Autoregressive to Anything: Time-Series Input Processes for Simulation

Marne C. Cario
Department of Industrial, Welding and Systems Engineering
The Ohio State University, Columbus, OH 43210, USA

Barry L. Nelson
Department of Industrial Engineering and Management Sciences
Northwestern University, Evanston, IL 60208, USA

January 31, 1996

Abstract

We develop a model for representing stationary time series with arbitrary marginal distributions and autocorrelation structures and describe how to generate data based upon our model for use in a simulation.

Keywords: simulation, time series, input modeling

1 Introduction

Dependent input processes arise naturally in many applications; for example, the interarrival-time gaps between opening, reading, writing or closing a computer file are sometimes autocorrelated because file activity tends to occur in bursts. Stochastic simulation is often used to analyze such systems. Due to the lack of general-purpose methods for representing and generating dependent processes, simulation practitioners frequently use independent processes to approximate them. This practice may result in output performance measures that are seriously in error (see, for example, Livny et al. [6]).

In this paper we present a model for representing a stationary time-series input process $\{Y_t; t = 1, 2, \ldots\}$ with an arbitrary marginal distribution and any feasible autocorrelation structure specified through lag

p. We use a transformation-oriented approach to represent $\{Y_t\}$. This approach takes a process with a known autocorrelation structure, the base process $\{Z_t\}$, and transforms it to achieve the desired marginal distribution for the input process, $\{Y_t\}$. The target autocorrelation structure of $\{Y_t\}$ is obtained by adjusting the autocorrelation structure of the base process. In our model, the base process is a standardized Gaussian autoregressive process of order p (AR(p)), so we refer to $\{Y_t\}$ as an ARTA (AutoRegressive To Anything) process.

Song et al. [10] use a transformation-oriented approach with an AR(1) base process. However, they attempt to match only the lag-1 autocorrelation, and they use simulation to find the lag-1 autocorrelation of the base process that gives the desired lag-1 autocorrelation of the input process. This approach becomes computationally prohibitive if extended to more than two or three autocorrelations. Our approach matches $p \geq 1$ autocorrelations, and is computationally more efficient.

TES (Melamed [7]) is another transformation-oriented approach that attempts to match any arbitrary marginal distribution and autocorrelation structure, and that is guaranteed to match any feasible lag-1 autocorrelation. TES, which is implemented in the computer software package TEStool (Melamed et al. [8]), uses a series of autocorrelated uniform random variables as the base process. Users of TEStool interactively adjust the base process until they achieve an input process that approximates the desired characteristics. However, the adjustment is applied to the distribution of the noise term in the base process, which indirectly results in an adjustment to the autocorrelation structure of the input process. We believe that our model is both theoretically simpler and easier to use than TES because we directly change the autocorrelations of the base process to adjust the corresponding autocorrelations of the input process.

A completely different approach, for which there is a large literature, is to construct a time-series process that exploits properties that are specific to the particular marginal distribution of interest for $\{Y_t\}$. An example is Lewis, et al. [5] who construct time series with gamma marginals. The primary shortcoming of this type of approach is that it is not general: a different model is required for each marginal distribution of interest. In addition, the sample paths of these processes, while adhering to the desired marginal distribution and autocorrelation structure, sometimes have unexpected features.

We present our model in Section 2. In Section 3 we develop some relationships between the AR(p) base process and the input process. We then discuss how to use these relationships to select the autocorrelations for the base process that give the desired autocorrelations for the input process. In Section 4 we describe how to generate time series based upon our model for use as simulation inputs and we present some sample output. Our conclusions appear in Section 5.

2 Model

The goal of our model is to define a stationary time series $\{Y_t\}$ with the following properties:

1. $Y_t \sim F_Y, t = 1, 2, \ldots$, where F_Y is an arbitrary cumulative distribution function (cdf); and
2. $(\mathrm{Corr}[Y_t, Y_{t+1}], \mathrm{Corr}[Y_t, Y_{t+2}], \ldots, \mathrm{Corr}[Y_t, Y_{t+p}])' = (\rho_1, \rho_2, \ldots, \rho_p)' = \boldsymbol{\rho}$,

where F_Y and $\boldsymbol{\rho}$ are given. We represent $\{Y_t\}$ as a transformation of a standardized Gaussian AR(p) process.

ARTA Process

1. Let $\{Z_t; t = 1, 2, \ldots\}$ be a stationary Gaussian AR(p) process

$$Z_t = \alpha_1 Z_{t-1} + \alpha_2 Z_{t-2} + \cdots + \alpha_p Z_{t-p} + \varepsilon_t,$$

   where $\{\varepsilon_t\}$ is a series of independent $\mathrm{N}(0, \sigma^2)$ random variables and σ^2 is selected so that the marginal distribution of the $\{Z_t\}$ process is $\mathrm{N}(0, 1)$. Specifically,

$$\sigma^2 = 1 - \alpha_1 r_1 - \alpha_2 r_2 - \cdots - \alpha_p r_p,$$

   where $r_h = \mathrm{Corr}[Z_t, Z_{t+h}]$.

2. Define the ARTA process $Y_t = F_Y^{-1}[\Phi(Z_t)], t = 1, 2, \ldots$, where Φ is the standard normal cdf.

The transformation $F_Y^{-1}[\Phi(\cdot)]$ ensures that $\{Y_t\}$ has the desired marginal distribution F_Y, since $U = \Phi(Z_t)$ is uniformly distributed on the interval $(0,1)$, implying that $Y_t = F_Y^{-1}[U]$ has distribution F_Y by well-known properties of the inverse cdf. Therefore, the central problem is to select the autocorrelation structure $\mathbf{r} = (r_1, r_2, \ldots, r_p)'$ for the AR(p) process, $\{Z_t\}$, that gives the desired autocorrelation structure $\boldsymbol{\rho}$ for the input process, $\{Y_t\}$.

3 Properties of ARTA Processes

The autocorrelation structure of the AR(p) base process $\{Z_t\}$ directly determines the autocorrelation structure of the input process $\{Y_t\}$, since $\mathrm{Corr}[Y_t, Y_{t+h}] = \mathrm{Corr}\left\{F_Y^{-1}[\Phi(Z_t)], F_Y^{-1}[\Phi(Z_{t+h})]\right\}$. To adjust this correlation, we can restrict attention to adjusting $\mathrm{E}[Y_t Y_{t+h}]$, since

$$\mathrm{Corr}[Y_t, Y_{t+h}] = \frac{\mathrm{E}[Y_t Y_{t+h}] - (\mathrm{E}[Y])^2}{\mathrm{Var}[Y]},$$

and $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$ are fixed by F_Y. Then, since (Z_t, Z_{t+h}) has a standard bivariate normal distribution with correlation $\mathrm{Corr}[Z_t, Z_{t+h}] = r_h$, we have

$$\begin{aligned} \mathrm{E}[Y_t Y_{t+h}] &= \mathrm{E}\left\{F_Y^{-1}[\Phi(Z_t)] F_Y^{-1}[\Phi(Z_{t+h})]\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_Y^{-1}[\Phi(z_t)] F_Y^{-1}[\Phi(z_{t+h})] \varphi_{r_h}(z_t, z_{t+h}) dz_t dz_{t+h}, \end{aligned} \tag{1}$$

where φ_{r_h} is the standard bivariate normal probability density function (pdf) with correlation r_h. We are only interested in processes for which this expectation exists.

Observe from Equation (1) that the lag-h autocorrelation of $\{Y_t\}$ is a function only of the lag-h autocorrelation of $\{Z_t\}$, which appears in the expression for φ_{r_h}. We denote the implied lag-h autocorrelation of $\{Y_t\}$ by the function $\rho(r_h)$. Thus, the problem of determining the autocorrelations for $\{Z_t\}$ that give the desired autocorrelations for $\{Y_t\}$ reduces to p independent problems: For each lag $h = 1, 2, \ldots, p$, find the value r_h for which $\rho(r_h) = \rho_h$. Unfortunately, it is not possible to find the r_h-values analytically; however, we establish some properties of the function $\rho(r_h)$ that enable us to perform an efficient numerical search to find the r_h-values to within any precision.

The first two properties concern the sign and the range of $\rho(r_h)$ for $-1 \le r_h \le 1$.

Proposition 1. *For any distribution F_Y, $\rho(0) = 0$, and $r_h \ge (\le)\, 0$ implies that $\rho(r_h) \ge (\le)\, 0$.*

Proof. If $r_h = 0$, then

$$\mathrm{E}[Y_t Y_{t+h}] = \mathrm{E}\left\{F_Y^{-1}[\Phi(Z_t)]F_Y^{-1}[\Phi(Z_{t+h})]\right\} = \mathrm{E}\left\{F_Y^{-1}[\Phi(Z_t)]\right\}\mathrm{E}\left\{F_Y^{-1}[\Phi(Z_{t+h})]\right\} = \mathrm{E}[Y_t]\mathrm{E}[Y_{t+h}]$$

since $r_h = 0$ implies that Z_t and Z_{t+h} are independent when they are bivariate normal. If $r_h \ge (\le)\, 0$, then $\mathrm{Cov}[g_1(Z_t, Z_{t+h}), g_2(Z_t, Z_{t+h})] \ge (\le)\, 0$ for all nondecreasing functions g_1 and g_2 such that the covariance exists (Tong [11], p. 20). Taking $g_1(Z_t, Z_{t+h}) \equiv F_Y^{-1}[\Phi(Z_t)]$ and $g_2(Z_t, Z_{t+h}) \equiv F_Y^{-1}[\Phi(Z_{t+h})]$, the result follows since $F_Y^{-1}[\Phi(\cdot)]$ is a nondecreasing function. □

It follows from the proof of Proposition 1 that taking $r_h = 0$ results in an input process in which Y_t and Y_{t+h} are independent, as well as uncorrelated. Proposition 2 shows that the maximum and minimum possible correlations are attainable.

Proposition 2. *Let $\overline{\rho}$ and $\underline{\rho}$ be the maximum and minimum feasible bivariate correlations, respectively, for random variables having marginal distribution F_Y (notice that $\overline{\rho} = 1$). Then, $\rho(1) = \overline{\rho}$ and $\rho(-1) = \underline{\rho}$.*

Proof. A correlation of 1 is the maximum possible for bivariate normal random variables. Therefore, taking $r_h = 1$ is equivalent (in distribution) to setting $Z_t \leftarrow \Phi^{-1}(U)$ and $Z_{t+h} \leftarrow \Phi^{-1}(U)$, where U is a $\mathrm{U}(0,1)$ random variable (Whitt [12]). But this definition of Z_t and Z_{t+h} implies that $Y_t \leftarrow F_Y^{-1}[U]$ and $Y_{t+h} \leftarrow F_Y^{-1}[U]$, from which it follows that $\rho(1) = \overline{\rho}$ by the same reasoning. Similarly, taking $r_h = -1$ is equivalent to setting $Y_t \leftarrow F_Y^{-1}[U]$ and $Y_{t+h} \leftarrow F_Y^{-1}[1-U]$, from which it follows that $\rho(-1) = \underline{\rho}$. □

Our next two results shed light on the shape of $\rho(r_h)$.

Theorem 1. *The function $\rho(r_h)$ is nondecreasing for $-1 \le r_h \le 1$.*

Proof. For $0 \le r_h \le 1$, the result follows directly from Tong [11], p. 119. See the Appendix for the case when $-1 \le r_h < 0$. □

Theorem 2. *If there exists $\epsilon > 0$ such that $\mathrm{E}[\mid Y_t Y_{t+h} \mid^{1+\epsilon}] < \infty$ for all values of $-1 \le r_h \le 1$, where Y_t, Y_{t+h} are defined by an ARTA process, then the function $\rho(r_h)$ is continuous for $-1 \le r_h \le 1$.*

Proof. See the Appendix. □

Since $\rho(r_h)$ is a continuous, nondecreasing function under the mild conditions stated in Theorem 2, any reasonable search procedure can be used to find r_h such that $\rho(r_h) \approx \rho_h$. Proposition 1 provides the initial bounds for such a procedure. Proposition 2 shows that the extremal values of ρ_h are attainable under our model. Furthermore, from Proposition 2, Theorem 2 and the Intermediate Value Theorem, any feasible bivariate correlation for F_Y is attainable under our model. Theorem 1 provides the theoretical basis for adjusting the values of r_h, and is the key to establishing convergence of a search procedure.

To illustrate how these results are useful, suppose that F_Y is the exponential distribution with mean 1.0 and we require the autocorrelation structure to be specified to $p = 2$ lags as $\boldsymbol{\rho} = (\rho_1, \rho_2)' = (0.9, 0.6)'$. We used a crude search procedure to find $\mathbf{r} = (r_1, r_2)' = (0.9, 0.65)'$, for which $\rho_1 = \rho(0.9) = 0.883$ and $\rho_2 = \rho(0.65) = 0.607$. All that is required for greater precision is a more stringent convergence criterion.

Throughout the previous discussion we assumed that there exists a stationary process with marginal distribution F_Y and autocorrelation structure $\boldsymbol{\rho}$. However, not all combinations of F_Y and $\boldsymbol{\rho}$ are feasible. Clearly, for $\boldsymbol{\rho}$ to be feasible we must have $\underline{\rho} \le \rho_h \le \overline{\rho}$ for each $h = 1, 2, \ldots, p$. In addition, the $(p+1) \times (p+1)$ autocorrelation matrix defined by

$$\begin{bmatrix} 1 & r_1 & \cdots & r_p \\ r_1 & 1 & \cdots & r_{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ r_p & r_{p-1} & \cdots & 1 \end{bmatrix}$$

must be nonnegative definite. These two conditions are necessary, but not sufficient, for the existence of a stationary $\{Y_t\}$ process with marginal distribution F_Y and autocorrelation structure $\boldsymbol{\rho}$. Our next result indicates that the input process $\{Y_t\}$ is stationary if the base AR(p) process $\{Z_t\}$ is.

Proposition 3. *If $\{Z_t\}$ is strictly stationary, then $\{Y_t\}$ is strictly stationary.*

Proof. The proof follows immediately from the definition of strict stationarity. □

Proposition 3 enables us to check the stationarity of the AR(p) process—which is easy to do (e.g., Cryer [3], pp. 80–81)—to determine the stationarity of $\{Y_t\}$.

4 Generating Simulation Input

Let $\mathbf{r}$ be a $p \times 1$ vector of r_h-values such that $\rho(r_h) \approx \rho_h$, for $h = 1, 2, \ldots, p$. Given $\mathbf{r}$, we can generate values of $\{Y_t\}$. We must first solve for the $p \times 1$ vector of AR parameters, $\boldsymbol{\alpha}$, using the Yule-Walker equations (e.g., Cryer [3], pp. 70–71)

$$\mathbf{r}' = \boldsymbol{\alpha}'\boldsymbol{\Psi} \tag{2}$$

where

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & r_1 & \cdots & r_{p-1} \\ r_1 & 1 & \cdots & r_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{p-1} & r_{p-2} & \cdots & 1 \end{bmatrix}.$$

We then use $\boldsymbol{\alpha}$ to check the stationarity of $\{Z_t\}$. If $\{Z_t\}$ is stationary, then we generate values of $\{Y_t\}$ using the following procedure.

ARTA Generation Procedure

1. Generate p initial values $\{Z_{p-1}, Z_{p-2}, \ldots, Z_0\}$ from a multivariate normal distribution with $\boldsymbol{\mu} = \mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Psi}$ (see, for example, Law and Kelton [4], pp. 505–506). Set $t \leftarrow p$.

2. Set $Z_t \leftarrow \alpha_1 Z_{t-1} + \alpha_2 Z_{t-2} + \cdots + \alpha_p Z_{t-p} + \varepsilon_t$ where ε_t is an independent $\mathrm{N}(0, \sigma^2)$ random variable with $\sigma^2 = 1 - \alpha_1 r_1 - \alpha_2 r_2 - \cdots - \alpha_p r_p$.

3. Return $Y_t \leftarrow F_Y^{-1}[\Phi(Z_t)]$.
4. Set $t \leftarrow t+1$ and go to step 2.

Continuing the exponential distribution example from the previous section (recall that $\boldsymbol{\rho} = (0.9, 0.6)'$ and $\mathbf{r} = (0.9, 0.65)'$), we solved (2) for the AR(2) parameters $\boldsymbol{\alpha} = (1.66, -0.842)'$. The resultant AR(2) process is stationary; hence, so is $\{Y_t\}$. A time series plot of 200 values of $\{Y_t\}$ generated by our procedure appears in Figure 1. Figures 2 and 3 are plots of Y_{t+1} vs. Y_t and Y_{t+2} vs. Y_t, respectively, for the same values. The sample autocorrelations for the $\{Z_t\}$ and $\{Y_t\}$ processes were $(0.893, 0.632)$ and $(0.882, 0.600)$, respectively. Figures 1–3 indicate that the generated values of $\{Y_t\}$ have the desired properties and exhibit plausible sample paths.

5 Conclusions

ARTA processes provide a straightforward method for representing time series with arbitrary marginal distributions and autocorrelation structures for use as simulation inputs. They also provide insight into the effect of transformations on time-series processes in general.

Software to fit and generate ARTA processes with standard choices for the marginal distribution, including the empirical cdf, is described in Cario [2]. Key features of the software are efficient implementation of the numerical integration for (1) and a fast search procedure. The Fortran code is available from the second author.

Although the focus of this paper is generating time series, the ideas also apply to generating finite-dimensional random vectors with correlated elements: The approach is to transform a $k \times 1$ standard multivariate normal (MVN) vector $\mathbf{Z}$ to obtain a $k \times 1$ input vector $\mathbf{X}$ with given marginal cdfs and correlation

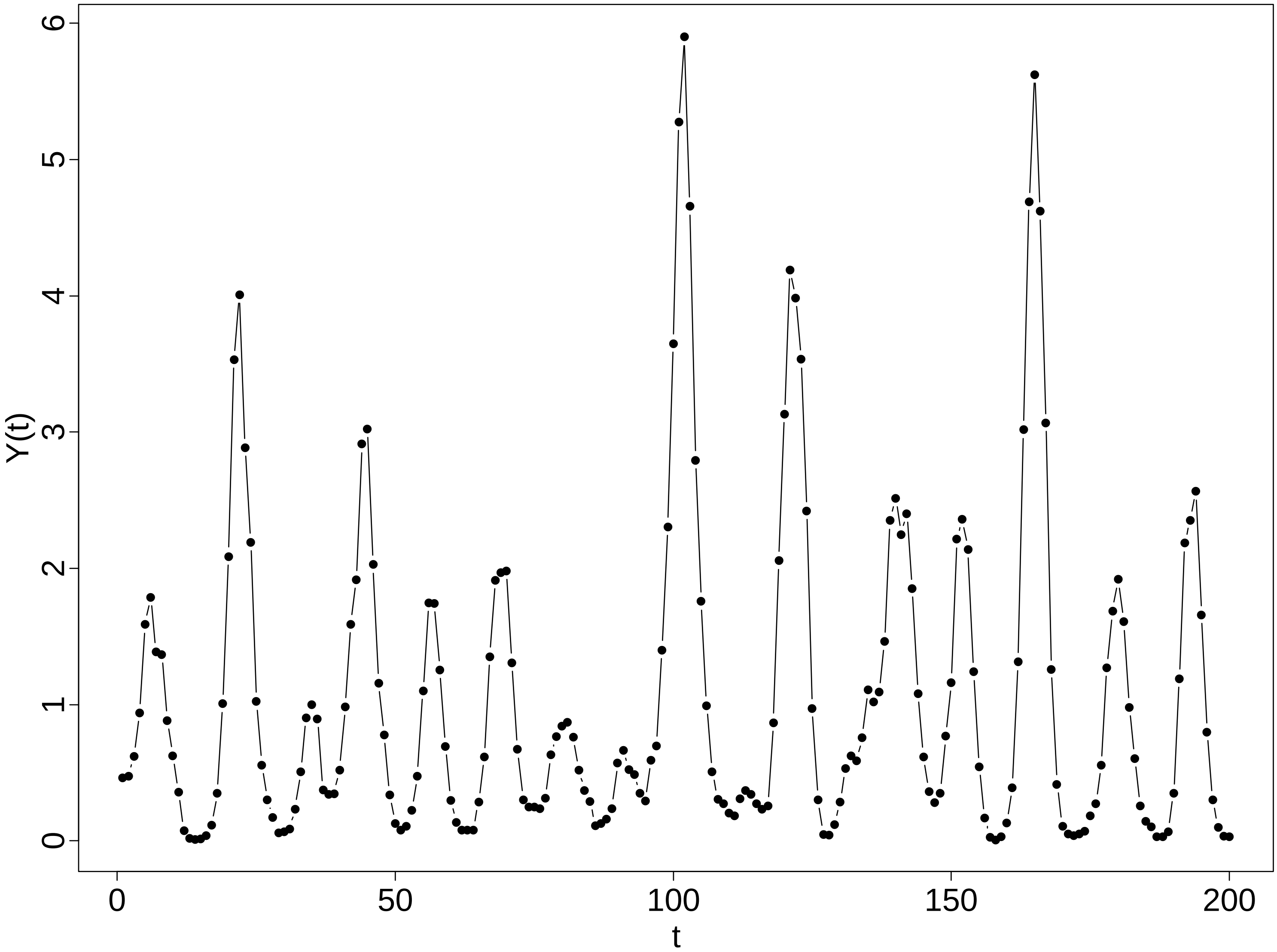

Figure 1: Sample path of an ARTA process with exponential marginals and autocorrelations $\rho_1 = 0.9$ and $\rho_2 = 0.6$.

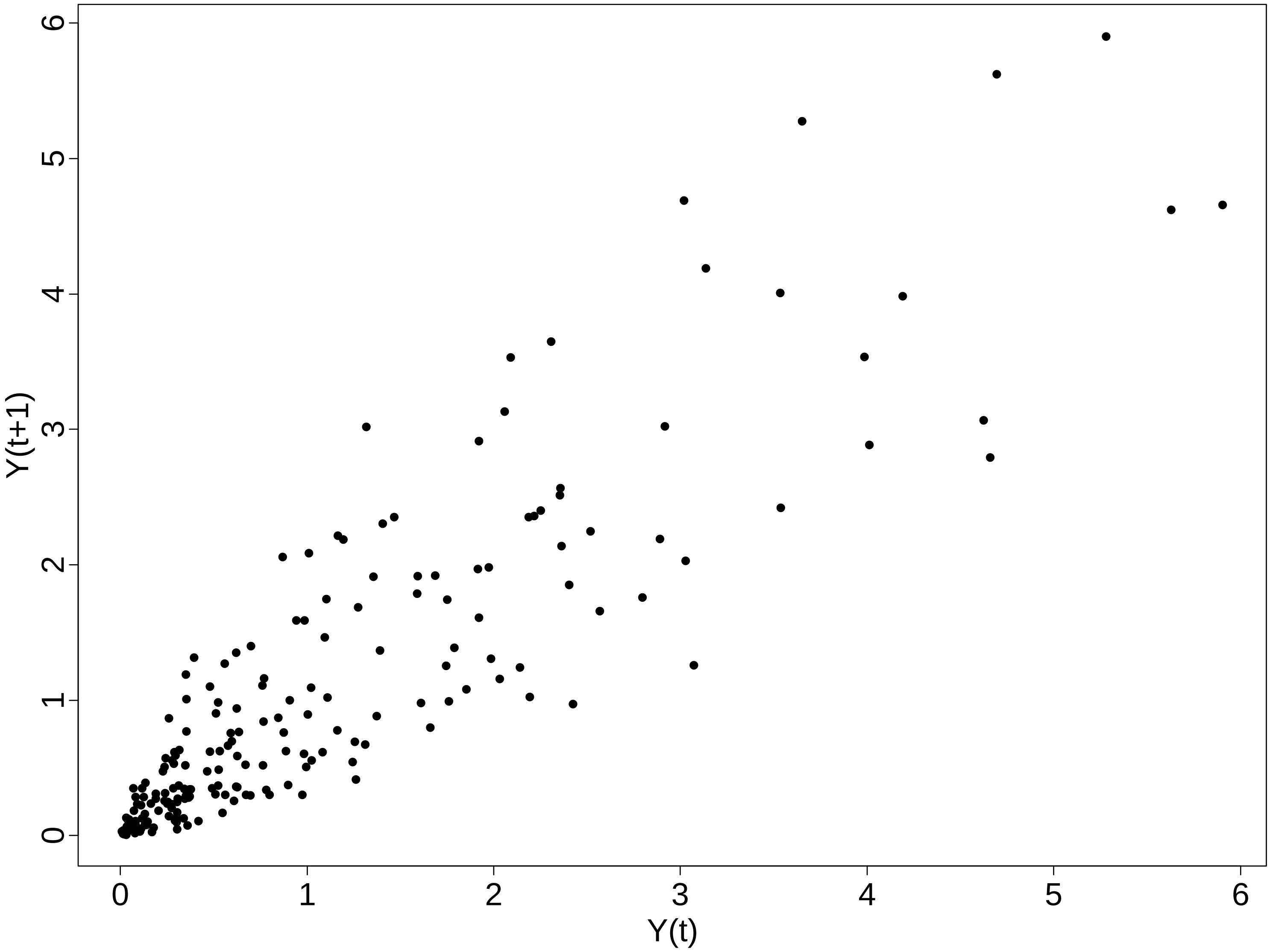

Figure 2: Scatterplot of (Y_t, Y_{t+1}) for the ARTA process with exponential marginals and lag-1 autocorrelation $\rho_1 = 0.9$.

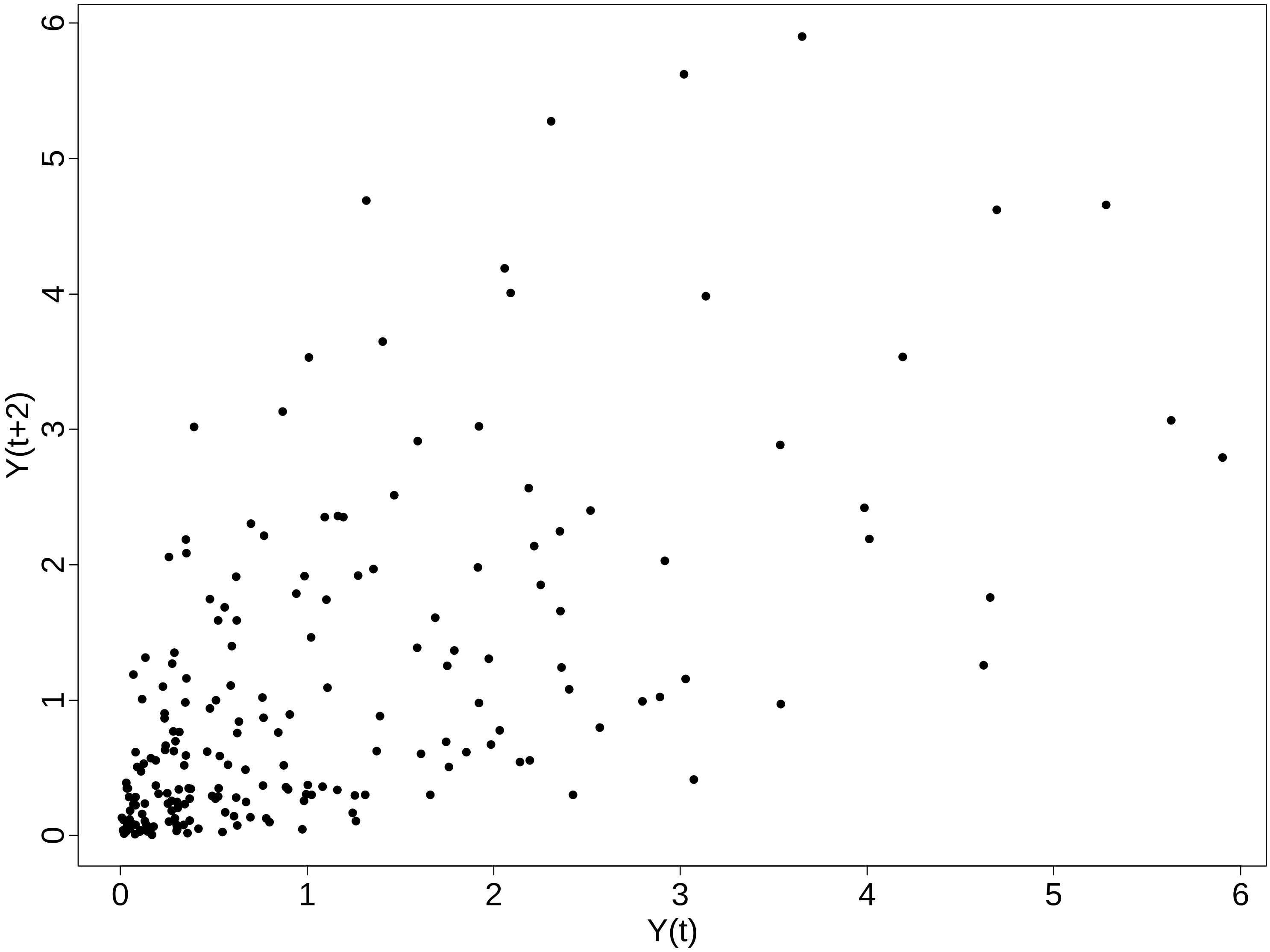

Figure 3: Scatterplot of (Y_t, Y_{t+2}) for the ARTA process with exponential marginals and lag-2 autocorrelation $\rho_2 = 0.6$.

matrix. Specifically,

$$\mathbf{X} = \begin{pmatrix} F_{X_1}^{-1}[\Phi(Z_1)] \\ F_{X_2}^{-1}[\Phi(Z_2)] \\ \vdots \\ F_{X_k}^{-1}[\Phi(Z_k)] \end{pmatrix}$$

where $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_k)'$ is a standard MVN vector with correlation matrix $\mathbf{\Sigma}$, and $F_{X_1}, F_{X_2}, \ldots, F_{X_k}$ are the desired marginal cdfs. Each off-diagonal element of the correlation matrix of the $\mathbf{X}$ vector is a function only of the corresponding element of $\mathbf{\Sigma}$. Cario [2] extends the results of Section 3 to the case of random vectors.

Appendix

Lemma 1. *Let* $\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $\boldsymbol{\Sigma}_{\rho_i} = \begin{pmatrix} 1 & \rho_i \\ \rho_i & 1 \end{pmatrix}$*, and let* $(X_1, X_2)'$ *and* $(Z_1, Z_2)'$ *be bivariate normal random variables with common mean* $\boldsymbol{\mu}$ *and variance-covariance matrices* $\boldsymbol{\Sigma}_{\rho_2}$ *and* $\boldsymbol{\Sigma}_{\rho_1}$*, respectively, where* $0 \le \rho_1 < \rho_2 < 1$*. Let* $g(x)$ *be a nondecreasing function of* x *for* $-\infty < x < \infty$*. Then for any* g *for which* $\mathrm{E}[g^2(X)]$ *exists,* $\mathrm{E}[g(X_1)g(-X_2)] \le \mathrm{E}[g(Z_1)g(-Z_2)]$.

Proof. The proof extends Tong [11], pp. 119–120.

Let T_1, T_2, V_1, V_2, and W be i.i.d. N(0,1) random variables. Then,

$$(X_1, -X_2) \stackrel{d}{=} \left(\sqrt{1-\rho_2}T_1 + \sqrt{\rho_2 - \rho_1}V_1 + \sqrt{\rho_1}W,\ -\sqrt{1-\rho_2}T_2 - \sqrt{\rho_2 - \rho_1}V_1 - \sqrt{\rho_1}W\right)$$

and

$$(Z_1, -Z_2) \stackrel{d}{=} \left(\sqrt{1-\rho_2}T_1 + \sqrt{\rho_2 - \rho_1}V_1 + \sqrt{\rho_1}W,\ -\sqrt{1-\rho_2}T_2 - \sqrt{\rho_2 - \rho_1}V_2 - \sqrt{\rho_1}W\right)$$

where $\stackrel{d}{=}$ denotes equality in distribution. Therefore,

$$\mathrm{E}[g(X_1)g(-X_2)] \quad = \quad \mathrm{E}\left[\mathrm{E}\left\{\mathrm{E}\left[g(\sqrt{1-\rho_2}T_1 + \sqrt{\rho_2-\rho_1}V_1 + \sqrt{\rho_1}W)\ \times\right.\right.\right.$$

$$
\begin{aligned}
& g(-\sqrt{1-\rho_2}T_2 - \sqrt{\rho_2-\rho_1}V_1 - \sqrt{\rho_1}W) \mid V_1, W\Big] \mid W\Big\}\Big] \\
= \quad & \mathrm{E}\left[\mathrm{E}\left\{\Psi_W(V_1)\Psi_{-W}(-V_1) \mid W\right\}\right]
\end{aligned}
$$

where

$$
\Psi_w(v_1) = \mathrm{E}\left[g(\sqrt{1-\rho_2}T + \sqrt{\rho_2-\rho_1}V_1 + \sqrt{\rho_1}W) \mid V_1 = v_1, W = w\right]
$$

and the expectation is with respect to T, an independent $\mathrm{N}(0,1)$ random variable.[1]

For g nondecreasing and fixed $W = w$, the function $\Psi_w(v_1)$ is nondecreasing in v_1. Similarly, $-\Psi_{-w}(v)$ is nonincreasing in v (where v is a dummy variable used only for clarity). Therefore,

$$
\mathrm{Var}\left[\Psi_w(V_1) - \{-\Psi_{-w}(V)\}\right] = \mathrm{Var}\left[\Psi_w(V_1)\right] + \mathrm{Var}\left[-\Psi_{-w}(V)\right] - 2\mathrm{Cov}\left[\Psi_w(V_1), -\Psi_{-w}(V)\right]
$$

is minimized with respect to all joint distributions of (V_1, V) with $\mathrm{N}(0,1)$ marginals when $V_1 = \Phi^{-1}(U)$ and $V = \Phi^{-1}(1-U)$, where U is a $\mathrm{U}(0,1)$ random variable (Rubinstein et al. [9], Proposition 1). For $\mathrm{N}(0,1)$ random variables this implies that $V = -V_1$. Therefore, $\mathrm{Cov}[\Psi_w(V_1), -\Psi_{-w}(V)]$ is maximized (equivalently, $\mathrm{Cov}[\Psi_w(V_1), \Psi_{-w}(V)]$ is minimized) by letting $V = -V_1$. Thus,

$$
\begin{aligned}
\mathrm{E}\{\Psi_w(V_1)\Psi_{-w}(-V_1) \mid W = w\} \quad & \leq \quad \mathrm{E}\{\Psi_w(V_1) \mid W = w\}\mathrm{E}\{\Psi_{-w}(-V_1) \mid W = w\} && (3) \\
& = \quad \mathrm{E}\{\Psi_w(V_1) \mid W = w\}\mathrm{E}\{\Psi_{-w}(-V_2) \mid W = w\} && (4)
\end{aligned}
$$

where (3) holds because the minimum expected value must be smaller than the expected value under independence, and (4) holds because V_1 and V_2 are identically distributed. Since (3) and (4) hold for any value of W, it follows that

$$
\mathrm{E}\left[\mathrm{E}\{\Psi_W(V_1)\Psi_{-W}(-V_1) \mid W\}\right] \leq \mathrm{E}\left[\mathrm{E}\{\Psi_W(V_1) \mid W\}\mathrm{E}\{\Psi_{-W}(-V_2) \mid W\}\right].
$$

[1] Notice that $T_1 \stackrel{d}{=} -T_2$, and they are independent.

But notice that

$$\begin{aligned}
\mathrm{E}\left[g(Z_1)g(-Z_2)\right] &= \mathrm{E}\left[\mathrm{E}\{\Psi_W(V_1)\Psi_{-W}(-V_2)\} \mid W\right] \\
&= \mathrm{E}\left[\mathrm{E}\{\Psi_W(V_1) \mid W\}\mathrm{E}\{\Psi_{-W}(-V_2) \mid W\}\right]
\end{aligned}$$

since V_1 and V_2 are independent. $\Box$

Corollary. *Let* $(X_1, X_2)'$ *and* $(Z_1, Z_2)'$ *have bivariate normal distributions with common mean* $\boldsymbol{\mu}$ *and variance-covariance matrices* $\boldsymbol{\Sigma}_{\rho_2}$ *and* $\boldsymbol{\Sigma}_{\rho_1}$, *respectively, where* $-1 < \rho_2 < \rho_1 \leq 0$. *Let* $g(x)$ *be a nondecreasing function of* x *for* $-\infty < x < \infty$. *Then,* $\mathrm{E}[g(X_1)g(X_2)] \leq \mathrm{E}[g(Z_1)g(Z_2)]$.

Proof. This follows from Lemma 1 since $(X_1, -X_2)'$ and $(Y_1, -Y_2)'$ have bivariate normal distributions with mean $\boldsymbol{\mu}$ and covariance matrices $\boldsymbol{\Sigma}_{-\rho_2}$ and $\boldsymbol{\Sigma}_{-\rho_1}$, respectively. $\Box$

Proof of Theorem 1. By taking $g \equiv F_Y^{-1}[\Phi(\cdot)]$ in the Corollary, it follows that $\rho(r_h)$ is nondecreasing for $-1 \leq r_h < 0$. $\Box$

Lemma 2. *For a given cdf* F_Y, *if there exists* $\epsilon > 0$ *such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sup_{r \in [-1,1]} \left\{\left|F_Y^{-1}[\Phi(z_1)]F_Y^{-1}[\Phi(z_2)]\right|^{1+\epsilon} \varphi_r(z_1, z_2)\right\} dz_1 dz_2 \quad < \quad \infty \tag{5}$$

then $\rho(r)$ *is a continuous function for* $-1 \leq r \leq 1$.

Proof. Let Z_1 and Z_3 be i.i.d. N(0,1) random variables. Let $r \in [-1,1]$ be fixed, and $\{r_n\}_{n=1}^{\infty}$ be any sequence such that $r_n \in [-1,1]$, for $n = 1, 2, \ldots$, and $r_n \to r$ as $n \to \infty$. For $n = 1, 2, \ldots$, define

$$Z_{1n} \equiv Z_1, \qquad Z_{2n} \equiv r_n Z_1 + \sqrt{1 - r_n^2}Z_3, \qquad Z_2 \equiv rZ_1 + \sqrt{1 - r^2}Z_3.$$

Further, let $Y_{in} \equiv F_Y^{-1}[\Phi(Z_{in})]$, for $i = 1, 2$, and $h \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv F_Y^{-1}[\Phi(z_1)]F_Y^{-1}[\Phi(z_2)]$. Since h is monotone in z_1 and z_2 individually, it has a countable number of discontinuities. Therefore, by the Continuous Mapping Theorem (Billingsley [1], Theorem 29.2)

$$h \begin{pmatrix} Z_{1n} \\ Z_{2n} \end{pmatrix} \stackrel{d}{\Rightarrow} h \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ as } n \to \infty,$$

since

$$\begin{pmatrix} Z_{1n} \\ Z_{2n} \end{pmatrix} \stackrel{d}{\Rightarrow} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ as } n \to \infty,$$

where $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution. Equivalently,

$$Y_{1n}Y_{2n} \quad \stackrel{d}{\Rightarrow} \quad Y_1Y_2 \text{ as } n \to \infty, \tag{6}$$

where $Y_i \equiv F_Y^{-1}[\Phi(Z_i)]$, for $i = 1, 2$. It follows from (5), (6), and Theorem 25.12 of [1], that $\mathrm{E}[Y_{1n}Y_{2n}] \to \mathrm{E}[Y_1Y_2]$ as $n \to \infty$; equivalently, $\rho(r_n) \to \rho(r)$ as $n \to \infty$. $\square$

Notice that condition (5) of Lemma 2 is equivalent to stating that $\mathrm{E}[\mid Y_tY_{t+h} \mid^{1+\epsilon}] < \infty$ for all values of $-1 \leq r_h \leq 1$, where Y_t, Y_{t+h} are defined by our transformation, which is the condition given in the statement of Theorem 2.

Proof of Theorem 2. Theorem 2 follows immediately from Lemma 2 with $Z_1 \equiv Z_t, Z_2 \equiv Z_{t+h}, Y_1 \equiv Y_t, Y_2 \equiv Y_{t+h}$, and $r \equiv r_h$. $\square$

BIBLIOGRAPHY

1. Billingsley, P. 1995. *Probability and Measure.* Third Edition. New York: John Wiley.
2. Cario, M. C. 1996. *Modeling and Generating Dependent Inputs for Discrete-Event Simulation.* Ph.D. dissertation. Department of Industrial, Welding and Systems Engineering, The Ohio State University, Columbus, Ohio.

3. Cryer, J. D. 1986. *Time Series Analysis.* Boston: PWS Publishers.

4. Law, A. M. and W. D. Kelton. 1991. *Simulation Modeling & Analysis.* Second Edition. New York: McGraw-Hill.

5. Lewis, P. A. W., McKenzie, E. and D. K. Hugus. 1989. Gamma Processes. *Communications in Statistics–Stochastic Models* **5**(1), 1–30.

6. Livny, M., Melamed, B. and A. K. Tsiolis. 1993. The Impact of Autocorrelation on Queuing Systems. *Management Science* **39**(3): 322–339.

7. Melamed, B. 1991. TES: A Class of Methods for Generating Autocorrelated Uniform Variates. *ORSA Journal on Computing* **3**(4): 317–329.

8. Melamed, B., Hill, J. R. and D. Goldsman. 1992. The TES Methodology: Modeling Empirical Stationary Time Series, in: J. J. Swain, D. Goldsman, R. C. Crain and J. R. Wilson (eds.), *Proceedings of the 1992 Winter Simulation Conference*, Institute of Electrical and Electronics Engineers, Piscataway, NJ, 135–144.

9. Rubinstein, R. Y., Samorodnitsky, G. and M. Shaked. 1985. Antithetic Variates, Multivariate Dependence and Simulation of Stochastic Systems. *Management Science* **31**(1): 66–77.

10. Song, W. T., Hsiao, L. and Y. Chen. 1995. Generation of Pseudo-random Time Series with Specified Marginal Distributions. *European Journal of Operational Research*, forthcoming.

11. Tong, Y. L. 1990. *The Multivariate Normal Distribution.* New York: Springer-Verlag.

12. Whitt, W. 1976. Bivariate Distributions with Given Marginals. *The Annals of Statistics* **4**(6): 1280–1289.