New Perspectives on Mixed-Integer Convex Optimization with Applications in Statistical Learning

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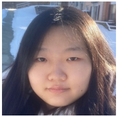

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Supported by NSF grants 2007814 and 2006762 (with A. Gómez)

Time travel

Before we talk about Mixed-Integer **Convex Quadratic** Programs, let's do an experiment to see how far we've come in Mixed-Integer **Linear** Programming (MILP)

Impact of cutting planes in Mixed-Integer **Linear** Programming (MILP) software

Without cuts

```
Explored 1933736 nodes (3989094 simplex iterations) in 38.70 seconds
Thread count was 8 (of 8 available processors)
```

With cuts

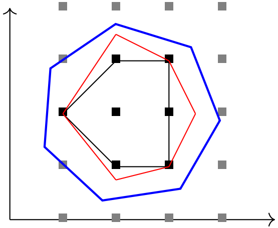
```
Cutting planes:
  Gomory: 4
  Implied bound: 22
  MIR: 19
  Flow cover: 35
  Flow path: 18
  Relax-and-lift: 2

Explored 1 nodes (426 simplex iterations) in 0.10 seconds
Thread count was 8 (of 8 available processors)
```

Extended formulation

```
Explored 0 nodes (238 simplex iterations) in 0.03 seconds
Thread count was 8 (of 8 available processors)
```

What's the Secret Sauce?

Polyhedral Theory for MILP



Original formulation
Stronger formulation
Ideal formulation (convex hull, facets)

- Structured cutting planes (Cover, flow cover, flow path, etc.)
- General-purpose cutting planes (Gomory, MIR, disjunctive, etc.)
- Presolve, heuristics, branching, ...

See, also, "Progress in Mathematical Programming Solvers from 2001 to 2020," Koch et al, 2021.

Agenda

MIQP with indicators

$$\min_{x,z} a^\top x + b^\top z + \frac{1}{2}x^\top Qx \qquad (Q \succeq 0)$$

$$\text{s.t. } x_j(1 - z_j) = 0, \quad j \in [n] := \{1, \dots, n\} \qquad (x \circ (\mathbf{1} - z) = \mathbf{0})$$

$$x \in \mathbb{R}^n,\ z \in Z \subseteq \{0,1\}^n$$

or equivalently, in its epigraph form,

$$\min_{x,z,t} a^\top x + b^\top z + \frac{1}{2}t$$

$$\text{s.t. } t \geq x^\top Qx, x \circ (\mathbf{1} - z) = \mathbf{0}, x \in \mathbb{R}^n,\ z \in Z$$

Alternative formulation of non-convex complementarity constraint

$$-Mz \leq x \leq Mz \qquad \text{(Big-M constraint)}$$

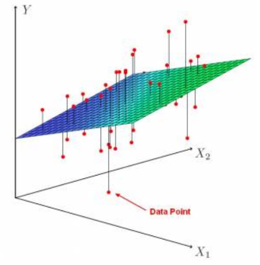
Weak continuous relaxation

Motivating Example: Best Subset Selection

Given model matrix $A_{m \times n}$ and response vector $y \in \mathbb{R}^m$

$$\min_{x : \|x\|_0 \leq k} \|y - Ax\|_2^2,$$

where $\|x\|_0 = \sum_{i=1}^n \mathbb{1}_{\{x_i \neq 0\}}$ is the "ℓ_0 norm," $k \in \mathbb{Z}$ is a given cardinality.



Here, $Z = \{z \in \{0,1\}^n : \sum_{i=1}^n z_i \leq k\}$, $Q = A^\top A$, $a = -y^\top A$

NP-hard. (Chen et al, 2017)

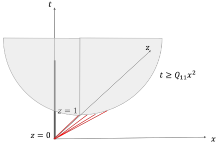
Other Applications

- Structured regression (e.g., Bertsimas et al, 2021; Hazimeh and Mazumder, 2020)
- Probabilistic graphical models (e.g., Küçükyavuz et al., 2020)
- Portfolio optimization (e.g., Bienstock, 1996)
- Power systems (e.g., Bacci et al., 2019)
- Machine scheduling (e.g., Aktürk et al., 2009)

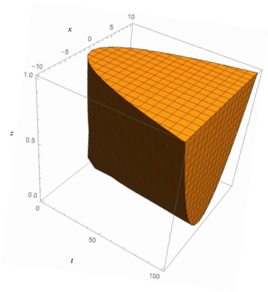
Agenda

Special Cases: $n = 1$ (or Q is diagonal)

$$R \equiv \{(z,x,t) \mid t \geq Q_{11}x^2,\ x \circ (\mathbf{1} - z) = \mathbf{0},\ z \in \{0,1\}\}$$



$$\mathrm{cl\ conv}(R) \equiv \{(z,x,t) \mid t \geq Q_{11}\frac{x^2}{z},\ z \in [0,1]\}$$ (Big-M free, SOCP)



Convention: $\frac{0}{0} = 0$.

Perspective reformulation: Ceria and Soares (1999), Frangioni and Gentile (2006), Aktürk et al. (2008), Günlük and Linderoth (2010)...

Why perspective? For convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ its perspective function $\phi(x,z) = zf\left(\frac{x}{z}\right) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is also convex.

Solution approaches leveraging perspective formulation

1. Find "good" diagonal matrix $D, D_{ii} \geq 0$ such that $Q - D \succeq 0$
   - Using minimum eigenvalue of Q (Frangioni, 2006)
   - Using SDP heuristics (Frangioni, 2007)
   - Using ridge regularization (Bertsimas and Van Parys, 2020)
   - Maximizing relaxation quality (Zheng et al., 2014; Dong et al., 2015)

2. Use branch-and-bound based on the perspective reformulation

$$\begin{aligned}
\min_{x,z}\ & a^\top x + b^\top z + \frac{1}{2}x^\top (Q - D)x + \frac{1}{2}\sum_{i=1}^{n} \frac{D_{ii}x_i^2}{z_i} \\
\text{s.t.}\ & -Mz \leq x \leq Mz \qquad \text{(Big-M constraint)} \\
& x \in \mathbb{R}^n,\ z \in Z \subseteq \{0,1\}^n
\end{aligned}$$

Research Questions

- Can we exploit matrix and constraint structure to obtain stronger relaxations? (Part 1)

- What does strong mean for MIQP?
  Can we leverage polyhedral theory for MIQP? (Part 2)

Special Cases: Rank-one convex function f

$X = \left\{(z,x,t) \in \{0,1\}^n \times \mathbb{R}^{n+1} \mid t \geq f(q^\top x), x \circ (\mathbf{1} - z) = \mathbf{0}, z \in Z\right\}$
Quadratic: $f(q^\top x) = (q^\top x)^2$ for a given vector $q \in \mathbb{R}^n$, i.e., $Q = qq^\top \succeq 0$.

Theorem (Wei, Gómez, Küçükyavuz, 2022)

If f is convex, $f(0) = 0$, and Z is 'connected', then

$$\text{cl conv}(X) = \left\{(z,x,t) | z \in \text{conv}(Z), t \geq f(q^\top x), t \geq (\pi^\top z) f\left(\frac{q^\top x}{\pi^\top z}\right), \forall \pi \in \mathcal{F}\right\},$$

where $\mathcal{F}$ is a family of strong separating inequalities for

$$\text{conv}(Z \setminus \{\mathbf{0}\}) \;=\; \text{conv}(Z) \bigcap \{z \in \mathbb{R}^n \;:\; \pi^\top z \geq 1,\; \forall \pi \in \mathcal{F}\}$$

- New perspectives
  Subsumes all related convexifications to date; first convexification for a logistic loss function.

How to characterize $\mathcal{F}$? Can use ideas in Angulo et al, "Forbidden Vertices," 2015.

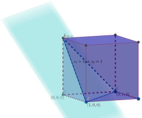
Special Case: $f(q^\top x) = (q^\top x)^2, Z = \{0,1\}^n$

$$R \equiv \{(z, x, t) \in \{0,1\}^n \times \mathbb{R}^{n+1} \ : \ t \geq (\mathbf{q}^\top x)^2, \ x \circ (\mathbf{1} - z) = \mathbf{0}\}.$$

Theorem (Atamtürk and Gómez, 2019)

$$\text{cl conv}(R) = \left\{(z, x, t) \in [0,1]^n \times \mathbb{R}^{n+1} \mid t \geq (\mathbf{q}^\top x)^2, \ t \geq \frac{(\mathbf{q}^\top x)^2}{\sum_{i\in[n]} z_i}\right\}$$

- $\sum_{i=1}^n z_i \geq 1$ is a strong inequality separating $\mathbf{0}$ from the set Z



$$\text{conv}(Z\backslash\{\mathbf{0}\}) = \{z \in [0,1]^n \ : \ \sum_{i=1}^n z_i \geq 1\}$$

Special Case: Diagonal Q, General $Z \subset \{0,1\}^n$

$$R \equiv \{(z,x,t) \mid t_i \geq Q_{ii}x_i^2, i \in [n],\ x \circ (\mathbf{1}-z) = \mathbf{0},\ z \in Z\}$$

Corollary (Wei, Gómez, Küçükyavuz, 2022)

$$\text{cl conv}(R) = \left\{(z,x,t) \middle| t_i \geq \frac{Q_{ii}x_i^2}{z_i}, i \in [n], z \in \text{conv}(Z)\right\}.$$

Xie and Deng (2020) show this for $Z = \{z \in \{0,1\}^n : \sum_{i=1}^n z_i \leq k\}$.

Numerical Results

Least squares regression with **strong hierarchy** constraints on pairwise interactions.

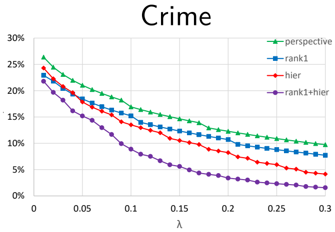
$$
\begin{aligned}
\min_{z,x} \ & \sum_{\ell=1}^{p}\left(y_\ell - \sum_{i=1}^{n} A_{\ell i}x_i - \sum_{i=1}^{n}\sum_{j=i}^{n} A_{\ell i}A_{\ell j}x_{ij}\right)^2 + \lambda\|x\|_2^2 + \mu\|z\|_1 \\
\text{s.t. } & x_i(1-z_i) = 0 \qquad \forall i \\
& x_{ij}(1-z_{ij}) = 0 \qquad i \le j \\
& z_{ii} \le z_i \qquad \forall i \\
& z_{ij} \le z_i,\ z_{ij} \le z_j \qquad i \le j \\
& z \in \{0,1\}^{\frac{n(n+3)}{2}}
\end{aligned}
$$

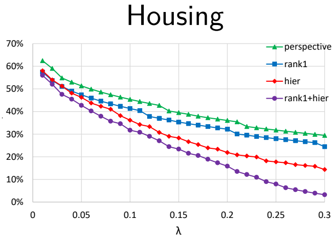
- $z_i + z_j - z_{ij} \ge 1$ is a strong inequality separating $\mathbf{0}$ from the set Z

Relaxation comparisons

- **Perspective**: Optimal perspective relaxation (Dong et al., 2015)
- **Rank1**: Rank-one relaxation (Atamtürk and Gómez, 2019)
- **Hier**: Hierarchical strengthening (the formulation we proposed)
- **Rank1 + Hier**: Combine these two methods

Optimality Gaps: Varying λ

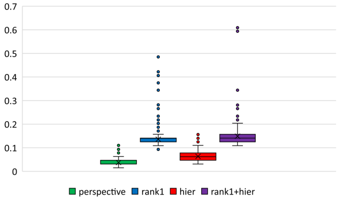




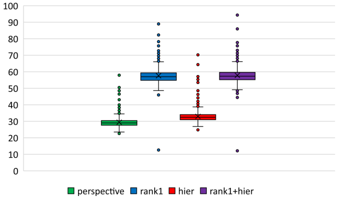
- **Hier** (vs. **Persp**): Significant improvement in lower bound
- **Rank1+Hier** (vs. **Rank1**): Gives the best optimality gap

Solution Time

Crime



Housing

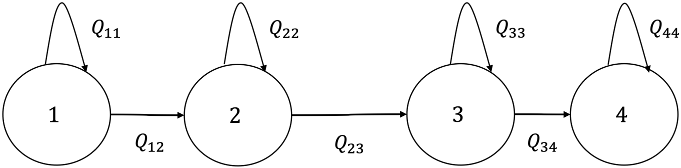


- **Hier** (vs. **Persp**): Only a slight increase in solution time
- **Rank1+Hier** (vs. **Rank1**): No increase in solution time

Special Case: Tridiagonal Q

$$Q = \begin{pmatrix} * & * & 0 & 0 & \dots & 0 \\ * & * & * & 0 & \dots & 0 \\ 0 & * & * & * & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \end{pmatrix}$$

Support graph of Q: Arc (i,j) for $i \le j$ with $Q_{ij} \neq 0$



Special Case: Tridiagonal Q

$$\begin{aligned} \min_{x,z}\ & a^\top x + b^\top z + \frac{1}{2} x^\top Q x \qquad\qquad (Q \succ 0,\ \text{tridiagonal}) \\ \text{s.t.}\ & x \circ (\mathbf{1} - z) = \mathbf{0} \\ & x \in \mathbb{R}^n,\ z \in \{0,1\}^n \end{aligned}$$

Suppose $z = \mathbf{1}$

- Optimality condition: Solve $Qx = -a$

- Thomas Algorithm for tridiagonal Q takes $O(n)$ time

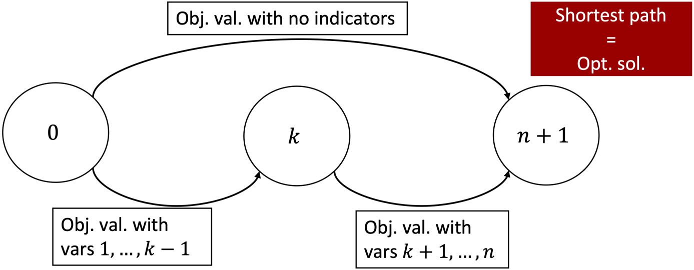
Single indicator: $z_k \in \{0, 1\}$

Now suppose $z_j = 1$ for $j \in [n] \setminus \{k\}$:

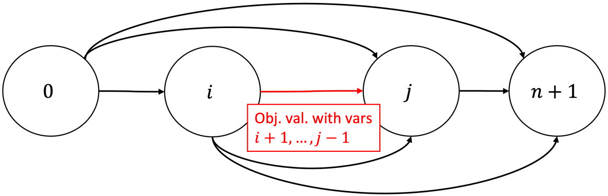
$$\min_{x, z_k} a^\top x + b_k z_k + \frac{1}{2} \sum_{i \in [n]} Q_{ii} x_i^2 + \sum_{i \in [n-1]} Q_{i,i+1} x_i x_{i+1}$$

$$\text{s.t. } x_k(1 - z_k) = 0$$

$$x \in \mathbb{R}^n, \ z_k \in \{0, 1\}$$



All indicators: $z \in \{0,1\}^n$



Proposition (Liu, Fattahi, Gómez, Küçükyavuz, 2022)

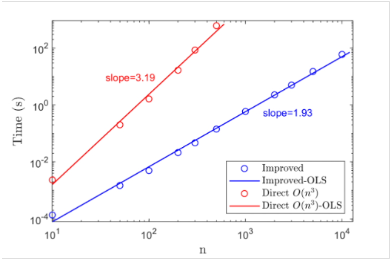
MIQP with tridiagonal matrices can be solved by solving a shortest path problem.

Complexity:

- Direct: $O(n^2)$ arcs $\times O(n)$ arc cost calculation $= O(n^3)$
- Improved: $O(n^2)$

Leads to a shortest path-based compact tight extended formulation.

Experiments



n	50	100	200	500	1000	10000
Improved	**<0.01s**	**<0.01s**	**<0.01s**	**0.14s**	**0.59s**	**59.6s**
Direct	0.16s	1.63s	18.9s	608.9s	>3600s	>3600s
Big-M	0.20s	214.1s	>3600s	>3600s	>3600s	>3600s

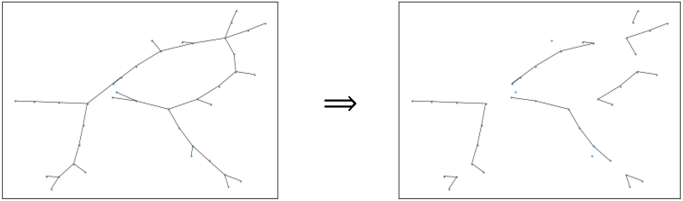
Can we leverage this efficient algorithm to solve the problem for non-tridiagonal $Q \succ 0$?

Sparse, Strictly Diagonally Dominant Matrix Q

Idea: Split Q into tridiagonal submatrices: $T_1, \ldots, T_\ell$ and a remainder R of off-tridiagonals

- Use convexification and Fenchel duality for off-tridiagonals

- Decomposes to path subproblems ($O(n^2)$ algorithm)



Convexification and Fenchel duality

Rewriting the problem

$$\min_{x,z,t} a^\top z + b^\top z + \frac{1}{2}\sum_{k=1}^{\ell} t_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n} |Q_{i,j}|(x_i \pm x_j)^2$$

$$\text{s.t. } t_k \geq x^\top T_k x, k = 1, \dots, \ell, x \circ (\mathbf{1} - z) = \mathbf{0}, x \in \mathbb{R}^n, z \in \{0,1\}^n, t \in \mathbb{R}^\ell.$$

Convexify the rank-one terms to obtain relaxation objective

$$\min_{x,z,t} a^\top x + b^\top z + \frac{1}{2}\sum_{k=1}^{\ell} t_k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n} |Q_{i,j}|\frac{(x_i \pm x_j)^2}{\min\{1, z_i + z_j\}}$$

Relax complicating terms via Fenchel dual to obtain relaxation

$$\zeta_p = \min_{x,z,t}\ \max_{\alpha,\beta}\ a^\top x + b^\top z + \frac{1}{2}\sum_{k=1}^{\ell} t_k$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n} |Q_{ij}|\Big(\alpha_{ij}(x_i \pm x_j) - \beta_{ij,i} z_i - \beta_{ij,j} z_j - f^*(\alpha_{ij}, \beta_{ij,i}, \beta_{ij,j})\Big)$$

Fenchel Decomposition

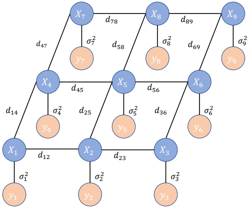
Strong duality holds, so

$$\zeta_p = \max_{\alpha,\beta} -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=i+2}^{n} |Q_{ij}| f^*(\alpha_{ij}, \beta_{ij,i}, \beta_{ij,j}) + \min_{x,z,t} \left\{\psi(x, z, t, \alpha, \beta)\right\}$$

- Inner min Independent tridiagonal problems
- Outer max Subgradient ascent

Computational Study

Inference with graphical models



- Given: noisy observations (orange)
- Goal: find true values (blue)
- Arc (i, j): connection between variables i, j with $Q_{ij} \neq 0$

Computational Results

$n = 100$

Noise	Low		High	
Method	Time(s)	Gap	Time(s)	Gap
Decomposition	**0.1**	**<1%**	**0.5**	**<1%**
Big-M	0.3	0.0%	3600	4.7%

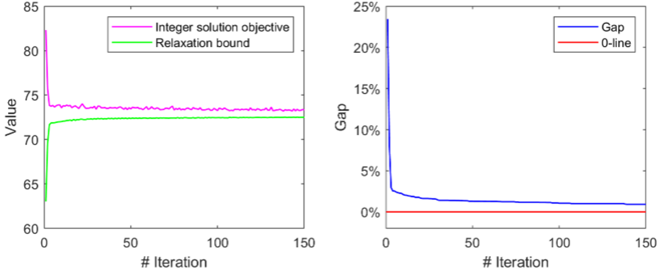
$n = 1600$

Noise	Low		High	
Method	Time(s)	Gap	Time(s)	Gap
Decomposition	**43.6**	**<1%**	**190.5**	**1.0%**
Big-M	3600	3.9%	3600	30.9%

Gap=(Upper Bound- Lower Bound)/Upper Bound

Decomposition Method

$n = 1600$, high noise



Research Questions

- Can we exploit matrix and constraint structure to obtain stronger relaxations? (Part 1) ✓

- What does strong mean for MIQP?
  Can we leverage polyhedral theory for MIQP? (Part 2)

Agenda

General $Q \succ 0$

A Combinatorial View

For a subset $S \in Z$, $z_i = 1$ if $i \in S$ and Q_S is the submatrix of Q indexed by S (similarly a_S, b_S, x_S)

$$\begin{aligned} \min_x \ & a^\top x + b_S + \frac{1}{2} x^\top Q x \quad = \quad \min_{x_S} a_S^\top x_S + b_S + \frac{1}{2} x_S^\top Q_S x_S \\ \text{s.t. } & x_i = 0, \ \forall i \notin S. \end{aligned}$$

- $x_S^* = -Q_S^{-1} a_S$.
- A combinatorial problem of selecting subset S

$$\min_{S \subseteq [n]} \ b_S - \frac{1}{2} a_S^\top Q_S^{-1} a_S$$

Notation

- Given $S \subseteq \{1, \dots, n\}$:
  - e_S = n-dimensional indicator vector of S
  - Q_S = $|S| \times |S|$ submatrix of Q induced by S
  - $\hat{Q}_S^{-1}$ = $n \times n$ matrix corresponding to Q_S^{-1} in the rows/columns of S, and 0 elsewhere

Example: $Q = \begin{pmatrix} d_1 & 1 \\ 1 & d_2 \end{pmatrix}$

$d_1 d_2 > 1$

S	e_S	Q_S	$\hat{Q}_S^{-1}$
$\emptyset$	$(0\ 0)$	$\emptyset$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\{1\}$	$(1\ 0)$	(d_1)	$\begin{pmatrix} 1/d_1 & 0 \\ 0 & 0 \end{pmatrix}$
$\{2\}$	$(0\ 1)$	(d_2)	$\begin{pmatrix} 0 & 0 \\ 0 & 1/d_2 \end{pmatrix}$
$\{1,2\}$	$(1\ 1)$	$\begin{pmatrix} d_1 & 1 \\ 1 & d_2 \end{pmatrix}$	$\frac{1}{d_1 d_2 - 1}\begin{pmatrix} d_2 & -1 \\ -1 & d_1 \end{pmatrix}$

Structure of the convex hull (extended formulation)

$$X \equiv \left\{(z,x,t) \in Z \times \mathbb{R}^{n+1} \mid t \geq x^\top Q x,\ x \circ (\mathbf{1} - z) = \mathbf{0}\right\}.$$

$$P \equiv \operatorname{conv}\left(\left\{(e_S, \hat{Q}_S^{-1})_{S \in Z}\right\}\right).$$

Theorem (Wei, Atamtürk, Gómez and Küçükyavuz, 2022)

If Q is positive definite, then

$$\operatorname{cl}\operatorname{conv}(X) = \left\{(z,x,t) \in [0,1]^n \times \mathbb{R}^{n+1} \mid \exists W \in \mathbb{R}^{n\times n},\ \begin{pmatrix} W & x \\ x^\top & t \end{pmatrix} \succeq \mathbf{0}, (z, W) \in P\right\}.$$

Can be extended to the psd/low rank case (a more compact extended formulation)

Preliminaries

Definition

Given a matrix $W \in \mathbb{R}^{p \times q}$, its pseudoinverse $W^\dagger \in \mathbb{R}^{q \times p}$ is the unique matrix satisfying: $WW^\dagger W = W$, $W^\dagger W W^\dagger = W^\dagger$, $(WW^\dagger)^\top = WW^\dagger$, $(W^\dagger W)^\top = W^\dagger W$.

Examples

- if W is invertible then $W^\dagger = W^{-1}$
- $W = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $W^\dagger = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix}$

Lemma (Generalized Schur Complement)

$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{12}^\top & U_{22} \end{pmatrix}$ with $U_{11} \in \mathcal{S}^{m \times m}$ and $U_{22} \in \mathcal{S}^{n \times n}$, and $U_{12} \in \mathbb{R}^{m \times n}$. Then $U \succeq \mathbf{0}$ if and only if $U_{11} \succeq \mathbf{0}$, $U_{11} U_{11}^\dagger U_{12} = U_{12}$ and $U_{22} - U_{12}^\top U_{11}^\dagger U_{12} \succeq \mathbf{0}$.

Recall

Theorem (Wei, Atamtürk, Gómez and Küçükyavuz, 2022)

If Q is positive definite, then

$$\text{cl conv}(X) = \left\{(z,x,t) \in [0,1]^n \times \mathbb{R}^{n+1} \mid \exists W \in \mathbb{R}^{n\times n} \ \begin{pmatrix} W & x \\ x^\top & t \end{pmatrix} \succeq \mathbf{0},\ (z,W) \in P\right\}.$$

Proof idea: Optimizing over cl conv(X) is equivalent to optimizing the original problem.

Proof of the Theorem

Optimizing over cl conv(X)

$$\begin{aligned} \min_{x,z,W} \quad & a^\top x + b^\top z + t \\ \text{s.t.} \quad & \begin{pmatrix} W & x \\ x^\top & t \end{pmatrix} \succeq \mathbf{0} \\ & (z, W) \in P \equiv \text{conv}\left(\left\{(e_S, \hat{Q}_S^{-1})_{S \in Z}\right\}\right) \end{aligned}$$

- $z = e_S$ for some $S \in Z$
- $W = \hat{Q}_S^{-1} = \begin{pmatrix} Q_S^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \succeq \mathbf{0}$ and $W^\dagger = \begin{pmatrix} Q_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$
- $\color{blue}{WW^\dagger x = x} \Leftrightarrow \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x_S \\ x_{[n]\setminus S} \end{pmatrix} = \begin{pmatrix} x_S \\ x_{[n]\setminus S} \end{pmatrix} \Leftrightarrow x_{[n]\setminus S} = \mathbf{0}$
- $\color{red}{t \geq x^\top W^\dagger x} \Leftrightarrow t \geq x_S^\top Q_S x_S$

Example: Quadratic with "Choose-one" constraint

$$X_{C_1} = \{(z,x,t) \in \{0,1\}^n \times \mathbb{R}^{n+1} \mid t \geq x^\top Q x,\ x \circ (\mathbf{1} - z) = \mathbf{0},\ \textstyle\sum_{i=1}^n z_i \leq 1\}$$

Corollary

$$\operatorname{cl}\operatorname{conv}(X_{C_1}) = \left\{(z,x,t) \in \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R} \mid t \geq \textstyle\sum_{i=1}^n Q_{ii} \frac{x_i^2}{z_i},\ \sum_{i=1}^n z_i \leq 1\right\}.$$

- $P = \operatorname{conv}\left(\left\{(0,\mathbf{0}), (e_{\{i\}}, \hat{Q}_{\{i\}}^{-1})_{i=1}^n\right\}\right)$
- $P = \{(z,W) \mid W_{ij} = 0, i \neq j, W_{ii} = \frac{z_i}{Q_{ii}}, i = 1, \ldots, n\}$

$$\begin{pmatrix} W & x \\ x^\top & t \end{pmatrix} \succeq \mathbf{0}, (z,W) \in P \Leftrightarrow \begin{pmatrix} \frac{z_1}{Q_{11}} & \ldots & 0 & x_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \frac{z_n}{Q_{nn}} & x_n \\ x_1 & \ldots & x_n & t \end{pmatrix} \succeq \mathbf{0} \Leftrightarrow t \geq \sum_{i=1}^n Q_{ii} \frac{x_i^2}{z_i}$$

Structure of the convex hull (original space)

Let

$$X = \left\{(x,z,t) \in \mathbb{R}^n \times Z \times \mathbb{R} : t \geq x^\top Q x,\ x \circ (\mathbf{1} - z) = \mathbf{0}\right\}$$

Suppose a minimal description of P is given by

$$\begin{aligned} \langle \Gamma_i, W \rangle - \gamma_i^\top z &\leq \beta_i, \quad i = 1, \ldots, m_1 \\ \langle \Gamma_i, W \rangle - \gamma_i^\top z &= \beta_i, \quad i = m_1 + 1, \ldots, m. \end{aligned}$$

Theorem (Wei, Atamtürk, Gómez, Küçükyavuz, 2022)

$(x,z,t) \in cl\ conv(X)$ iff $z \in \mathrm{conv}(Z)$, $t \geq 0$ and

$$t \geq \frac{x^\top \left(\sum_{i=1}^m \Gamma_i s_i\right) x}{\beta^\top s + \left(\sum_{i=1}^m \gamma_i s_i\right)^\top z}$$

for all $s \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m-m_1}$ such that $\sum_{i=1}^m \Gamma_i s_i \succeq 0, \sum_{i=1}^m Tr(\Gamma_i) s_i \leq 1$.

Observations

- Semi-infinite conic quadratic program, but "finitely" generated by $(\Gamma_i, \gamma_i, \beta_i), i \in [m]$.
- The strongest conic quadratic inequality is given by

$$t \geq \max_{s \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m-m_1}} \frac{x^\top \left(\sum_{i=1}^m \Gamma_i s_i\right) x}{\beta^\top s + \left(\sum_{i=1}^m \gamma_i s_i\right)^\top z}$$

$$\text{s.t.} \sum_{i=1}^m \Gamma_i s_i \succeq 0, \ \sum_{i=1}^m \mathrm{Tr}(\Gamma_i) s_i \leq 1.$$

- How to work with $P = \mathrm{conv}\left(\left\{(e_S, \hat{Q}_S^{-1})_{S \in Z}\right\}\right)$ in practice?
  We give an MILP formulation for $\left\{(e_S, \hat{Q}_S^{-1})_{S \in Z}\right\}$.
  Preliminary tests show that this MILP is faster than perspective for some instances.

Research Questions

- Can we exploit matrix and constraint structure to obtain stronger relaxations? (Part 1) ✓

- What does strong mean for MIQP?
  Can we leverage polyhedral theory for MIQP? (Part 2) ✓

Agenda

Conclusions

- We characterize the convex hulls of MIQPs with indicators
- Convexification reduces to finding a facial description of a polytope
- We can use any tools from MILP to do so
- We can use polyhedral theory to understand strength of convexifications
- Offers insights into design of algorithms

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