A Tale of Two Topics: (i) SAA Review and (ii) Testbed Update.

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PART I

Problem Statement

$$\begin{aligned} \text{minimize} \quad & g(x) \\ \text{subject to} \quad & h(x) \geq 0, \\ & x \in \mathcal{D} \subset \mathbb{R}^q, \end{aligned}$$

where

- $g : \mathcal{D} \to \mathbb{R}$ can only be estimated using the "black box" estimator G_m, where $G_m(x) \Rightarrow g(x)$ for all $x \in \mathcal{D}$ and m is some measure of simulation effort;
- $h : \mathcal{D} \to \mathbb{R}^n$ can only be estimated using the "black box" estimator H_m, where $H_m(x) \Rightarrow g(x)$ for all $x \in \mathcal{D}$ and m is some measure of simulation effort;
- $\mathcal{D} \subseteq \mathbb{R}^q$ is a known set, e.g., the non-negative orthant.

Notes and Some Notation

- The case of known h has been studied far more.
- The feasible region resulting from the constraints h and the region $\mathcal{D}$ are usually assumed to be closed and convex.
- Denote (π^*, v^*) as the set of global minima and the global minimum value corresponding to the problem. Denote λ^* as the set of local minima (appropriately defined) of the problem.
- Usually an element of π^* or an element of λ^* is requested.

Sample Average Approximation (SAA)

Logic:

1. "Generate" a sample-path problem with sample size m.
2. Use a procedure to "solve" the sample-path problem

$$\begin{aligned} \text{minimize} \quad & G_m(x) \\ \text{subject to} \quad & h(x) \geq 0, \\ & x \in \mathcal{D} \subset \mathbb{R}^q. \end{aligned}$$

Algorithm Parameters:

(i) procedure for solving the sample-path problems;

(ii) sample size m;

(iii) if sample-path problem can only be solved numerically, the error-tolerance ϵ to within which the sample-path problem should be solved.

SAA Refinement — Retrospective Approximation (RA)

Logic:

1. "Generate" kth sample-path problem with sample-size m_k.
2. Use a procedure to solve the kth sample-path problem to within error-tolerance ϵ_k. Obtain a retrospective solution X_k.
3. Weight obtained solutions to get

$$\overline{X}_k = \sum_{j=1}^{k} w_j X_j, \, w_j \geq 0, \sum_{j=1}^{k} w_j = 1.$$

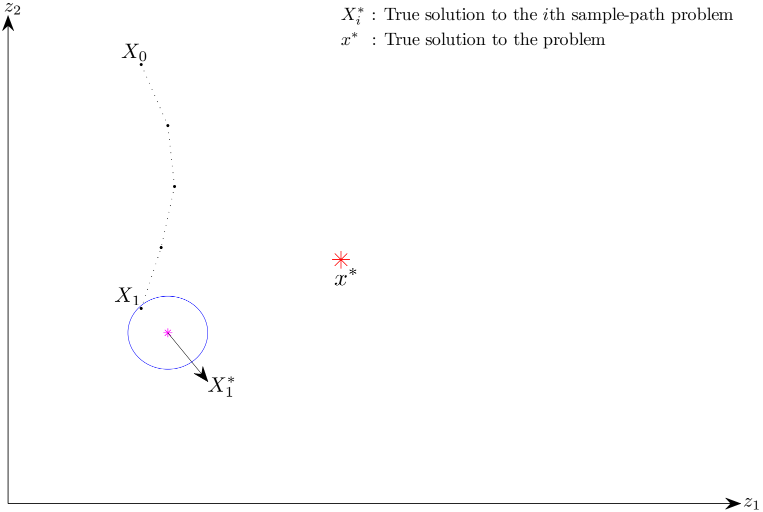
4. Update $k = k + 1$ and goto Step 1.

Algorithm Parameters:

(i) procedure for solving the sample-path problems;
(ii) sample-size sequence $\{m_k\}$;
(iii) error-tolerance sequence $\{\epsilon_k\}$.

Retrospective Approximation

X_i : Approx. solution to the ith sample-path problem
X_i^* : True solution to the ith sample-path problem
x^* : True solution to the problem



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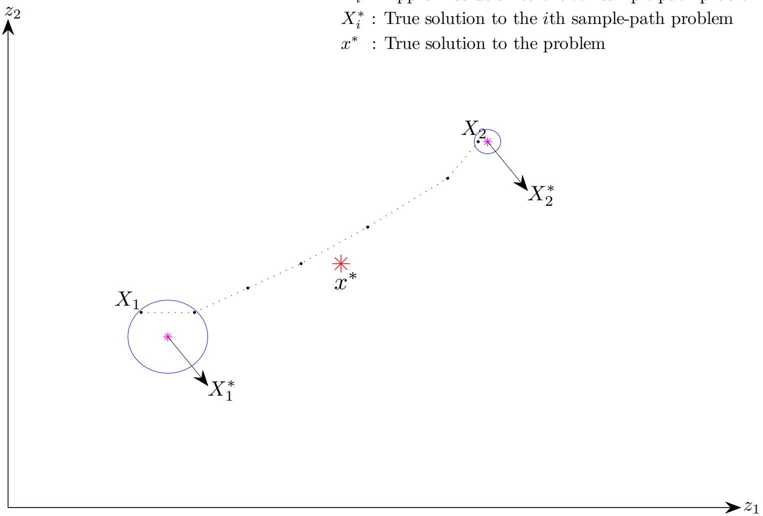
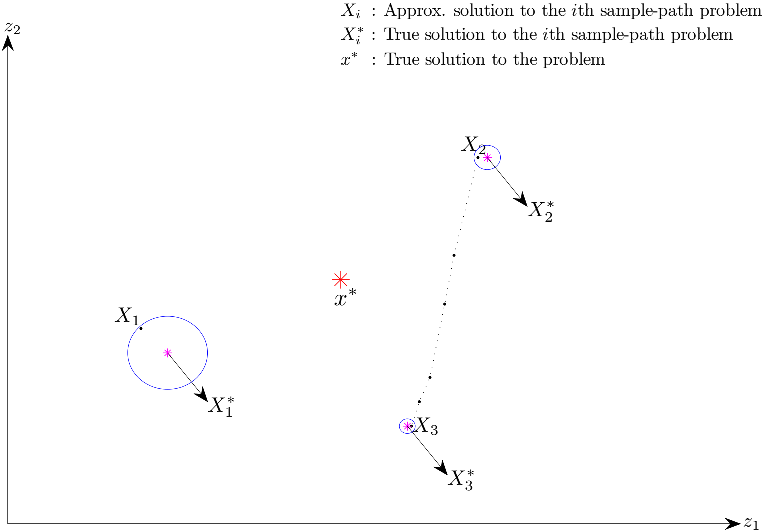


Retrospective Approximation



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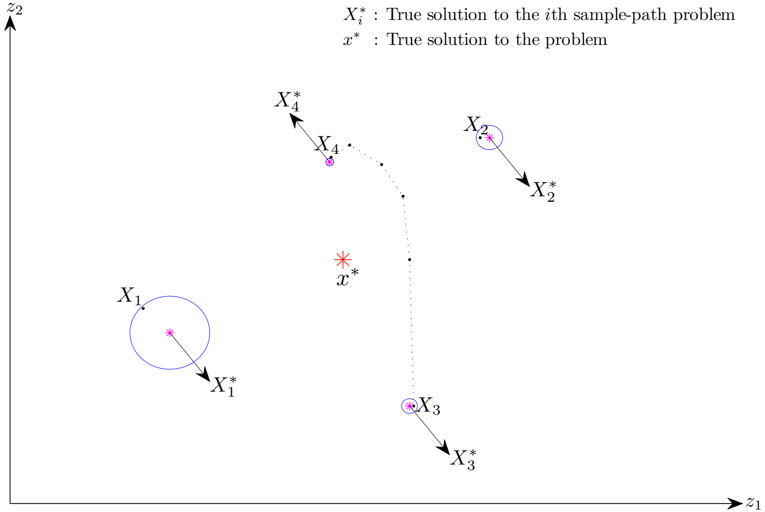


SAA and RA — When?

Advantages

1. When sample-path problems have structure so that generic search procedures are guaranteed to work well.
2. When sample-path problems have special structure that is known and can be utilized for efficiency. (Surprising counterexample provided by Nemirovski et al. [9].)
3. Advances in deterministic math. programming at our disposal, in principle.

Disadvantages

1. When the user cannot be expected to choose an appropriate procedure to solve sample-path problems.
2. Sample-paths are poorly behaved or have no known structure, and so choice of procedure is unclear.
3. Incorporation of variance reduction techniques can be difficult.

An Outline of Key Results

1. Consistency
   - Convergence of optimal value (SAA and RA).
   - Convergence of optimal solution (SAA and RA).
2. Speed of Convergence
   - CLT-type results for optimal value (SAA and RA).
   - CLT-type results for optimal solutions (SAA and RA).
   - Results under special conditions.
3. Algorithmic Results
   - Minimum sample size results (SAA).
   - Quality of solution/confidence interval type results (SAA).
   - Parameter choice results (RA).
4. Results relating to stochastic feasible regions.

Consistency (SAA and RA)

Theorem (Shapiro [14])

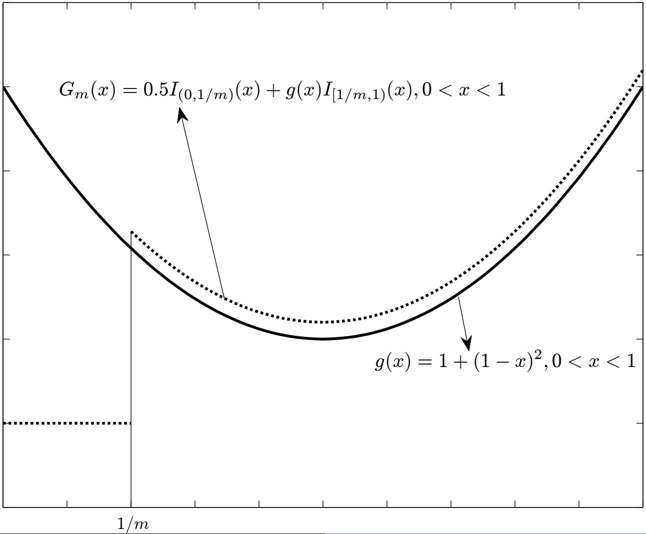
1. If $\lim_{m\to\infty}\sup_{x\in\mathcal{D}}|g(x)-G_m(x)|=0$ wp1, then $V_m^*\to v^*$ wp1.
2. If $\lim_{m\to\infty}\sup_{x\in\mathcal{D}}|g(x)-G_m(x)|=0$ wp1, $\mathcal{D}$ is compact, and g is continuous, then $\lim_{m\to\infty}\text{dist}(\pi^*,\Pi_m^*)\to 0$ wp1.

– Results carry over to the RA context in a straightforward manner.

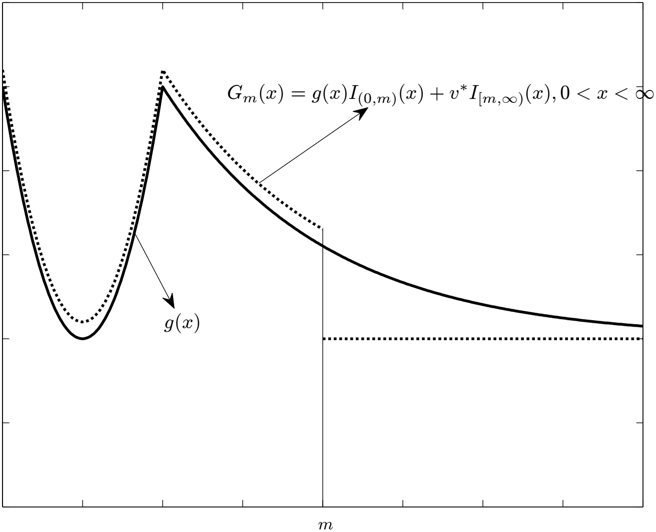
– Corresponding results for local minima provided by Bastin et al. [1]

– Results on epiconvergence provided by Dupačová and Wets [4], and Robinson [12].

Consistency — Optimal Value



Consistency — Optimal Solution



Speed of Convergence (SAA and RA) — Optimal Value

Theorem (Pflug [11], Shapiro [14])

Suppose $\beta(m)$ is a function satisfying $\lim_{m\to\infty}\beta(m) = \infty$ such that $\beta(m)(G_m - g) \Rightarrow Y(x) \in \mathcal{C}(\mathcal{D})$, where $\mathcal{C}(\mathcal{D})$ is the linear space of continuous functions on $\mathcal{D}$. Then,

$$\beta(m)(V_m^* - v^*) \Rightarrow \min_{x\in\pi^*} Y(x).$$

- Rate of convergence of "black box" estimator transferred over to optimal value.
- When g is an expectation, the functional CLT condition is satisfied with a Lipschitz condition on G_m, where the Lipschitz constant has finite second moment.
- Corresponding CLT on optimal solution can be found in King and Rockefellar [5], and Shapiro [13].

Speed of Convergence — Important Special Cases

Theorem (Kleywegt et al. [6])

Let $\mathcal{D}$ be finite and $g(x) = E[G(x)]$, $G_m(x) = \sum_{i=1}^{m} Y_i(x)$ where $Y_1, Y_2, \ldots$ are iid copies of a random variable $Y(x)$. Denote $\pi^*(\epsilon) = \{x : g(x) - v^* \le \epsilon\}$ and $\Pi_m^*(\delta) = \{x : G_m(x) - V_m^* \le \delta\}$. Then, if $\delta \le \epsilon$,

$$\Pr\{\Pi_m^*(\delta) \not\subseteq \pi^*(\epsilon)\} \le |\pi^* \setminus \pi^*(\epsilon)| \exp\{-m\gamma(\delta, \epsilon)\},$$

where $\gamma(\delta, \epsilon) = \min_{x \in \pi^* \setminus \pi^*(\epsilon)} I_x(-\delta)$, $I_x(\cdot)$ being the rate function associated with the sequence $\{G_m(x)\}$.

- The probability of not obtaining an ϵ-optimal solution drops exponentially.
- This result forms the essence of most minimum sample size results [7, 14].

Speed of Convergence — Important Special Cases

Theorem (Shapiro and Homem-de-Mello [15])

Let g be a finite-valued function having a sharp minimum, i.e., g satisfies $g(x) \geq g(x^*) + c\|x - x^*\|$ for $x \in \mathcal{D}$, where c is a positive constant and x^* is the unique minimum. Let $g(x) = E[G(x)]$ for $G_m(x) = \sum_{i=1}^{m} Y_i(x)$, where $Y_1(x), Y_2(x), \ldots$ are iid copies of a random variable $Y(x)$ having fixed finite support. Then, if G is convex, and the set $\mathcal{D}$ is closed and convex, $\Pi_m^* = \{x^*\}$ for large enough m wp1.

- The result is easily extended to contexts where π^* is not a singleton.
- The earlier result on exponential convergence holds in this special case as well.

Results for Solution Quality

Theorem (Mak et al. [8])

Let $g(x) = E[G(x)]$ for $G_m(x) = \sum_{i=1}^{m} Y_i(x)$, where $Y_1(x), Y_2(x), \ldots$ are iid copies of a random variable $Y(x)$. Then,

1. $E[V_m^*] \leq v^*$;
2. $E[V_{m+1}^*] \geq E[V_m^*]$;
3. If $x \in \mathcal{D}$, $0 \leq g(x) - v^* \leq g(x) - E[V_m^*]$.

– The result is very general, e.g., $\mathcal{D}$ need not be convex. (See also Birge [3] for related results.)
– Mak et al. [8] use the above result to construct confidence intervals on the optimality gap of a candidate solution.
– Bayraksan and Morton [2] extend this to a sequential procedure for constructing confidence intervals.

Parameter Choice in RA

How to choose the sequence of sample sizes $\{m_k\}$, and the sequence of error tolerances $\{\epsilon_k\}$ in RA? Consider the following three conditions.

C.1. When the numerical procedure used to solve sample-path problems exhibits

(a) linear convergence: $\liminf_{k\to\infty} \epsilon_k \sqrt{m_{k-1}} > 0$;

(b) polynomial convergence: $\liminf_{k\to\infty} \frac{\log\left(1/\sqrt{m_{k-1}}\right)}{\log(\epsilon_k)} > 0$.

C.2. $\limsup_{k\to\infty} \left(\sum_{j=1}^{k} m_j\right) \epsilon_k^2 < \infty$.

C.3. $\limsup_{k\to\infty} \left(\sum_{j=1}^{k} m_j\right) m_k^{-1} < \infty$.

Parameter Choice in RA

Theorem (Pasupathy [10])

If the sequences $\{\epsilon_k\}$, $\{m_k\}$ satisfy conditions C.1, C.2, and C.3, and $\pi^* = \{x^*\}$, then $W_k\|X_k - x^*\|^2 = O_p(1)$.

Theorem (Pasupathy [10])

If even one of the conditions C.1, C.2, or C.3 is violated, and $\pi^* = \{x^*\}$, $W_k\|X_k - x^*\|^2 \xrightarrow{p} \infty$.

	Exp. Growth ($m_k = e^{1.1 m_{k-1}}$)	Pol. Growth ($m_k = m_{k-1}^{1.1}$)	Lin. Growth ($m_k = 1.1 m_{k-1}$)
Pol. Conv.	N	Y	Y
Lin. Conv.	N	N	Y
S-Lin. Conv.	N	N	NA

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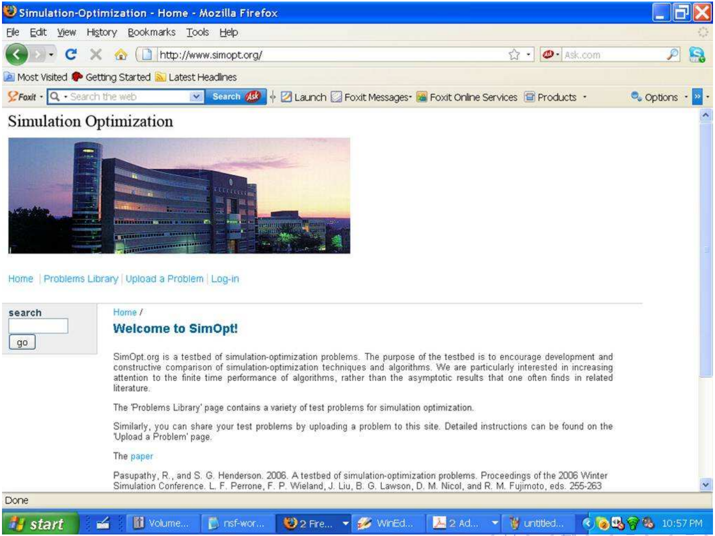
PART II

Introducing ... an SO Testbed

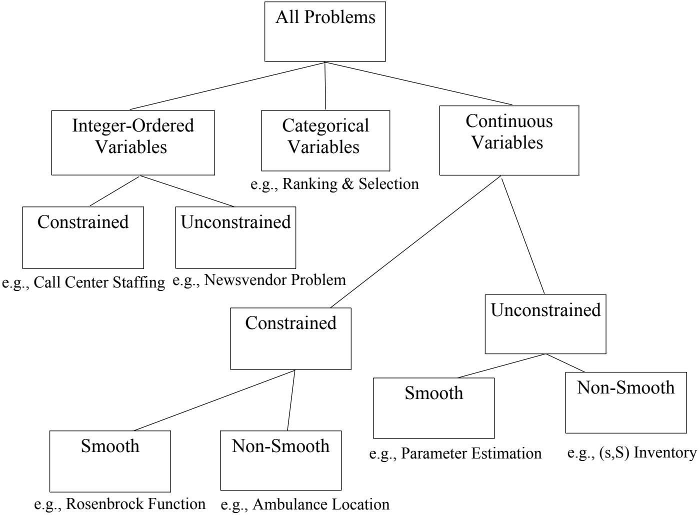


Recall Objectives

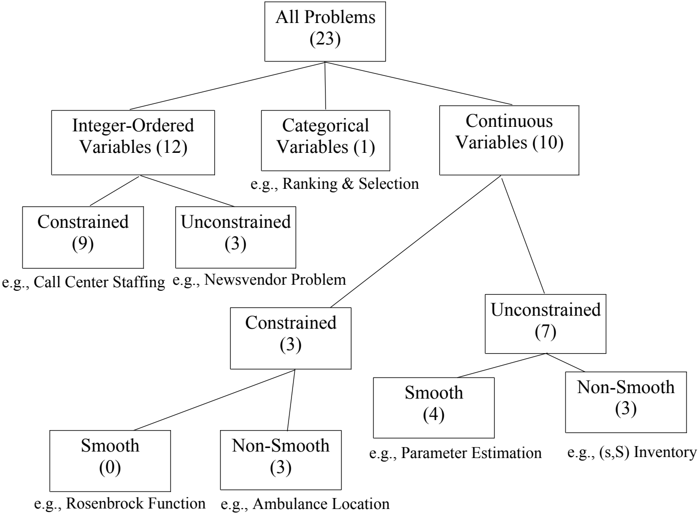
- Fill the stated and yet unfulfilled need for a carefully designed testbed of SO problems.

  (The stochastic programming community has a few of its own libraries, e.g., SIPLIB for stochastic integer programs, POSTS for linear recourse problems.)
- Actively draw attention to finite-time performance of algorithms, through the use of finite-time performance measures.
- Identify particular problem types that defy efficient solution.
- Increase visibility and usage of SO formulation and solution.

Problem Organization Within Testbed



Where do we stand?



Where do we stand and what's next?

1. Good: The testbed is close to having a critical mass of problems.
2. Bad: Not as much variety as we would like to see.
3. Bad: Small number of continuous-variable and categorical-variable problems.

1. One PhD and one undergrad comb past WSC proceedings.
2. Large source of potential continuous-variable problems: Approximate DP.

Group Discussion

1. What does the apparent dearth of problems tell us?
2. Should particular categories be coalesced?
3. Is it time yet to launch the testbed?
4. To submit or simply view, point your browser to www.simopt.org.