Stochastically Constrained Optimization via Simulation

Seong-Hee Kim

Georgia Institute of Technology

December 4, 2010

Problem

Notation

- $\mathbf{x}$: a solution with discrete decision variables.
- Θ: the set of possible solutions $\mathbf{x}$.

Want to find:

$$\begin{aligned} &\text{argmin}_{\mathbf{x}\in\Theta} g(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}[Y(\mathbf{x})] \\ &\text{subject to } h(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}[C(\mathbf{x})] \geq q \end{aligned}$$

- $g(\mathbf{x})$ and $h(\mathbf{x})$ are estimated only through simulation.
- Existing OvS algorithms are designed without consideration for stochastic constraints.

Set Up

- One replication generates one observation $Y(\mathbf{x})$ and $C(\mathbf{x})$.
- At search iteration k, a number of solutions are generated and a number of replications are made for each generated solution.
- Let $n_k(\mathbf{x})$ represent the total number of replications made up to search iteration k for $\mathbf{x}$.
- Let $v_k(\mathbf{x})$ represent the number of visits to $\mathbf{x}$ made up to search iteration k.

- $\widehat{g}_{v_k}(\mathbf{x})$ and $\widehat{h}_{v_k}(\mathbf{x})$ are sample averages of all available observations up to v_k visits (i.e., sample averages of n_k observations).
- Define the penalty factor, $\lambda_{v_k}(\mathbf{x})$ as

$$\lambda_{v_k}(\mathbf{x}) = \begin{cases} \lambda_{v_k-1}(\mathbf{x}) \times \theta_1, & \text{if } \widehat{h}_{v_k}(\mathbf{x}) \geq q; \\ \lambda_{v_k-1}(\mathbf{x}) \times \theta_2, & \text{if } \widehat{h}_{v_k}(\mathbf{x}) < q, \end{cases}$$

where $\lambda_0(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Theta$, $0 < \theta_1 < 1$, and $\theta_2 > 1$.

Penalized Objective

The penalized objective at search iteration k is defined as

$$Z_{v_k}(\mathbf{x}) = \widehat{g}_{v_k}(\mathbf{x}) + \lambda_{v_k}(\mathbf{x}) \times \max\{0, q - \widehat{h}_{k_v}(\mathbf{x})\} \times P,$$

where P is a penalty constant and can be one.

Why does it work?

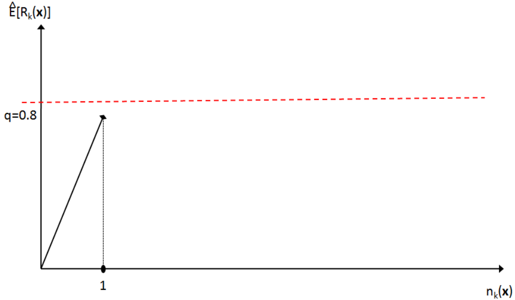
- If an OvS algorithms satisfy $v_k(\mathbf{x}) \to \infty$ and $n_k(\mathbf{x}) \to \infty$ as $k \to \infty$, then

  $Z_{v_k}(\mathbf{x}) \to g(\mathbf{x})$ w.p. 1, if $\mathbf{x}$ is strictly feasible; and
  $Z_{k_k}(\mathbf{x}) \to \infty$ w.p. 1, otherwise.

Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

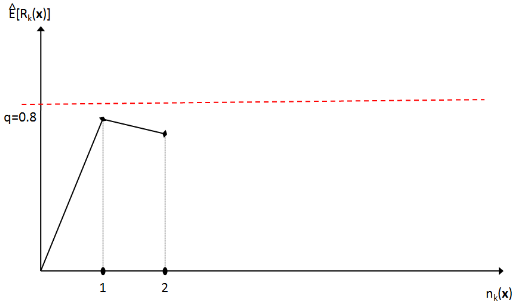
- Observe $\widehat{h}_1(\mathbf{x}) = 0.75$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_1(\mathbf{x}) = \lambda_0(\mathbf{x}) \times \theta_2 = 1 \times 10 = 10$.
- $\lambda_1(\mathbf{x}) \times \max\{0, q - \widehat{h}_1(\mathbf{x})\} = 10 \times \max\{0, 0.10\} = 1$.



Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

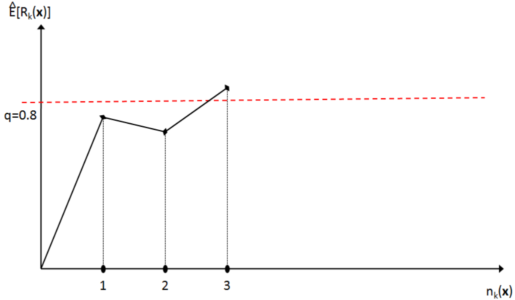
- Observe $\widehat{h}_2(\mathbf{x}) = 0.70$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_2(\mathbf{x}) = \lambda_1(\mathbf{x}) \times \theta_2 = 10 \times 10 = 100$.
- $\lambda_2(\mathbf{x}) \times \max\{0, q - \widehat{h}_2(\mathbf{x})\} = 100 \times \max\{0, 0.10\} = 10$.



Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

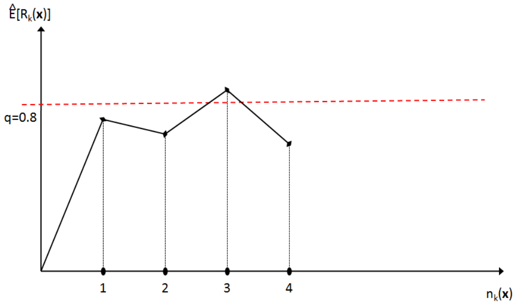
- Observe $\widehat{h}_3(\mathbf{x}) = 0.84$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_3(\mathbf{x}) = \lambda_2(\mathbf{x}) \times \theta_1 = 100 \times 0.1 = 10$.
- $\lambda_3(\mathbf{x}) \times \max\{0, q - \widehat{h}_3(\mathbf{x})\} = 10 \times \max\{0, -0.04\} = 0$.



Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

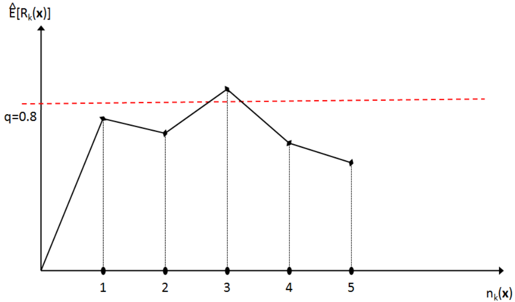
- Observe $\widehat{h}_4(\mathbf{x}) = 0.65$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_4(\mathbf{x}) = \lambda_3(\mathbf{x}) \times \theta_2 = 10 \times 10 = 100$.
- $\lambda_4(\mathbf{x}) \times \max\{0, q - \widehat{h}_4(\mathbf{x})\} = 100 \times \max\{0, 0.15\} = 15$.



Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

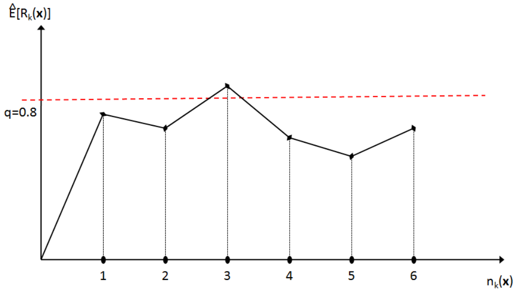
- Observe $\widehat{h}_5(\mathbf{x}) = 0.60$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_5(\mathbf{x}) = \lambda_4(\mathbf{x}) \times \theta_2 = 100 \times 10 = 1000$.
- $\lambda_5(\mathbf{x}) \times \max\{0, q - \widehat{h}_5(\mathbf{x})\} = 1000 \times \max\{0, 0.20\} = 200$.



Example

Constraint: Contaminant spill detection reliability $h(\mathbf{x}) \geq 0.8$.
Consider a solution with $h(\mathbf{x}) = 0.65$ and set $P = 1$.

- Observe $\widehat{h}_6(\mathbf{x}) = 0.65$ ($\theta_1 = 0.1$ and $\theta_2 = 10$).
- $\lambda_6(\mathbf{x}) = \lambda_5(\mathbf{x}) \times \theta_2 = 1000 \times 10 = 10000$.
- $\lambda_6(\mathbf{x}) \times \max\{0, q - \widehat{h}_6(\mathbf{x})\} = 10000 \times \max\{0, 0.15\} = 1500$.



Multiple Constraints

$$\text{argmin}_{\mathbf{x}\in\Theta} g(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}[Y(\mathbf{x})],$$
$$\text{subject to } h^i(\mathbf{x}) = \mathbf{E}_{\mathbf{x}}[C_i(\mathbf{x})] \geq q_i \quad i = 1, 2, \ldots, s$$

$$\lambda^i_{v_k}(\mathbf{x}) = \begin{cases} \lambda^i_{v_k-1}(\mathbf{x}) \times \theta_1, & \text{if } \widehat{h}^i_{v_k}(\mathbf{x}) \geq q^i; \\ \lambda^i_{v_k-1}(\mathbf{x}) \times \theta_2, & \text{if } \widehat{h}^i_{v_k}(\mathbf{x}) > q^i, \end{cases}$$

where $\lambda^i_0(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Theta$, $0 < \theta_1 < 1$, and $\theta_2 > 1$.

$$Z_{v_k}(\mathbf{x}) = \widehat{g}_{v_k}(\mathbf{x}) + \sum_{i=1}^{s} \lambda^i_{v_k}(\mathbf{x}) \times \max\{0, q - \widehat{h}^i_{v_k}(\mathbf{x})\} \times P^i.$$

Issues

- What if $h^i(\mathbf{x}) = q_i$?
- Choice of θ_1 and θ_2: Make it as a function of n?
  - $0 < \theta_1 < 1$: Start with a large number then decrease it as n increases.
  - $\theta_2 > 1$: Start with a small number and increase it as n increases.
- Give some tolerance for feasibility decision?
  - feasible if $\widehat{h}_{v_k}(\mathbf{x}) \geq q - \epsilon$; and
  - infeasible if $\widehat{h}_{v_k}(\mathbf{x}) \leq q + \epsilon$.
- Incorporate constrained ranking and selection?
- Other issues?