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Analytic and External Control Variates for Queueing Network Simulation

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While external control variates (ECVs) have long been advocated for reducing the variance of simulation estimators, the lack of tractable external control systems and the almost doubling of the computational burden has hindered their use. This paper presents preliminary work on a new type of control variate, the analytic control variate (ACV), that has the advantages of the ECV but requires much less additional computation. Use of an ACV is demonstrated for open queueing networks.

Key words: control variates, queueing network, simulation, variance reduction

INTRODUCTION

Computer simulation is commonly employed for the analysis of stochastic systems, such as queueing networks. Simulation analysis, however, introduces a host of concerns found in sampling experiments. One of these, estimator variability, is addressed in this paper.

Simulation estimators are random variables. Reducing the variance of these estimators can increase the computational efficiency of a simulation experiment, meaning that a specified precision can be achieved for less computation cost, or increased precision can be achieved for the same computation cost. Variance reduction techniques (VRTs) have been developed for this purpose. This paper examines a class of VRTs known as control variates (CVs), and specifically two types of CVs: the external control variate and the new analytic control variate. The analytic-control-variate approach breaks new ground in the class of CV estimators, and may have the potential to simplify the selection and implementation of CVs for simulation analysis. The research reported here represents a start in assessing this potential in terms of both ease of application and performance.

The remainder of the paper is organized into five sections. The first reviews CV estimators and introduces the new analytic control variate. The second and third sections describe the particular control variate and experiments considered here, and the fourth section presents the experiment results. The paper concludes with a section discussing open research questions.

CONTROL VARIATES

The idea behind CVs is to exploit the correlation between two or more random variables to produce a variance reduction. Let Y be a scalar random variable with expectation θ, an unknown scalar parameter of interest. Let C be another (possibly multivariate) random variable having known expectation ϕ. Control variate estimators attempt to use the difference $C - \phi$ to counter the unknown estimation error $Y - \theta$. Nelson[1] gives a characterization of CV estimators in terms of five properties, and discusses several estimators that exhibit these properties. This paper, however, will focus only on the linear CV estimator for univariate C, which is described next.

Linear control-variate estimator

The *linear CV estimator* is

$$Y(\mathrm{b}) = Y - \mathrm{b}(C - \phi), \tag{1}$$

where $Y(\mathrm{b})$ is the CV estimator, b is a constant called the *control coefficient*, and C is a univariate control with known expectation ϕ. The variance of $Y(\mathrm{b})$ is

$$\mathrm{Var}[Y(\mathrm{b})] = \mathrm{Var}[Y] + \mathrm{b}^2\mathrm{Var}[C] - 2\mathrm{b}\mathrm{Cov}[Y, C]. \tag{2}$$

The value of b that minimizes Var[Y(b)], which can be found by differentiating (2) with respect to b, is

$$b^* = \frac{\text{Cov}[Y, C]}{\text{Var}[C]}. \tag{3}$$

Substituting b* into (1) yields the minimum variance CV estimator $Y(b^*)$ with variance

$$\text{Var}[Y(b^*)] = (1 - \rho^2)\text{Var}[Y], \tag{4}$$

where ρ is the correlation coefficient of Y and C. Equation (4) indicates that the greater the square of the correlation between Y and C, the greater the variance reduction relative to Var[Y].

Equation (3) specifies the optimal value for the control coefficient b for a univariate control. Unfortunately, Cov[Y, C] is usually unknown. While the value of b could be arbitrarily chosen (for example, b = 1), (2) shows that variance increases are possible for some values of b. Thus, b* is usually estimated in the following manner.[2] Generate K independent and identically distributed replications of Y and C, (Y_k, C_k), $k = 1, 2, \ldots, K$, then replace Cov[Y, C] and Var[C] in (3) with their sample equivalents. We denote this estimator by $\hat{b}^*$, and note that it is algebraically the same as the slope estimator in a least-squares regression of Y on $C - \phi$. The CV point estimator of θ then becomes

$$\bar{Y}(\hat{b}^*) = \bar{Y} - \hat{b}^*(\bar{C} - \phi), \tag{5}$$

where $\bar{Y}$ and $\bar{C}$ are sample means.

Since $\hat{b}^*$ is a function of the C_k, $\hat{b}^*$ and $\bar{C}$ are not generally independent. Thus, $\bar{Y}(\hat{b}^*)$ is generally biased for θ, even if C is unbiased for ϕ and Y is unbiased for θ; the key condition for unbiasedness is that $E[Y|C] = \theta + b(C - \phi)$. The bias does decrease as K increases, however. Kleijnen[2] and others recommend splitting or jack-knifing to reduce the bias.

The linear CV estimator can be extended in a straightforward manner to a Q-variate control vector; see, for instance, Lavenberg and Welch.[3] While we shall not consider $Q > 1$ in this paper, an important result in Lavenberg and Welch, specialized to $Q = 1$, is the variance ratio

$$\frac{\text{Var}[\bar{Y}(\hat{b}^*)]}{\text{Var}[\bar{Y}]} = \frac{K-2}{K-3}(1 - \rho^2). \tag{6}$$

Equation (6) indicates that the number of replications K should not be too small, otherwise the variance reduction due to the correlation between Y and C will be significantly diminished. Equation (6) quantifies the degradation from the minimum achievable variance (4) due to estimating b*.

Selecting control variates

One difficulty in applying CVs is the need to identify an effective control, C, meaning strongly correlated with Y. *Internal CVs (ICVs)* use input random variables as controls. *Inputs* are random variables with user-specified probability distributions that describe the randomness in a stochastic model, e.g. the interarrival times and service times in a queueing system. ICVs are available in any simulation, but it is sometimes difficult to predict which input random variables, if any, will be strongly correlated with the output random variables of interest.

External control variates (ECVs) are another source of CVs. ECVs depend on the existence of a second system that is similar to the system of interest, but which allows for analytical calculation or numerical approximation of ϕ, where ϕ is the parameter of the second system that corresponds to θ for the system of interest. We call the second system the *control system*. When both the system of interest and the control system are simulated using common random numbers (see, for instance, Law and Kelton[4]), it is hoped that C, the estimator of ϕ from the control system, will be strongly correlated with Y, the estimator of θ for the system of interest. Since ECVs require a second simulation, considerable variance reduction must be achieved to warrant the additional computational expense.

Nelson[1] suggests another method for choosing the control. Let δ be a vector of parameters of the input distributions for a control system, and suppose that $\phi = g(\delta)$, where g is a function and ϕ is again the parameter of the control system corresponding to θ for the system of interest.

The method is applicable if the system of interest has the same input parameters, δ, but differs from the control system in other ways. The key step is to simulate the system of interest to obtain Y and $\hat{\delta}$, where $\hat{\delta}$ is an estimator of the known quantity δ. Rather than using the difference $\hat{\delta} - \delta$ as an ICV, however, the *analytic control variate (ACV)* $Y(\mathrm{b}) = Y - \mathrm{b}(g(\hat{\delta}) - g(\delta)) = Y - \mathrm{b}(g(\hat{\delta}) - \phi)$ is formed. If K replications $(Y_k, \hat{\delta}_k)$, $k = 1, 2, \ldots, K$ are generated, an estimator analogous to (5) is

$$\bar{Y}(\hat{\mathrm{b}}^*) = \bar{Y} - \hat{\mathrm{b}}^* \, (\bar{g}(\hat{\delta}) - \phi), \tag{7}$$

where $\hat{\mathrm{b}}^*$ estimates $\mathrm{b}^* = \mathrm{Cov}[Y, g(\hat{\delta})]/\mathrm{Var}[g(\hat{\delta})]$. Note that $g(\hat{\delta})$ may not be an unbiased estimator of ϕ, but should be made to be a consistent estimator of ϕ (as in the example below).

In contrast to the ECV, the ACV does not require a second simulation. Also, ACVs are not sensitive to the method of random variate generation used to realize the inputs, while ECVs usually require the inverse cumulative distribution-function method (see, for instance, Law and Kelton[4]) to ensure that using common random numbers induces correlation between the outputs of the system of interest and the control system. What is required for ACVs is a control system, the function g (which may be a numerical procedure as well as a closed-form function), and the vector of realized input parameters $\hat{\delta}$. The ACV retains the advantage of the ECV that the control variate is always the estimator of the parameter of the control system corresponding to the unknown parameter of the system of interest. The remainder of the paper presents an example.

JACKSON NETWORK CONTROL VARIATES

The difficult requirement for both ECVs and ACVs is finding a system model to serve as a source of controls. It has long been folklore in simulation that, for general queueing networks, the Jackson network[5] could serve as a source of ECVs. Briefly, a Jackson network is composed of N possibly multiple-server, infinite-capacity queues with exponentially distributed service times and probabilistic routeing of customers between queues. In the open Jackson networks considered here, customers arrive to one or more of the queues according to independent Poisson arrival processes. For the Jackson model, several long-run (sometimes called 'steady-state') performance measures can be analytically derived.

The following notation will be used to characterize a Jackson network:

N is the number of queues in the network.

$\lambda = (\lambda_1, \ldots, \lambda_N)$ is the external arrival rate at queues 1 to N.
$\mu = (\mu_1, \ldots, \mu_N)$ is the service rate of an individual server at queues 1 to N.
$s = (s_1, \ldots, s_N)$ is the number of servers at queues 1 to N.
$r = \{r_{ij}\}$ is the matrix of probabilities r_{ij} that a customer departing from queue i next goes to queue j.
$e = (e_1, \ldots, e_N)$ is the effective arrival rate at queues 1 to N, which is a function of λ, μ, s and r.

Let $\delta = (\lambda, \mu, r)$. For fixed N and s, the following performance measures are functions of δ:

$(\phi_{11}, \ldots, \phi_{1N}) = g_1(\delta)$, the long-run utilization of the servers at queues 1 to N.

$(\phi_{21}, \ldots, \phi_{2N}) = g_2(\delta)$, the long-run customer wait in queue at queues 1 to N.

Equations for determining e, $(\phi_{11}, \ldots, \phi_{1N})$ and $(\phi_{21}, \ldots, \phi_{2N})$ can be found in texts such as Giffin.[6]

EXPERIMENT METHODOLOGY

Three different queueing networks were investigated experimentally, each network meeting the requirements of the Jackson model, except possibly for the distribution of service time. The service-time distributions investigated were the exponential, the Weibull and the uniform. ECVs and ACVs were used to estimate $(\theta_{11}, \ldots, \theta_{1N})$ and $(\theta_{21}, \ldots, \theta_{2N})$, the long-run server utilization and long-run wait in queue, respectively, for each queue in the network. Note that additional performance measures, such as expected queue length, can be obtained from these performance measures via Little's formula.[7] From here on, let (θ_1, θ_2) and (ϕ_1, ϕ_2) be generic for the server utilization and queue time at any queue in the network of interest and the corresponding Jackson network, respectively.

Application of control variates

The ECV estimator requires two simulations. The first is a simulation of the network of interest, and it estimates θ_1 and θ_2; denote the estimators by Y_1 and Y_2, respectively, which are averages over a single replication. The second simulation is of a similar Jackson network that approximates the network of interest and estimates (the known parameters) ϕ_1 and ϕ_2; denote the estimators by C_1 and C_2, respectively, which are also averages over a single replication. Since the exponential service-time distribution yields the Jackson model itself, ECVs were only obtained for the Weibull and uniform cases. For these two distributions the second simulation was run using common random numbers, using the inverse cumulative distribution-function method to generate input realizations, and using the means of the Weibull and uniform distributions as the means of the exponential service-time distributions of the Jackson model. Thus, both the network of interest and the corresponding Jackson network share the same parameters N, s and δ.

The ACV requires only the simulation of the network of interest. Additional code was added to the simulation to record the vector of observed parameters, $\hat{\delta}$. Both the observed $\hat{\delta}$ and the known δ were substituted into the Jackson network equations g_1 and g_2 to obtain $g_i(\hat{\delta})$ and $\phi_i = g_i(\delta)$, $i = 1, 2$, for the ACV. The extra effort required to use the ACV is the additional computation time needed to record $\hat{\delta}$. This effort is insignificant relative to the cost of the second simulation required for the ECV. Once the controls or parameters needed to compute the controls have been obtained, the computational effort to compute both types of CV estimators is the same. However, in general we need only that the network of interest and the Jackson network share some parameters.

One drawback of the Jackson ACV is that, as the traffic intensity at, say, queue n (which is $e_n/s_n\mu_n$) increases, obtaining a realization of $\hat{\delta}$ that violates the Jackson model assumption that the traffic intensity is less than 1 becomes more likely, even though the actual traffic intensity is less than 1. The Jackson network results are invalid when this occurs. As the length of a replication increases, the probability that this occurs goes to zero, but there is always a positive probability for finite-length replications. A satisfactory solution to this problem has not yet been obtained. Another drawback is that the ACVs are biased estimators. However, because we used method of moments estimators for δ, they are consistent. Determination of the severity of the bias is open to further study, since reduced variance at the expense of significantly larger mean-squared error is unacceptable. In the experiments reported here the bias was not significant.

Networks considered

To evaluate the effectiveness of the Jackson-based CVs, three networks with different structure and complexity were simulated. Network I consists of two queues, each with its own external Poisson arrival process. Customers completing service at each queue may be routed to the other queue or may depart from the system entirely (see Figure 1).

Network II is a three-queue, tandem, acyclic network. An external Poisson arrival process occurs only at queue 1, where customers complete service and move to the second and then the third queues for service. Departure from the network occurs only when service is completed at the third queue (see Figure 2).

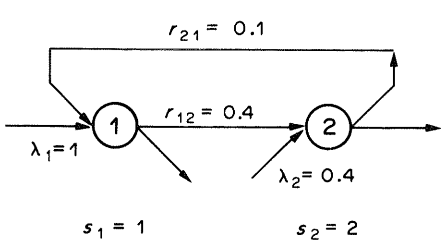

FIG. 1. *Network I.*

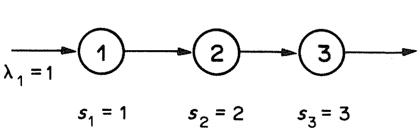

FIG. 2. *Network II.*

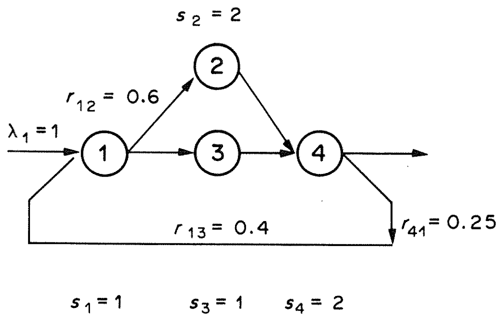

FIG. 3. *Network III.*

Network III consists of four queues, with an external Poisson arrival process at queue 1. Customers completing service at the first queue are routed to either the second or third queues, and then to the fourth. Customers completing service at the fourth queue may return to the first queue or depart from the system entirely (see Figure 3).

Three service-time distributions were studied: the exponential, the Weibull and the uniform. This selection provides three references for studying the Jackson-based CVs: the exponential, the requisite for the Jackson model, infinite in the right tail and highly variable; the Weibull, for which the exponential is a special case, but hump-shaped in our examples; and the uniform, a finite-range distribution with considerably less variability. Each network contains single and multiple-server queues, and parameters were chosen so that the traffic intensity at each queue in the network is the same. While it is hoped that ACVs can be applied to networks that depart even further (than just service-time distribution) from the Jackson network assumptions, these three networks were considered adequate for the initial investigation.

Experiment design

The CV estimators investigated were (5) for the ECV and (7) for the ACV, both requiring K replications of the simulation. However, since measures of interest are long-run measures, and since pilot experiments indicated the potential for excessively long initial-transient periods, a single replication divided into approximately independent batches of time length t was used rather than independent replications; see, for instance, Schmeiser[8] for the method of batch means. The number of batches per estimator was set at $K = 25$, since that number yields a loss ratio $(K-2)/(K-3)$ of only 5% from (6). A test of independence[9] was used to determine a batch size, t, such that batch means were approximately independent. Table 1 shows the selected batch size for each network.

All simulations were initialized with no customers in the network. An initial transient period of 10,000 time units was deleted from the beginning of each replication. This period was chosen based on pilot experiments on the exponential service-time networks, and was conservatively long in hopes of eliminating initial-condition effects from the experiment results. Following the initial period of 10,000 time units, 250 batches were collected. This allowed $J = 10$ ECV and ACV estimates to be calculated, each based on $K = 25$ batches. To summarize, the simulation of a given network involved deleting an initial period of 10,000 time units, collecting 250 batches of length t, and then computing $J = 10$ CV estimators, each based on 25 batches. The summary statistics from the J estimators are reported in the next section.

TABLE 1. *Selected batch lengths*

Network	t (time units)
I	150
II	200
III	300

RESULTS

The results listed in Tables 2, 3 and 4 are for the ACV estimates of θ_1 and θ_2 for networks I, II and III, respectively, at the 0.9 traffic intensity for each queue in the network. Variance reductions for the 0.5 traffic intensity followed a similar pattern: substantial reductions for θ_1 (approx. 98%) and somewhat less substantial reductions for θ_2 (usually in the 15%–75% range). There was one variance increase for a parameter of network III. The ECVs were consistently less effective than the ACVs, and did not warrant the expenditure of the second simulation. A complete listing of results can be found in Sharon.[10]

For the networks considered, the true value of θ_2 is available for the networks with exponential service times. Seven of the nine crude estimators for θ_2 fell within one standard deviation of the known value. This provides a check on the coding of the simulation model. The estimates are not equal to the true values because of sampling variability and any remaining initial-condition bias.

The most prominent feature of the results is the performance of the ACVs for θ_1, the long-run server utilization, relative to the ACVs for θ_2, the long-run wait time. In every case the utilization estimator yielded greater variance reductions. One reason for this may be that the analytic control for utilization is a simple ratio, as opposed to the complex function for the queue time.[6] While θ_1 can be calculated for all the networks, it was included in this study to investigate the performance of the ACV approach.

Another feature of the results is that the ACV estimate of θ_2 is consistently lower than the crude estimate. One possible cause of the difference is the introduction of bias in the ACV estimator, in

TABLE 2. *Experiment results for network I with $\mu_l = 1.2$ and $\mu_2 = 0.46$, giving traffic intensity 0.9*

Parameter	Crude	Variance	ACV	Variance	% Reduction
Exponential service time					
θ_{11}	0.8872	0.0008	0.9025	0.0001	84.5
θ_{12}	0.9076	0.0006	0.9130	0.0002	67.6
θ_{21}	5.9196	2.1802	5.0694	1.1656	46.5
θ_{22}	9.6711	4.3639	8.4098	3.3497	23.1
Weibull service time					
θ_{11}	0.8934	0.0005	0.9010	0.0001	87.2
θ_{12}	0.9033	0.0003	0.9034	<0.00005	85.1
θ_{21}	4.1025	0.8210	3.7813	0.5469	33.4
θ_{22}	4.9167	1.0489	4.1912	0.5425	48.8
Uniform service time					
θ_{11}	0.8991	0.0002	0.9023	<0.00005	77.2
θ_{12}	0.9023	0.0002	0.9017	<0.00005	76.1
θ_{21}	3.7309	0.5230	3.3113	0.3808	27.2
θ_{22}	3.9522	0.4220	3.4828	0.3020	28.5

TABLE 3. *Experiment results for network II with $\mu_l = 1.11$, $\mu_2 = 0.56$ and $\mu_2 = 0.37$, giving traffic intensity 0.9*

Parameter	Crude	Variance	ACV	Variance	% Reduction
Exponential service time					
θ_{11}	0.8872	0.0005	0.9008	<0.00005	92.4
θ_{12}	0.8947	0.0001	0.9010	<0.00005	63.8
θ_{13}	0.8906	0.0004	0.8985	0.0001	67.8
θ_{21}	6.3363	1.1857	5.6212	0.2973	74.9
θ_{22}	7.3719	4.4346	6.4935	1.8028	59.3
θ_{23}	6.1086	3.0807	5.6702	2.5672	17.1
Weibull service time					
θ_{11}	0.8941	0.0002	0.8997	<0.00005	93.7
θ_{12}	0.8939	0.0001	0.8999	<0.00005	87.3
θ_{13}	0.8918	0.0002	0.8978	<0.00005	81.3
θ_{21}	4.2966	0.4055	3.8640	0.1924	52.6
θ_{22}	2.9552	0.4286	2.7459	0.3320	22.5
θ_{23}	2.2526	0.2985	2.1985	0.2648	11.3
Uniform service time					
θ_{11}	0.8964	0.0001	0.9015	<0.00005	84.8
θ_{12}	0.8946	0.0001	0.8995	<0.00005	88.2
θ_{13}	0.8950	0.0001	0.8996	<0.00005	84.7
θ_{21}	4.1722	0.3479	3.8384	0.4355	43.5
θ_{22}	1.0114	0.0388	0.9515	0.0314	18.9
θ_{23}	0.5184	0.0072	0.4910	0.0054	24.5

TABLE 4. *Experiment results for network III with $\mu_1 = 1.48$, $\mu_2 = 0.44$, $\mu_3 = 0.59$ and $\mu_4 = 0.74$, giving traffic intensity 0.9*

Parameter	Crude	Variance	ACV	Variance	% Reduction
Exponential service time					
θ_{11}	0.8912	0.0005	0.8989	0.0001	87.8
θ_{12}	0.8999	0.0003	0.9053	<0.00005	82.0
θ_{13}	0.8874	0.0006	0.9044	0.0001	79.6
θ_{14}	0.8972	0.0003	0.9019	0.0001	64.8
θ_{21}	6.0750	7.5294	5.4466	4.2202	43.9
θ_{22}	9.0400	8.1951	7.8507	4.6498	43.3
θ_{23}	13.1571	12.4821	10.8947	6.9876	44.0
θ_{24}	5.4977	0.5474	5.0025	0.7746	−41.5
Weibull service time					
θ_{11}	0.9320	0.0003	0.9132	0.0001	81.7
θ_{12}	0.8943	0.0001	0.9007	<0.00005	77.0
θ_{13}	0.8943	0.0003	0.9047	0.0001	71.5
θ_{14}	0.8963	0.0001	0.8990	<0.00005	67.5
θ_{21}	6.5081	6.2433	5.7123	4.1207	34.0
θ_{22}	4.3736	0.7746	4.0249	0.5064	34.6
θ_{23}	7.9743	4.4438	7.2919	3.0851	30.5
θ_{24}	1.9755	0.0951	1.8792	0.0878	7.7
Uniform service time					
θ_{11}	0.9353	0.0002	0.9143	0.0001	46.4
θ_{12}	0.8939	0.0002	0.9019	<0.00005	91.7
θ_{13}	0.8946	0.0004	0.9043	0.0001	85.2
θ_{14}	0.8935	0.0002	0.8977	<0.00005	75.4
θ_{21}	6.0980	7.2565	5.2239	4.9088	32.3
θ_{22}	2.8475	0.2466	2.5843	0.1150	53.4
θ_{23}	5.3534	1.0206	4.5992	0.6964	31.7
θ_{24}	0.5270	0.0056	0.5125	0.0053	5.3

addition to the initial-condition bias that both estimators face. As previously mentioned, bias can arise from the need to estimate b*, and because $E[g(\hat{\delta})] \neq g(\delta)$ [although $g(\hat{\delta})$ is a consistent estimator of $g(\delta)$]. These are important effects to investigate, because variance reduction at the expense of increased bias may not be acceptable.

DISCUSSION

This research indicates the promise of ACVs, and highlights areas requiring further study. These areas fall into two categories: the identification of parametric models to serve as sources of control variates, and the development of appropriate output-analysis methodology.

A ready supply of appropriate control models would greatly reduce the effort and uncertainty involved in selecting controls. This also holds a degree of promise for automating or embedding the ACV approach in existing simulation-support packages, since only a class of models rather than a particular control variate would have to be identified.

In our experiments, the Jackson model served well for the estimation of long-run utilization; however, its application to the queue-time measure was not as effective. Further study is required to determine whether this poorer performance was due to limitations of the Jackson model or other factors, discussed below. A modification that still uses the Jackson model, and might be more effective, is to observe parameters other than δ; for example, observing the effective arrival rate, e, and the probability of an empty queue, p, then computing the Jackson values from these parameters.[6] Extensions to closed queueing networks are already in progress. Also, other approximate queueing models besides the Jackson models should be considered, and extensions to other types of systems (e.g. inventory) should be possible.

The second category, the output-analysis methodology, offers many unanswered questions. The effects due to the method employed to eliminate initialization bias and the use of batch means both have an impact on the performance of the CV estimators. To a large extent, our approach was dictated by cost considerations. It is possible that some aspects of the ACVs' performance were confounded with the performance of the output-analysis method. Recently, Nelson[11] has analytically investigated the combined effects of simultaneously applying multivariate CVs and batching. Finally, procedures such as jack-knifing could be used to reduce the bias of ACV estimators.

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