j)} + d_{jl}^2) \eta_1^1 & \\ - d_{jl}^2 \eta_1^2 + d_{jl}^2 \eta_j^2 + e_{g(j)}^1 + e_j^2 & \text{if } j \in T_3 \cap [2, k], r(j) = l, \\ u_0 - d_{jl}^2 \eta_1^1 - d_{jl}^2 \eta_1^2 + d_{jl}^2 \eta_{k+1}^1 + d_{jl}^2 \eta_j^2 + e_j^2 & \text{if } j \in T_3 \cap [k+1, l]. \end{cases}$$

$$\bar{u}_j^2 = \begin{cases} u_j^2 + \epsilon \eta_j^2 - \epsilon \eta_{r(j)+1}^2 & \text{if } j \in T_3 \cap [2, k], r(j) < k, \\ u_j^2 + \epsilon \eta_{g(j)}^1 + \epsilon \eta_j^2 - \epsilon \eta_{l+1}^2 - \epsilon \eta_{k+1}^1 & \text{if } j \in T_3 \cap [2, k], r(j) = l, \\ u_j^2 + \epsilon \eta_j^2 - \epsilon \eta_{l+1}^2 & \text{if } j \in T_3 \cap [k+1, l], \end{cases}$$

where $0 < \epsilon < d_{r(j)+1}^2$ if $j \in T_3 \cap [2, k]$, $r(j) < k$; $0 < \epsilon < d_{l+1}^2$ if $j \in T_3 \cap [2, k]$, $r(j) = l$, or $j \in T_3 \cap [k+1, l]$. Note that $j \neq 1$ from facet condition (2).

For $j \in T_2 \setminus T_3$, consider the points

$$u_j^2 = \begin{cases} u_0 + (\phi_{g(j)} + d^2_{j,r(j)})\eta^1_{g(j)} - (\phi_{g(j)} + d^2_{j,r(j)})\eta^1_1 - d^2_{jl}\eta^2_1 & \\ +d^2_{j,r(j)}\eta^2_j + d^2_{r(j)+1,l}\eta^2_{r(j)+1} + e^1_{g(j)} + e^2_j + e^2_{r(j)+1} & \text{if } j \in (T_2 \setminus T_3) \cap [2,k], r(j) < k, \\ u_0 + (\phi_{g(j)} + d^2_{jl})\eta^1_{g(j)} - (\phi_{g(j)} + d^2_{jl})\eta^1_1 & \\ -d^2_{jl}\eta^2_1 + d^2_{jl}\eta^2_j + e^1_{g(j)} + e^2_j & \text{if } j \in (T_2 \setminus T_3) \cap [2,k], r(j) = l \\ u_0 - d^2_{jl}\eta^1_1 - d^2_{jl}\eta^2_1 + d^2_{jl}\eta^1_{k+1} + d^2_{jl}\eta^2_j + e^2_j & \text{if } j \in (T_2 \setminus T_3) \cap [k+1,l], \end{cases}$$

and $\bar{u}^2_j = u_0 + e^2_j$. Note that $j \neq 1$ from facet condition (2) and if $j \in (T_2 \setminus T_3) \cap [k+1,l]$, then $r(j) = l$.

(1) If $l \neq n$, three more points, u^1_{k+1}, u^1_{l+1} and $\bar{u}^1_{l+1}$, are to be considered. Let $\bar{q} := \max\{j \in [p^1,k] : j \notin T_2\}$ and $q := \max\{j \in T_1 : j \leq \bar{q}\}$. By facet condition (8), $\bar{q}$ exists.

(a) If $k = l < n$.

$$\begin{aligned} u^1_{k+1} = \quad & u_0 + (\phi_q + d^1_{k+1} + d^2_{k+1})\eta^1_q - \phi_q\eta^1_1 - d^2_{r(q)+1,k}\eta^2_1 \\ & + d^2_{r(q)+1,k}\eta^2_{r(q)+1} - (d^1_{k+1,n} + d^2_{k+1,n})\eta^1_{k+1} - d^2_{k+2,n}\eta^2_{k+1} \\ & + (d^1_{k+2,n} + d^2_{k+2,n})\eta^1_{k+2} + d^2_{k+2,n}\eta^2_{k+2} + e^1_q + e^2_{r(q)+1} - e^1_{k+1} + e^1_{k+2} + e^2_{k+2}, \\ u^2_{l+1} = \quad & u_0 + (\phi_q + d^2_{l+1})\eta^1_q - \phi_q\eta^1_1 - d^2_{r(q)+1,l}\eta^2_1 + d^2_{r(q)+1,l+1}\eta^2_{r(q)+1} - d^2_{l+1}\eta^1_{k+1} \\ & - d^2_{l+1,n}\eta^2_{l+1} + d^2_{l+2,n}\eta^2_{l+2} + e^1_q + e^2_{r(q)+1} - e^2_{l+1} + e^2_{l+2}, \\ \bar{u}^2_{l+1} = \quad & u^2_{l+1} + e^2_{l+1}. \end{aligned}$$

(b) If $k < l < n$.

$$\begin{aligned} u^1_{k+1} = \quad & u_0 + (\phi_q + d^1_{k+1})\eta^1_q - \phi_q\eta^1_1 - d^2_{r(q)+1,l}\eta^2_1 + d^2_{r(q)+1,l}\eta^2_{r(q)+1} \\ & - (d^1_{k+1,n} + d^2_{l+1,n})\eta^1_{k+1} + (d^1_{k+2,n} + d^2_{l+1,n})\eta^1_{k+2} + e^1_q + e^2_{r(q)+1} - e^1_{k+1} + e^1_{k+2}, \\ u^2_{l+1} = \quad & u_0 + (\phi_q + d^2_{l+1})\eta^1_q - \phi_q\eta^1_1 - d^2_{r(q)+1,l}\eta^2_1 + d^2_{r(q)+1,l+1}\eta^2_{r(q)+1} - d^2_{l+1}\eta^1_{k+1} \\ & - d^2_{l+1,n}\eta^2_{l+1} + d^2_{l+2,n}\eta^2_{l+2} + e^1_q + e^2_{r(q)+1} - e^2_{l+1} + e^2_{l+2}, \\ \bar{u}^2_{l+1} = \quad & u^1_{k+1} + e^1_{l+1}. \end{aligned}$$

(2) If $k < l = n$, one more point u^1_{k+1} is to be considered.

$$u^1_{k+1} = \quad u_0 + (\phi_{w^1} + d^1_{k+1})\eta^1_{w^1} - \phi_{w^1}\eta^1_1 - d^2_{r(w^1)+1,n}\eta^2_1 + d^2_{r(w^1)+1,n}\eta^2_{r(w^1)+1} - d^1_{k+1,n}\eta^1_{k+1} + d^1_{k+2,n}\eta^1_{k+2} + e^1_{w^1} + e^1_{r(w^1)+1} - e^1_{k+1} + e^1_{k+2}.$$

Next, for the case of $k = l = n$ with $|T_1| = 1$ ($T_1 = \{p^1\}$), we show that the $4n-4$ points, $\{u_0, \{u^1_j, \bar{u}^1_j, u^2_j, \bar{u}^2_j\}_{j\in[2,n]} \setminus \{\bar{u}^1_{p^1}\}\}$, are affinely independent. For all other cases, we show that the $4n-4$ points, $\{u_0, \{u^1_j, \bar{u}^1_j, u^2_j, \bar{u}^2_j\}_{j\in[2,n]} \setminus \{\bar{u}^1_{k+1}\}\}$, are affinely independent.

We assume that the $4n-4$ points associated with a particular choice of (T_1, T_2, T_3, k, l) lie on the hyperplane $\sum_{j=1}^n(\lambda^1_j x^1_j + \lambda^2_j x^2_j + \theta^1_j y^1_j + \theta^2_j y^2_j) = \pi_0$.

(1) For the case of $k = l = n$ with $T_2 = \emptyset$, by facet condition (9), we have $|T_1| = 1$. Comparing u_0 with $\bar{u}^1_j$ for $j \in [2,n] \setminus T_1$ and $\bar{u}^2_i$ for $i \in [2,n]$, we get $\theta^1_j = \theta^2_i = 0$ for $j \in [2,n] \setminus T_1$ and $i \in [2,n]$. Comparing u^1_j and $\bar{u}^1_j$ for $j \in [2,n] \setminus T_1$, we get $\lambda^1_1 = \lambda^1_j$ for $j \in [2,n] \setminus T_1$. Similarly, $\lambda^2_1 = \lambda^2_j$ for $j \in [2,n]$. Comparing u_0 and u^1_j, $j \in T_1$, we get $\theta^1_j = \phi_j(\lambda^1_1 - \lambda^1_j)$.

(2) Now consider the cases $k = l = n$ with $T_2 \neq \emptyset$, or $k < l = n$, or $k \leq l < n$. Comparing u_0 with $\bar{u}^1_j$ for $j \in ([2,k] \setminus T_1) \cup [k+2,n]$ and $\bar{u}^2_i$ for $i \in ([2,l] \setminus T_3) \cup [l+2,n]$, we get $\theta^1_j = \theta^2_i = 0$ for $j \in ([2,k] \setminus T_1) \cup [k+2,n]$ and $i \in ([2,l] \setminus T_3) \cup [l+2,n]$. Comparing u_0 with $\bar{u}^2_j$ for $j \in [k+2,n]$, we get $\lambda^1_{k+1} = \lambda^1_j$ for $j \in [k+2,n]$. Similarly, we have $\lambda^2_{l+1} = \lambda^2_j$ for $j \in [l+2,n]$; $\lambda^1_1 = \lambda^1_j$ for $j \in [2,k] \setminus T_1$; $\lambda^2_1 = \lambda^2_j$ for $j \in [2,l] \setminus T_2$. If $k < n$, comparing u^1_j and $\bar{u}^1_j$ for $j \in T_1$, we get $\lambda^1_j = \lambda^1_{k+1}$ for $j \in T_1$ with $k < n$. Comparing u^1_{k+1} and u^1_q, we get $\theta^1_{k+1} = 0$. Comparing u^2_{l+1} and $\bar{u}^2_{l+1}$, we get $\theta^2_{l+1} = 0$. Hence, we have $\theta^1_j = \theta^2_i = 0$ for $j \in [2,n] \setminus T_1$, $i \in [2,n] \setminus T_3$. For $j \in T_1$, comparing u_0 and u^1_j, we get $\theta^1_j = \phi_j(\lambda^1_1 - \lambda^1_j)$ for $j \in T_1$. Then, Comparing $\bar{u}^1_j$ for all $j \in T_1$, we get $\lambda^1_j = \lambda^1_{p^1}$ for $j \in T_1 \cup [k+1,n]$. Comparing u^2_j and $\bar{u}^2_j$ for $j \in (T_2 \setminus T_3) \cap [k+1,l]$, u^2_j and $\bar{u}^1_{g(j)}$ for $j \in (T_2 \setminus T_3) \cap [2,k]$, we get $\lambda^2_j = \lambda^1_1 + \lambda^2_1 - \lambda^1_{p^1}$ for $j \in T_2 \setminus T_3$. Comparing u^2_j and $\bar{u}^2_j$ for $j \in T_3$, u^2_{l+1} and u^1_q, we get $\lambda^2_j = \lambda^2_1$ for $j \in ([2,n] \setminus T_2) \cup T_3$. Finally, comparing u^2_j and $u^1_{g(j)}$ for $j \in T_3$, we get $\theta^2_j = \psi_j(\lambda^1_1 - \lambda^1_{p^1})$. Finally, from u_0, we get $\pi_0 = (d^1_{1k} + d^2_{1l})\lambda^1_1 + d^2_{1l}\lambda^2_1 + d^1_{k+1,n}\lambda^1_{k+1} + d^2_{l+1,n}\lambda^2_{l+1} + \theta^1_1 + \theta^2_1$.

Therefore, the hyperplane is of the form

$$\lambda_1^1 \sum_{j\in[1,k]\setminus T_1} x_j^1 + \lambda_{p^1}^1 \sum_{j\in T_1\cup[k+1,n]} x_j^1 + \lambda_1^2 \sum_{j\in([1,n]\setminus T_2)\cup T_3} x_j^2 + (\lambda_1^1+\lambda_1^2-\lambda_{p^1}^1) \sum_{j\in T_2\setminus T_3} x_j^2$$
$$+\theta_1^1 y_1^1 + \theta_1^2 y_1^2 + (\lambda_1^1 - \lambda_{p^1}^1)(\sum_{j\in T_1} \phi_j y_j^1 + \psi_j \sum_{j\in T_3} y_j^2)$$
$$=(d_{1k}^1 + d_{1l}^2)\lambda_1^1 + d_{1l}^2\lambda_1^2 + d_{k+1,n}^1\lambda_{k+1}^1 + d_{l+1,n}^2\lambda_{l+1}^2 + \theta_1^1 + \theta_1^2.$$

Hence these points define the 2-echelon inequality (16) up to a multiple θ_1^1 of $y_1^1 = 1$; a multiple θ_2^1 of $y_1^2 = 1$; a multiple $\lambda_{p^1}^1$ of $\sum_{i=1}^n x_i^1 = d_{1n}^1$; and a multiple λ_1^2 of $\sum_{i=1}^n x_i^2 = d_{1n}^2$. In addition, if $k = 0$, then by facet condition (10), $T_3 \neq \emptyset$, and by facet condition (3), $1 \notin T_3$, thus the point $(d_{1n}^1 + d_{1n}^2)\eta_1^1 + d_{1n}^2\eta_1^2 + e_1^1 + e_1^2 + \sum_{j\in T_3} e_j^2$ is not on the face defined by the 2-echelon inequality. If $k = l = n$, by facet conditions (1) and (10), $1 \notin T_1 \neq \emptyset$, then the point $(d_{1n}^1 + d_{1n}^2)\eta_1^1 + d_{1n}^2\eta_1^2 + e_1^1 + e_1^2 + \sum_{j\in T_1} e_j^1$ is not on the face defined by the 2-echelon inequality. For other cases, we have $1 \le k < n$ or $1 \le l < n$, and the point $(d_{1n}^1 + d_{1n}^2)\eta_1^1 + d_{1n}^2\eta_1^2 + e_1^1 + e_1^2$ is not on the face defined by the 2-echelon inequality. Hence, the face is proper.

□

Appendix C. Proof of Proposition 5

We prove a series of lemmas to prove Proposition 5. Lemmas (12) and (13) characterize the non-negative extreme rays of the projection cone defined by constraints (33)–(35) where each component is either 0 or 1. Lemmas (14)–(18) give conditions under which such an inequality is a non-dominated inequality (32).

Lemma 12. *Let* $(\sigma^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)$ *be a non-negative extreme ray of the projection cone. Then* $\gamma_{ut}^{11} = (\alpha_t^1 - \sigma_u^1)^+$, $\gamma_{ut}^{12} = \left(\sum_{j=u}^t \rho_{jt} - \sigma_u^1\right)^+$ *and* $\gamma_{ut}^{22} = \left(\alpha_t^2 - \sum_{j=u}^t \rho_{jt} - \sigma_u^2\right)^+$ *for* $1 \le u \le t \le n$.

Let $(\sigma^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)$ be a non-negative extreme ray of the projection cone such that the entries are equal to 0 or 1. Let $A_1 = \{t \in [1,n] : \alpha_t^1 = 1\}$, $S_1 = \{u \in [1,n] : \sigma_u^1 = 1\}$, $A_2 = \{t \in [1,n] : \alpha_t^2 = 1\}$, $S_2 = \{u \in [1,n] : \sigma_u^2 = 1\}$, $G_{11} = \{(u,t) : 1 \le u \le t \le n, \gamma_{ut}^{11} = 1\}$, $G_{12} = \{(u,t) : 1 \le u \le t \le n, \gamma_{ut}^{12} = 1\}$, $G_{22} = \{(u,t) : 1 \le u \le t \le n, \gamma_{ut}^{22} = 1\}$, and for $t \in [1,n]$, $R = \{(j,t) : 1 \le j \le t \le n, \rho_{jt} = 1\}$. Lemma 12 implies that $(u,t) \in G_{11}$ if and only if $1 \le u \le t \le n$, $t \in A_1$ and $u \notin S_1$.

Lemma 13. *Let* $(\sigma^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)$ *be a non-negative extreme ray of the projection cone such that the entries are equal to 0 or 1. Then for each* $t \in [1, n]$*, we must have* $|R \cap \{(j, t) : 1 \leq j \leq t\}| \leq 1$*. Let* $j(t) \in [0, t]$ *for* $t \in [1, n]$*, then* $R = \{(j(t), t) : j(t) \neq 0, t \in [1, n]\}$*.*

Proof. Let $\omega = (\sigma^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)$ be a non-negative extreme ray of the projection cone such that the entries are equal to 0 or 1. Let $\epsilon > 0$ be a very small number and consider the following rays $\hat{\omega} = (\hat{\sigma}^1, \hat{\sigma}^2, \hat{\gamma}^{11}, \hat{\gamma}^{12}, \hat{\gamma}^{22}, \hat{\alpha}^1, \hat{\alpha}^2, \hat{\rho})$ and $\tilde{\omega} = (\tilde{\sigma}^1, \tilde{\sigma}^2, \tilde{\gamma}^{11}, \tilde{\gamma}^{12}, \tilde{\gamma}^{22}, \tilde{\alpha}^1, \tilde{\alpha}^2, \tilde{\rho})$ where $\hat{\alpha}^1_t = \alpha^1_t + \epsilon$, $\tilde{\alpha}^1_t = \alpha^1_t - \epsilon$ for $t \in A^1$, $\hat{\alpha}^1_t = \alpha^1_t = \tilde{\alpha}^1_t = 0$ for $t \in [1, n] \setminus A^1$, $\hat{\alpha}^2_t = \alpha^2_t + \epsilon$, $\tilde{\alpha}^2_t = \alpha^2_t - \epsilon$ for $t \in A^2$, $\hat{\alpha}^2_t = \alpha^2_t = \tilde{\alpha}^2_t = 0$ for $t \in [1, n] \setminus A^2$, $\hat{\sigma}^1_t = \sigma^1_t + \epsilon$, $\tilde{\sigma}^1_t = \sigma^1_t - \epsilon$ for $t \in S^1$, $\hat{\sigma}^1_t = \sigma^1_t = \tilde{\sigma}^1_t = 0$ for $t \in [1, n] \setminus S^1$, $\hat{\sigma}^2_t = \sigma^2_t + \epsilon$, $\tilde{\sigma}^2_t = \sigma^2_t - \epsilon$ for $t \in S^2$ and $\hat{\sigma}^2_t = \sigma^2_t = \tilde{\sigma}^2_t = 0$ for $t \in [1, n] \setminus S^2$. For $t \in [1, n]$, define $j(t)$ to be the largest index j with $\rho_{jt} = 1$ (if none exists, then let $j(t) = 0$). Let $\hat{\rho}_{jt} = \tilde{\rho}_{jt} = \rho_{jt}$ for all j and t such that $j \neq j(t)$ and $\hat{\rho}_{j(t),t} = \rho_{j(t),t} + \epsilon$ and $\tilde{\rho}_{j(t),t} = \rho_{j(t),t} - \epsilon$ for $t \in [1, n]$. For $1 \leq u \leq t \leq n$, $\hat{\gamma}^{11}_{ut} = \gamma^{11}_{ut} + \epsilon$, $\tilde{\gamma}^{11}_{ut} = \gamma^{11}_{ut} - \epsilon$ if $(u, t) \in G_{11}$, $\hat{\gamma}^{11}_{ut} = \gamma^{11}_{ut} = \tilde{\gamma}^{11}_{ut} = 0$ otherwise, $\hat{\gamma}^{12}_{ut} = \gamma^{12}_{ut} + \epsilon$, $\tilde{\gamma}^{12}_{ut} = \gamma^{12}_{ut} - \epsilon$ if $(u, t) \in G_{12}$, $\hat{\gamma}^{12}_{ut} = \gamma^{12}_{ut} = \tilde{\gamma}^{12}_{ut} = 0$ otherwise, $\hat{\gamma}^{22}_{ut} = \gamma^{22}_{ut} + \epsilon$, $\tilde{\gamma}^{22}_{ut} = \gamma^{22}_{ut} - \epsilon$ if $(u, t) \in G_{22}$ and $\hat{\gamma}^{22}_{ut} = \gamma^{22}_{ut} = \tilde{\gamma}^{22}_{ut} = 0$ otherwise. Now, these two rays $\hat{\omega}$ and $\tilde{\omega}$ ($\hat{\omega} \neq \tilde{\omega} \neq \omega$) are in the projection cone and we have $\omega = \hat{\omega}/2 + \tilde{\omega}/2$. As ω is an extreme ray, both rays $\hat{\omega}$ and $\tilde{\omega}$ should be multiples of it. Therefore, we cannot have $\hat{\rho}_{jt} = \tilde{\rho}_{jt} = 1$ for any $j < j(t)$. Hence $|R \cap \{(j, t) : 1 \leq j \leq t\}| \leq 1$ for all $t \in [1, n]$. □

As a result, we can conclude that $(u, t) \in G_{12}$ if and only if $1 \leq u \leq t \leq n$, $u \leq j(t)$ and $u \notin S_1$ and $(u, t) \in G_{22}$ if and only if $1 \leq u \leq t \leq n$, $t \in A_2$, $u \notin S_2$ and $u > j(t)$.

The projection inequalities corresponding to the non-negative extreme rays with equal positive entries are of the form

$$\sum_{u \in S_1} x^1_u + \sum_{u \in S_2} x^2_u + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d^1_t + \sum_{t:(u,t) \in G_{12}} d^2_t \right) y^1_u + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d^2_t y^2_u \geq \sum_{t \in A_1} d^1_t + \sum_{t \in A_2} d^2_t \tag{39}$$

It is easy to see that it is of no use to make $j(t) > 0$ for $t \notin A_2$. So we are interested in the case with $j(t) = 0$ for $t \notin A_2$.

Let $t^i = \max_{t \in A_i} t$ if $A_i \neq \emptyset$ and $t^i = 0$ otherwise for $i = 1, 2$.

Lemma 14. *If* $A_1 \neq \emptyset$ *and there exists* $\hat{t} < t^1$ *with* $\hat{t} \notin A_1$*, then inequality (39) is dominated by other inequalities (39).*

Proof. Suppose that $A_1 \neq \emptyset$ and there exists $\hat{t} < t^1$ with $\hat{t} \notin A_1$. Then we would like to show that the projection inequality defined by sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ is dominated. Consider the projection inequalities (39) for sets $(A_1^1, A_2, S_1, S_2, R, G_{11}^1, G_{12}, G_{22})$ and $(A_1^2,\ A_2,\ S_1,\ S_2,\ R,\ G_{11}^2,\ G_{12},\ G_{22})$ where $A_1^1 = A_1 \cup \{\hat{t}\}$, $A_1^2 = A_1 \setminus \{t^1\}$, $G_{11}^1 = G_{11} \cup \{(u, \hat{t}) : u \notin S_1, u \leq \hat{t}\}$ and $G_{11}^2 = G_{11} \setminus \{(u, t^1) : u \notin S_1, u \leq t^1\}$. The first inequality is

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2 + d_{\hat{t}}^1 \left(1 - \sum_{u \notin S_1, u \leq \hat{t}} y_u^1 \right)$$

and the second inequality is

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2 - d_{t^1}^1 \left(1 - \sum_{u \notin S_1, u \leq t^1} y_u^1 \right).$$

Multiplying the first inequality with $d_{t^1}^1$, the second with $d_{\hat{t}}^1$, and dividing the sum by $d_{t^1}^1 + d_{\hat{t}}^1$, we obtain

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2 + \frac{d_{\hat{t}}^1 d_{t^1}^1}{d_{\hat{t}}^1 + d_{t^1}^1} \left(1 - \sum_{u \notin S_1, u \leq \hat{t}} y_u^1 \right) - \frac{d_{\hat{t}}^1 d_{t^1}^1}{d_{\hat{t}}^1 + d_{t^1}^1} \left(1 - \sum_{u \notin S_1, u \leq t^1} y_u^1 \right).$$

As $t^1 > \hat{t}$, we have $\sum_{u \notin S_1, u \leq t^1} y_u^1 \geq \sum_{u \notin S_1, u \leq \hat{t}} y_u^1$. As a result, the above inequality dominates the projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$.

□

Lemma 15. *If $A_2 \neq \emptyset$ and there exists $\hat{t} < t^2$ with $\hat{t} \notin A_2$, then inequality (39) is dominated by other inequalities (39).*

Proof. Consider the projection inequality defined by sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ and suppose that $A_2 \neq \emptyset$ and there exists $\hat{t} < t^2$ with $\hat{t} \notin A_2$.

Let $A_2^1 = A_2 \cup \{\hat{t}\}$, $A_2^2 = A_2 \setminus \{t^2\}$, $R^1 = R \cup \{(\min\{j(t^2), \hat{t}\}, \hat{t})\}$, $R^2 = R \setminus \{(j(t^2), t^2)\}$, $G_{12}^1 = G_{12} \cup \{(u, \hat{t}) : u \leq \hat{t}, (u, t^2) \in G_{12}\}$, $G_{22}^1 = G_{22} \cup \{(u, \hat{t}) : u \leq \hat{t}, (u, t^2) \in G_{22}\}$, $G_{12}^2 = G_{12} \setminus \{(u, t^2) : u \leq t^2\}$, $G_{22}^2 = G_{22} \setminus \{(u, t^2) : u \leq t^2\}$. First observe that sets $(A_1, A_2^1, S_1, S_2, R^1, G_{11}, G_{12}^1, G_{22}^1)$ and $(A_1, A_2^2, S_1, S_2, R^2, G_{11}, G_{12}^2, G_{22}^2)$ give valid projection inequalities. The projection inequality for $(A_1, A_2^1, S_1, S_2, R^1, G_{11}, G_{12}^1, G_{22}^1)$ is

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2 + d_{\hat{t}}^2 \left(1 - \sum_{u:(u,t^2) \in G_{12}, u \leq \hat{t}} y_u^1 - \sum_{u:(u,t^2) \in G_{22}, u \leq \hat{t}} y_u^2 \right)$$

and the projection inequality for $(A_1, A_2^2, S_1, S_2, R^2, G_{11}, G_{12}^2, G_{22}^2)$ is

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2 - d_{t^2}^2 \left(1 - \sum_{u:(u,t^2) \in G_{12}, u \leq t^2} y_u^1 - \sum_{u:(u,t^2) \in G_{22}, u \leq t^2} y_u^2 \right).$$

Again, multiplying the first inequality with $d_{t^2}^2$, the second with $d_{\hat{t}}^2$, and dividing the sum by $d_{t^2}^2 + d_{\hat{t}}^2$, we obtain

$$\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \notin S_1} \left(\sum_{t:(u,t) \in G_{11}} d_t^1 + \sum_{t:(u,t) \in G_{12}} d_t^2 \right) y_u^1 + \sum_{u \notin S_2} \sum_{t:(u,t) \in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t \in A_1} d_t^1 + \sum_{t \in A_2} d_t^2$$
$$+ \frac{d_{\hat{t}}^2 d_{t^2}^2}{d_{\hat{t}}^2 + d_{t^2}^2} \left(\sum_{u:(u,t^2) \in G_{12}, u \leq t^2} y_u^1 + \sum_{u:(u,t^2) \in G_{22}, u \leq t^2} y_u^2 - \sum_{u:(u,t^2) \in G_{12}, u \leq \hat{t}} y_u^1 - \sum_{u:(u,t^2) \in G_{22}, u \leq \hat{t}} y_u^2 \right).$$

As $\sum_{u:(u,t^2) \in G_{12}, u \leq t^2} y_u^1 + \sum_{u:(u,t^2) \in G_{22}, u \leq t^2} y_u^2 - \sum_{u:(u,t^2) \in G_{12}, u \leq \hat{t}} y_u^1 - \sum_{u:(u,t^2) \in G_{22}, u \leq \hat{t}} y_u^2$ is non-negative, the above inequality dominates the projection inequality (39) for (A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22}). □

These two lemmas imply that undominated projection inequalities have sets A_1 and A_2 of the form $A_1 = [1, t^1]$ and $A_2 = [1, t^2]$.

Lemma 16. *If $t^1 > t^2$, then inequality (39) is dominated by other inequalities (39).*

Proof. Consider the projection inequality defined by sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ with $t^1 > t^2$.

The projection inequality for $(A_1^1, A_2, S_1, S_2, R, G_{11}^1, G_{12}, G_{22})$ where $A_1^1 = A_1 \setminus \{t^1\}$, $G_{11}^1 = G_{11} \setminus \{(u,t^1) : u \notin S_1, u \leq t^1\}$ is

$$\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2 \right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 - d_{t^1}^1 \left(1 - \sum_{u\notin S_1, u\leq t^1} y_u^1 \right).$$

The projection inequality for $(A_1, A_2^1, S_1, S_2, R^1, G_{11}, G_{12}^1, G_{22})$ where $A_2^1 = A_2 \cup \{t^2+1\}$, $R^1 = R \cup \{(t^2+1, t^2+1)\}$ and $G_{12}^1 = G_{12} \cup \{(u, t^2+1) : u \notin S_1, u \leq t^2+1\}$ is

$$\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2 \right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 + d_{t^2+1}^2 \left(1 - \sum_{u\notin S_1, u\leq t^2+1} y_u^1 \right).$$

Now, we multiply the first inequality with $d_{t^2+1}^2$, the second inequality with $d_{t^1}^1$, add them up, and divide by $d_{t^2+1}^2 + d_{t^1}^1$ to obtain

$$\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2 \right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2$$
$$\geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 + \frac{d_{t^1}^1 d_{t^2+1}^2}{d_{t^2+1}^2 + d_{t^1}^1} \left(\sum_{u\notin S_1, u\leq t^1} y_u^1 - \sum_{u\notin S_1, u\leq t^2+1} y_u^1 \right).$$

This inequality dominates the projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ since $\sum_{u\notin S_1, u\leq t^1} y_u^1 - \sum_{u\notin S_1, u\leq t^2+1} y_u^1 \geq 0$.

□

Now we limit our investigation to the projection inequalities defined by sets A_1 and A_2 of the form $A_1 = [1, t^1]$ and $A_2 = [1, t^2]$ with $t^2 \geq t^1 \geq 0$. Note that if S_1 or S_2 has an element larger than t^2, then the resulting inequality is dominated. Hence, $S_1 \subseteq A_2$ and $S_2 \subseteq A_2$. The projection inequalities under consideration have the form

$$\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\in A_2\setminus S_1} \hat{\phi}_u y_u^1 + \sum_{u\in A_2\setminus S_2} \hat{\psi}_u y_u^2 \geq d_{1t^1}^1 + d_{1t^2}^2 \tag{40}$$

where $\hat{\phi}_u = d_{ut^1}^1 + \sum_{t:(u,t)\in G_{12}} d_t^2 = d_{ut^1}^1 + \sum_{t\in A_2: u\leq j(t)} d_t^2$ for $u \in A_2 \setminus S_1$ and $\hat{\psi}_u = \sum_{t:(u,t)\in G_{22}} d_t^2 = \sum_{t\in A_2: j(t)<u\leq t} d_t^2$ for $u \in A_2 \setminus S_2$.

Lemma 17. *If there exists $\hat{t} \in A_2$ with $j(\hat{t}) > t^1$, then inequality (40) is dominated by other inequalities (40).*

Proof. If there exists $\hat{t} \in A_2$ with $j(\hat{t}) > t^1$, then consider the projection inequalities defined by $(A_1^1, A_2, S_1, S_2, R, G_{11}^1, G_{12}, G_{22})$ where $A_1^1 = A_1 \cup \{j(\hat{t})\}$ and $G_{11}^1 = G_{11} \cup \{(u, j(\hat{t})) : u \notin S_1, u \leq j(\hat{t})\}$ and $(A_1, A_2^1, S_1, S_2, R^1, G_{11}, G_{12}^1, G_{22}^1)$ where $A_2^1 = A_2 \setminus \{\hat{t}\}$, $R^1 = R \setminus \{(j(\hat{t}), \hat{t}\}$, $G_{12}^1 = G_{12} \setminus \{(u, \hat{t}) : u \notin S_1, u \leq \hat{t}\}$ and $G_{22}^1 = G_{22} \setminus \{(u, \hat{t}) : u \notin S_2, u \leq \hat{t}\}$. These inequalities are

$$\begin{aligned}\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2\right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2 \\ \geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 + d_{j(\hat{t})}^1 \left(1 - \sum_{u\notin S_1, u\leq j(\hat{t})} y_u^1\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2\right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2 \\ \geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 - d_{\hat{t}}^2 \left(1 - \sum_{u:(u,\hat{t})\in G_{12}} y_u^1 - \sum_{u:(u,t^2)\in G_{22}} y_u^2\right).\end{aligned}$$

We multiply the first inequality with $d_{\hat{t}}^2$, the second inequality with $d_{j(\hat{t})}^1$, add them and divide by $d_{j(\hat{t})}^1 + d_{\hat{t}}^2$ to obtain

$$\begin{aligned}\sum_{u\in S_1} x_u^1 + \sum_{u\in S_2} x_u^2 + \sum_{u\notin S_1} \left(\sum_{t:(u,t)\in G_{11}} d_t^1 + \sum_{t:(u,t)\in G_{12}} d_t^2\right) y_u^1 + \sum_{u\notin S_2} \sum_{t:(u,t)\in G_{22}} d_t^2 y_u^2 \\ \geq \sum_{t\in A_1} d_t^1 + \sum_{t\in A_2} d_t^2 + \frac{d_{j(\hat{t})}^1 d_{\hat{t}}^2}{d_{j(\hat{t})}^1 + d_{\hat{t}}^2} \left(- \sum_{u\notin S_1, u\leq j(\hat{t})} y_u^1 + \sum_{u:(u,\hat{t})\in G_{12}} y_u^1 + \sum_{u:(u,t^2)\in G_{22}} y_u^2\right).\end{aligned}$$

Now since $\{u : (u,\hat{t}) \in G_{12}\} = \{u \notin S_1 : u \leq j(\hat{t})\}$, $-\sum_{u \notin S_1, u \leq j(\hat{t})} y^1_u + \sum_{u:(u,\hat{t}) \in G_{12}} y^1_u + \sum_{u:(u,t^2) \in G_{22}} y^2_u = \sum_{u:(u,t^2) \in G_{22}} y^2_u \geq 0$. Hence, this inequality dominates the projection inequality (40) for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$.

□

If $j(t) \leq t^1$ for all $t \in A_2$, then for $u \in A_2 \setminus A_1$, $\hat{\phi}_u = 0$. Hence the projection inequality (40) simplifies to inequality (36) with $\hat{\phi}_u = d^1_{ut^1} + \sum_{t \in A_2 : u \leq j(t)} d^2_t$ for $u \in A_1 \setminus S_1$ and $\hat{\psi}_u = \sum_{t \in A_2 : j(t) < u \leq t} d^2_t$ for $u \in A_2 \setminus S_2$. Finally, observe that if there exists $u \in S_1$ with $u > t^1$, as $j(t) \leq t^1$ for all $t \in A_2$, removing u from S_1 yields a stronger inequality. As a result, the interesting projection inequalities are defined by $0 \leq t^1 \leq t^2 \leq n$, $A_1 = [1, t^1]$, $A_2 = [1, t^2]$, $S_1 \subseteq A_1$, $S_2 \subseteq A_2$ and $j(t) \in [0, \min\{t, t^1\}]$ for $t \in A_2$.

Lemma 18. *In a nondominated projection inequality (36),* $j(1) \in [0, 1]$ *and* $j(t+1) \in \{j(t), t+1\}$ *for all* $t \in A_2$ *with* $t \leq n-1$*.*

Proof. Suppose that $0 \leq t^1 \leq t^2 \leq n$, $A_1 = [1, t^1]$, $A_2 = [1, t^2]$, $S_1 \subseteq A_1$ and $S_2 \subseteq A_2$ are given. Define $\Gamma(t, j) = \sum_{u \in A_1 \setminus S_1 : u \leq j} y^1_u + \sum_{u \in A_2 \setminus S_2 : j < u \leq t} y^2_u$. Then the left hand side of inequality (40) is equal to $\sum_{u \in S_1} x^1_u + \sum_{u \in S_2} x^2_u + \sum_{u \in A_1 \setminus S_1} d^1_{ut^1} y^1_u + \sum_{t \in A_2} d^2_t \Gamma(t, j(t))$. So for a given vector $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2})$ and fixed A_1, A_2, S_1 and S_2, the best $j(t)$ choices are those with minimum $\Gamma(t, j(t))$ values for each $t \in A_2$. Now let $t \in A_2$ with $t \leq n-1$ and observe that for a given $j \in [0, t]$, $\Gamma(t+1, j) = \Gamma(t, j) + \sum_{u \in A_2 \setminus S_2 : u = t+1} y^2_u$. This implies that $\operatorname{argmin}_{j \in [0,t]} \Gamma(t+1, j) = \operatorname{argmin}_{j \in [0,t]} \Gamma(t, j)$. Hence $j(t+1) \in \{j(t), t+1\}$. □

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