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The $[Ph_t/Ph_t/\infty]^K$ Queueing System: Part II—The Multiclass Network

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We demonstrate a numerically exact method for evaluating the time-dependent mean, variance, and higher-order moments of the number of entities in the multiclass $[Ph_t/Ph_t/\infty]^K$ queueing network system, as well as at the individual network nodes. We allow for multiple, independent, time-dependent entity classes and develop time-dependent performance measures by entity class at the nodal and network levels. We also demonstrate a numerically exact method for evaluating the distribution function and moments of virtual sojourn time through the network for virtual entities, by entity class, arriving to the system at time t. We include an example using software that we have developed and have put in downloadable form in the Online Supplement to this paper on the journal's website.

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1. Background

We generalize computational results for the $Ph_t/Ph_t/\infty$ queueing system in Nelson and Taaffe (2004) to the case of the multiclass $[Ph_t/Ph_t/\infty]^K$ queueing network. This network consists of K nodes, each of which has an infinite number of servers offering time-dependent, *phase-type* service. Each of the entity classes has its distinct independent, time-dependent, phase-type, network-arrival process or processes. Entities of each class circulate through the network via class-specific, time-dependent Markov routing.

Much has been written about infinite-server queues in the last several years. One reason for this is because infinite-server queues and infinite-server queueing networks have proven to be useful tools for analyzing mobile cellular telephone networks; see, for instance, Boucherie and van Dijk (2000) and Lee (1989). Kella and Whitt (1999) consider fluid network models, and Massey and Whitt (1993) present a thorough analysis of networks of infinite-server queues with nonstationary Poisson input. In their work, Massey and Whitt consider models having nonstationary *Poisson* input processes. Little has been written about nonstationary, infinite-server queueing networks having *non*-Poisson nonstationary input, and that is the focus of this paper.

In Nelson and Taaffe (2004) we developed efficient algorithms for computing time-dependent performance measures for the $Ph_t/Ph_t/\infty$ queueing system based on moment differential, partial-moment differential, and marginal-moment differential equations. The key result is that collectively this set of differential equations is *closed*; i.e., the moment, partial-moment, and marginal-moment differential equations contain no state probabilities on their right-hand sides. This *closure* property implies that time-dependent performance measures can be computed numerically to machine precision without evaluation of the entire (infinite) set of Kolmogorov forward equations. In that paper we also demonstrated that the time-dependent state distribution was *not* a time-dependent Poisson distribution—unlike the case of infinite-server queues and queueing networks having nonstationary Poisson input where the time-dependent state distributions *are* nonstationary Poisson when the initial conditions are empty and idle.

In this paper we generalize the single-node case to the K-node network case where we have R independent, time-dependent, phase-type arrival processes, perhaps representing different entity classes, and time-dependent, class-specific, Markov routing among the K nodes.

The two key results presented here are that (1) the single-class K-node network of time-dependent, phase-type service nodes having time-dependent Markov routing among those nodes is mathematically equivalent to a single-class, *single-node system* with a number of service phases equal to the total number of service phases in the network of service nodes; and (2) an R-class $[Ph_t/Ph_t/\infty]^K$ queueing network with class-specific, independent, time-dependent, phase-type arrival processes and Markov routing is mathematically equivalent to R independent $[Ph_t/Ph_t/\infty]^K$ queueing networks. In other words, we do *not* see an exponential increase in the computing effort for obtaining numerically exact moments when going to the network case, even though the *state space* is increasing in dimension and thus "exploding." For most other queueing networks, as the number of nodes or the number of classes increases, either the computational effort to get numerically exact results increases exponentially, or a decomposition-type *approximation* is needed to reduce the computing effort to a nonexponential increase. Without use of *any* approximations we can decompose the network and nodal time-dependent performance measures by class with computing effort that is *linear* in the number of nodes and *linear* in the number of classes.

Key results 1 and 2 are not particularly surprising, and similar observations have been made for other queueing systems (e.g., a network of $M_t/M_t/\infty$ queues is equivalent to a single $M_t/Ph_t/\infty$ queue, as implied by results in Whitt 1982, for instance). Therefore, the central contribution of this paper is to derive the network-to-single-node construction, and actually to implement it in a computationally useful form.

The paper is organized as follows. Section 2 presents the definitions and notation for the time-dependent arrival, service, and nodal-routing processes for the K-node, R-class network. Section 3 considers the simplified case of a single-arrival process network and proves the equivalence of the network system to the single-node system. Section 4 describes the multiple-entity-class network and presents a simple decomposition that requires no approximations for its evaluation. Section 5 briefly discusses time-dependent network-sojourn time distributions and moments. A detailed example is presented in §6, where diagrams illustrate the equivalence of the network to a single-node system. Section 7 is a brief summary and conclusion. A description of our `MAPLE` code, the code itself, and a detailed example of its use may be found in the Online Supplement to this paper on the journal's website.

2. The $[Ph_t/Ph_t/\infty]^K$ Queueing Network

In this section we define the multiclass $[Ph_t/Ph_t/\infty]^K$ queueing network notation.

2.1. The Arrival Processes

We represent the class-r time-dependent, (w_r+1)-dimensional, phase-type arrival process, $(\boldsymbol{\mathcal{A}}^{[r]}(t), \boldsymbol{\alpha}^{[r]}(t))$, by its underlying Markov chain, the vector of time-dependent Poisson rates associated with each (nonabsorbing) state, and the vector of initial arrival-state probabilities. Each of the arrival processes may represent the arrival process for a different class of entity, and any particular entity class may have multiple independent arrival processes. Let

$$\boldsymbol{\mathcal{A}}^{[r]}(t) \equiv \left(\begin{array}{c|c} \boldsymbol{\mathcal{A}}_1^{[r]}(t) & \boldsymbol{\mathcal{A}}_2^{[r]}(t) \\ \hline \boldsymbol{\alpha}^{[r]}(t)^T & 0 \end{array}\right)$$

be the one-step transition matrix of the Markov chain underlying the w_r-phase arrival process for class r, where

$$\boldsymbol{\mathcal{A}}_1^{[r]}(t) \equiv \begin{pmatrix} a_{11}^{[r]}(t) & \cdots & a_{1w_r}^{[r]}(t) \\ \vdots & \vdots & \vdots \\ a_{w_r1}^{[r]}(t) & \cdots & a_{w_rw_r}^{[r]}(t) \end{pmatrix}$$

is the matrix of transient-to-transient state transition probabilities, and

$$\boldsymbol{\mathcal{A}}_2^{[r]}(t) \equiv \begin{pmatrix} a_{1,\,w_r+1}^{[r]}(t) \\ \vdots \\ a_{w_r,\,w_r+1}^{[r]}(t) \end{pmatrix}$$

is the vector of transition probabilities from transient states to state w_r+1, the instantaneous absorbing state representing an entity arriving to the network via arrival process r. The vector $\boldsymbol{\alpha}^{[r]}(t) \equiv [\alpha_1^{[r]}(t), \ldots, \alpha_{w_r}^{[r]}(t)]^T$ contains the initial arrival-phase probabilities for the next entity to start through the arrival process.

Let $\boldsymbol{\lambda}^{[r]}(t) \equiv [\lambda_1^{[r]}(t), \ldots, \lambda_{w_r}^{[r]}(t)]^T$ be the vector of real-valued, integrable rate functions for transient states of the arrival process. Thus, the (w_r+1)-dimensional rate vector for the entire class-r phase arrival process is $[\boldsymbol{\lambda}^{[r]}(t)^T, \infty]$. The infinite rate corresponds to the instantaneous absorbing state.

Finally, let $\{A^{[r]}(t);\ t \geq 0\}$ be the random process representing the arrival phase of the next arrival to the network at time t from arrival process r, where $A^{[r]}(t) \in \{1, \ldots, w_r\}$. The instantaneous absorbing states need not be explicitly represented.

2.2. The Service Processes

We define the node-k (v_k+1)-dimensional phase-type service process, $(\boldsymbol{\mathcal{B}}^{[k]}(t), \boldsymbol{\beta}^{[k]}(t))$, in a manner similar to the definition of the arrival processes.

Let

$$\boldsymbol{\mathcal{B}}^{[k]}(t) \equiv \begin{pmatrix} \boldsymbol{\mathcal{B}}_1^{[k]}(t) & \boldsymbol{\mathcal{B}}_2^{[k]}(t) \\ \boldsymbol{\beta}^{[k]}(t)^T & 0 \end{pmatrix},$$

where

$$\boldsymbol{\mathcal{B}}_1^{[k]}(t) \equiv \begin{pmatrix} b_{11}^{[k]}(t) & \cdots & b_{1v_k}^{[k]}(t) \\ \vdots & \vdots & \vdots \\ b_{v_k1}^{[k]}(t) & \cdots & b_{v_kv_k}^{[k]}(t) \end{pmatrix}$$

is the underlying Markov chain one-step transition matrix for transient-to-transient service-phase transitions at node k, and

$$\boldsymbol{\mathcal{B}}_2^{[k]}(t) \equiv \begin{pmatrix} b_{1,\,v_k+1}^{[k]}(t) \\ \vdots \\ b_{v_k,\,v_k+1}^{[k]}(t) \end{pmatrix}$$

is the matrix of transition probabilities from transient service phases to the instantaneous absorbing state $v_k + 1$, which represents a service completion (a departure from node k). The vector $\boldsymbol{\beta}^{[k]}(t)^T \equiv [\beta_1^{[k]}(t), \ldots, \beta_{v_k}^{[k]}(t)]^T$ contains the initial service-phase probabilities for an entity entering service node k.

Let $\boldsymbol{\mu}^{[k]}(t) \equiv [\mu_1(t), \ldots, \mu_{v_k}(t)]^T$ be the vector of real-valued integrable rate functions for the transient states of the service process, so that $[\boldsymbol{\mu}^{[k]}(t)^T, \infty]$ is the (v_k+1)-dimensional rate vector for the entire node-k phase service process.

2.3. The Class-r Time-Dependent Markov Node-to-Node Routing Process

The class-r time-dependent Markov node-to-node routing probabilities are defined by

$$\mathbf{P}^{[r]}(t) \equiv \left(\begin{array}{c|c} \mathbf{P}_1^{[r]}(t) & \mathbf{P}_2^{[r]}(t) \\ \hline \mathbf{P}_0^{[r]}(t) & 0 \end{array}\right),$$

where

$$\mathbf{P}_1^{[r]}(t) \equiv \begin{pmatrix} p_{11}^{[r]}(t) & \cdots & p_{1K}^{[r]}(t) \\ \vdots & \ddots & \vdots \\ p_{K1}^{[r]}(t) & \cdots & p_{KK}^{[r]}(t) \end{pmatrix},$$

$$\mathbf{P}_2^{[r]}(t) \equiv \begin{pmatrix} p_{10}^{[r]}(t) \\ \vdots \\ p_{K0}^{[r]}(t) \end{pmatrix},$$

and

$$\mathbf{P}_0^{[r]}(t) \equiv \left(p_{01}^{[r]}(t), \ldots, p_{0K}^{[r]}(t)\right),$$

and where node 0 is the exit/entrance "node" of the network.

Notice that we specify a different time-dependent arrival process and node-to-node Markov routing process for each of the entity classes. We could also index the service-phase process parameters for each of the R entity classes, as well as for each of the K nodes in the network, by $(\boldsymbol{\mathcal{B}}^{[r,k]}(t), \boldsymbol{\mu}^{[r,k]}(t))$. By doing so we would be representing different time-dependent service processes (or distributions in the case of stationary systems) for each of the entity classes at each of the nodes. Our choice of notation for this presentation does not explicitly differentiate nodal service processes by entity class. This notation choice allows for less cumbersome network representation, and, as we will show later, the R entity-class network is equivalent to R independent single-class networks, so we can evaluate network performance measures by entity class via separate evaluations of the single-class networks, changing nodal-service-process parameters for each different evaluation.

3. The $[Ph_t/Ph_t/\infty]^K$ Single Arrival-Process Network

We start the description of the state of the multiclass $[Ph_t/Ph_t/\infty]^K$ process at time t with the case of K time-dependent, phase-type service nodes, and only one time-dependent, phase-type arrival process. Later we generalize to the case of R independent, phase-type arrival processes. In this section we omit the superscript $^{[r]}$ that indicates entity class for ease of reading.

The state of the process at time t is given by

$$\begin{aligned}[\mathbf{N}(t), A(t)] &\equiv [\mathbf{N}^{[1]}(t), \ldots, \mathbf{N}^{[K]}(t), A(t)] \\ &\equiv \left[\left(N_1^{[1]}(t), \ldots, N_{v_1}^{[1]}(t)\right), \ldots, \right. \\ &\qquad \left. \left(N_1^{[K]}(t), \ldots, N_{v_K}^{[K]}(t)\right), A(t)\right], \qquad (1)\end{aligned}$$

where $\{N_i^{[k]}(t);\ t \geq 0\}$ for $i = 1, \ldots, v_k$, and $k = 1, \ldots, K$, is the random process representing the number of entities who, at time t, are in the ith phase of their service at node k. Let the random variable representing the total number of entities in service at node k at time t be

$$N^{[k]}(t) \equiv \sum_{i=1}^{v_k} N_i^{[k]}(t),$$

and let the total number of entities in service in the entire network at time t be

$$N(t) \equiv \sum_{k=1}^{K} N^{[k]}(t).$$

The instantaneous absorbing states in the service processes need not be explicitly represented.

3.1. The $[Ph_t/Ph_t/\infty]^K$ and $Ph_t/Ph_t/\infty$ Equivalence

In this section we develop a key equivalence result. We continue to discuss the case of a network having a single, time-dependent, phase-type arrival process and the ordering of the state space as indicated in (1). We show that the *network* of time-dependent service nodes can be viewed as a *single* time-dependent service node. This observation leads to the result that a single-arrival-process $[Ph_t/Ph_t/\infty]^K$ network is equivalent to a single-arrival-process $Ph_t/Ph_t/\infty$ system; thus, all of the mathematical and computational performance measures developed in Nelson and Taaffe (2004) can be directly applied to the network system as easily as to a single-node, single-arrival-process system.

In the single-node, single-arrival-process notation of Nelson and Taaffe (2004) we need to construct $(\mathcal{B}(t), \boldsymbol{\mu}(t))$, where:

$$\mathcal{B}(t) \equiv \left(\begin{array}{c|c} \mathcal{B}_1(t) & \mathcal{B}_2(t) \\ \hline \boldsymbol{\beta}(t)^T & 0 \end{array}\right),$$

and $\boldsymbol{\mu}(t)$ represents the time-dependent rates at each transient phase of the service process. In constructing the single-node equivalent $(\mathcal{B}(t), \boldsymbol{\mu}(t))$ we must also include consideration of the time-dependent Markov routing among the nodes of the network.

Let

$$\mathcal{B}_1(t) \equiv \mathbf{P}_1(t) \cdot \mathbf{G}(t) + \mathbf{Diag}\big[\mathcal{B}_1^{[i]}(t)\big], \tag{2}$$

where

$$\mathbf{G}(t) \equiv [G_{ij}(t)],$$

$$G_{ij}(t) \equiv \begin{pmatrix} \big(b^{[i]}_{1,\,v_i+1}(t)\beta^{[j]}(t)\big)^{\mathrm{T}} \\ \vdots \\ \big(b^{[i]}_{v_i,\,v_i+1}(t)\beta^{[j]}(t)\big)^{\mathrm{T}} \end{pmatrix},$$

$$\mathbf{X} \cdot \mathbf{Y} \equiv [x_{ij}y_{ij}],$$

and $\cdot$ indicates the matrix Hadamard product. Thus

$$\mathcal{B}_1(t) = \begin{pmatrix} p_{11}(t)G_{11}(t) + \mathcal{B}_1^{[1]}(t) & p_{12}(t)G_{12}(t) & \cdots & p_{1K}(t)G_{1K}(t) \\ p_{21}(t)G_{21}(t) & p_{22}(t)G_{22}(t) + \mathcal{B}_1^{[2]}(t) & \vdots & p_{2K}(t)G_{2K}(t) \\ \vdots & & \ddots & \\ p_{K1}(t)G_{K1}(t) & \cdots & \cdots & p_{KK}(t)G_{KK}(t) + \mathcal{B}_1^{[K]}(t) \end{pmatrix}$$

is the underlying Markov chain one-step transition matrix for transient-to-transient state transitions in the single-node equivalent representation of the network.

Let

$$\mathcal{B}_2(t) \equiv \begin{pmatrix} p_{10}(t)B_2^{[1]}(t) \\ \vdots \\ p_{K0}(t)B_2^{[K]}(t) \end{pmatrix}$$

be the single-node equivalent matrix of transition probabilities from transient states to the instantaneous absorbing state $(\sum_{k=1}^{K} v_k) + 1$, which represents a departure from the network.

Finally, let

$$\boldsymbol{\beta}(t) \equiv \big[p_{01}(t)\beta^{[1]}(t), \ldots, p_{0K}(t)\beta^{[K]}(t)\big]^T$$

contain the initial service-phase probabilities for an entity completing its arrival process, and thus entering into service.

The total number of (nonabsorbing) service phases in the single-node equivalent representation of the network is $\sum_{k=1}^{K} v_k$. Let the vector of rates associated with the set of service phases be $\boldsymbol{\mu}(t)^T \equiv [\boldsymbol{\mu}^{[1]}(t), \ldots, \boldsymbol{\mu}^{[K]}(t)]$ where, as before, $\boldsymbol{\mu}^{[k]}(t) \equiv [\boldsymbol{\mu}_1^{[k]}(t), \ldots, \boldsymbol{\mu}_{v_k}^{[k]}(t)]^T$.

3.2. Construction

In this section we construct and interpret the matrix components of $\mathcal{B}(t)$. See §6 for a detailed example.

- $\mathcal{B}_1(t)$ for $i \neq j$: The (i, j)th component of $\mathcal{B}_1(t)$, itself a matrix, represents transitions from some terminal phase of node i to some initial phase of node j. For this transition to occur an entity must complete its time in a terminal phase of service at node i *and* be routed to node j. The conditional probability of an "absorption" or end of service, at phase s of node i, given that the entity is ending its service at phase s of node i at time t, is $b^{[i]}_{s,\,v_i+1}(t)$. The conditional probability that an entity ending its service at node i proceeds to some initial phase of service at node j at time t is $p_{ij}(t)$. The probability that an entity that enters node j at time t enters it at phase l is $\beta_l^{[j]}(t)$. Thus the (i, j)th entry of $\mathcal{B}_1(t)$ is a matrix of probabilities and the (s, l)th entry of that matrix is $b^{[i]}_{s,\,v_i+1}(t)p_{ij}(t)\beta_l^{[j]}(t)$, which represents the conditional probability of an entity that is finishing service in phase s of node i proceeding to leave node i and enter node j in phase l at time t.

- $\mathcal{B}_1(t)$ for $i = j$: There are two terms that compose each element of the diagonal entries in $\mathcal{B}_1(t)$ for $i = j$. They are $p_{ii}(t)G_{ii}(t)$ and $\mathcal{B}_1^{[i]}(t)$, and they have the following interpretation:

The first term, $p_{ii}(t)G_{ii}(t)$, is a matrix whose elements are $b^{[i]}_{s,\,v_i+1}(t)p_{ii}(t)\beta_l^{[i]}(t)$. This is the case of immediate feedback from node i to node i, similar to the description above.

The second term, $\mathcal{B}_1^{[i]}(t)$, contains the conditional probabilities guiding state transitions from a node i transient phase to another node i transient phase *without* passing through an instantaneous end-of-service or "absorption" phase at node i. Therefore the (s, l)th entry of $\mathcal{B}_1^{[i]}(t)$, for $l \neq v_i + 1$, is $b_{sl}^{[i]}(t)$, the conditional probability that an entity exiting phase s of node i proceeds directly to phase l of node i without having completed its service at node i.

• $\mathcal{B}_2(t)$: The ith component of $\mathcal{B}_2(t)$, itself a column vector, represents network departures from node i at time t. The ith vector entry in $\mathcal{B}_2(t)$ contains the conditional probabilities of an entity exiting node i, given that the entity is finishing service at some phase of node i at time t, and that this entity then leaves the network. Specifically, the sth element of the ith row of $\mathcal{B}_2(t)$ is $b_{s,0}^{[i]}(t)p_{i0}(t)$, the conditional probability that an entity completing service at phase s of node i exits the network at time t.

• $\boldsymbol{\beta}(t)$: The ith element of the vector $\boldsymbol{\beta}(t)$, $p_{0i}(t)\boldsymbol{\beta}^{[i]}(t)^T$, is a vector of conditional probabilities representing the entry node, and the entry phase within a node, of a newly arriving entity to the network. The $p_{0i}(t)$ factor represents the probability that node i is the entry node for an arrival at time t. The conditional probability of an arriving entity entering node i at phase s, given that node i is the entry node, is the sth element of the vector $\boldsymbol{\beta}^{[i]}(t)$.

Thus, the single-arrival process $[Ph_t/Ph_t/\infty]^K$ queueing network is equivalent to a $Ph_t/Ph_t/\infty$ queue. Next we show why a network having R independent, time-dependent, phase-type arrival processes is equivalent to R independent, single-arrival-process networks.

4. Multiple Entity Classes

We now generalize the model to include several entity classes. Each entity class may have multiple independent, time-dependent, phase-type arrival processes. In this section we will simply refer to "multiple arrival sources" or "multiple arrival processes" to include both multiple entity classes and entity classes having more than one arrival process.

In the simple Jackson network (either the stationary or the time-dependent case) when there is more than one (time-dependent) Poisson network-arrival process, the superposition property of independent Poisson processes results in a composite (time-dependent) Poisson network-arrival process. Likewise, the (time-dependent) Markov splitting of a (time-dependent) Poisson process results in (time-dependent) independent Poisson processes. This is not the case for any other continuous-time stochastic point process. In fact, in the stationary case, the splitting or superposition of independent renewal (but non-Poisson) processes results in a process or processes that are *not* renewal processes (Kao 1997). As a result, we cannot superpose several independent Ph_t processes and represent the overall network-arrival process as a single Ph_t process. We therefore present a generalization of the network model to include the case of having multiple independent entity-arrival sources.

Consider the R-class $[Ph_t/Ph_t/\infty]^K$ network. The state of the process at time t is given by

$$[\mathbf{N}(t), \mathbf{A}(t)] \equiv \big[\big(\mathbf{N}^{[1,1]}(t), \ldots, \mathbf{N}^{[1,K]}(t)\big), \ldots, \big(\mathbf{N}^{[R,1]}(t), \ldots, \mathbf{N}^{[R,K]}(t)\big), \mathbf{A}(t)\big],$$

where

$$\mathbf{N}^{[r,k]}(t) \equiv \big(N_1^{[r,k]}(t), \ldots, N_{v_k}^{[r,k]}(t)\big),$$

and

$$\mathbf{A}(t) \equiv \big(A^{[1]}(t), \ldots, A^{[R]}(t)\big).$$

Let $A^{[r]}(t)$ be the state of arrival process r at time t, and $\mathbf{A}(t)$ be the R-dimensional vector containing the $A^{[r]}(t)$ terms. Let

$$\mathbf{N}^{[k]}(t) \equiv \sum_{r=1}^{R} \mathbf{N}^{[r,k]}(t) = \big(N_1^{[k]}(t), \ldots, N_{v_k}^{[k]}(t)\big)$$

be the vector of random variables describing the total number of entities across all entity classes in each of the service phases at node k at time t. Let

$$N^{[r,k]}(t) \equiv \sum_{i=1}^{v_k} N_i^{[r,k]}(t)$$

be the random variable describing the total number of entities of entity class r at node k at time t, and let

$$N^{[k]}(t) \equiv \sum_{r=1}^{R} N^{[r,k]}(t)$$

be the random variable describing the total number of entities across all entity classes at node k at time t. Likewise, let

$$N^{[r,\cdot]}(t) \equiv \sum_{k=1}^{K} N^{[r,k]}(t)$$

be the random variable describing the total number of class-r entities in the network at time t.

The $[Ph_t/Ph_t/\infty]^K$ network for systems having R arrival sources (or entity classes) can be represented by R single-arrival-source $[Ph_t/Ph_t/\infty]^K$ networks. The key idea is that the entities never contend with one another for access to the servers because there is an infinite number of servers at every node; i.e., the entity classes are stochastically independent. Because of this independence, the single-node network model (and software) can be used to analyze each of the entity classes separately (at the nodal and network levels). Furthermore, because of the independence, overall network performance measures across all entity classes are also simple (mathematically) to compute. For instance,

$$\mathrm{E}[N(t)] = \mathrm{E}\left[\sum_{r=1}^{R} N^{[r,\cdot]}(t)\right] = \sum_{r=1}^{R} \mathrm{E}\big[N^{[r,\cdot]}(t)\big],$$

and

$$\mathrm{Var}[N(t)] = \mathrm{Var}\left[\sum_{r=1}^{R} N^{[r,\cdot]}(t)\right] = \sum_{r=1}^{R} \mathrm{Var}\big[N^{[r,\cdot]}(t)\big].$$

5. Network Virtual Sojourn Time

Let $W_t^{[r]}$ be the sojourn time of a virtual entity emanating from network arrival source r (or entity class r), at time t. Because there is an infinite number of servers at each node, the virtual sojourn time through the entire network has no impact on the sojourn times of other entities in the network before or after time t, regardless of the entity's class or network-arrival source.

Because of the equivalence of the R-arrival-process $[Ph_t/Ph_t/\infty]^K$ network to R single-arrival-process $[Ph_t/Ph_t/\infty]^K$ networks, and because of the equivalence of the $[Ph_t/Ph_t/\infty]^K$ network to the $Ph_t/Ph_t/\infty$ node, the *time-dependent virtual sojourn time* for a single-class, single-node system from Nelson and Taaffe (2004) is also the method required to compute the time-dependent, class-specific, virtual sojourn time for the network system.

In Nelson and Taaffe (2004) a method is described for evaluating the *time-dependent virtual sojourn time* for a single-class, single-node $Ph_t/Ph_t/\infty$ queueing system based on equivalence of the time-dependent virtual-waiting-time distribution at time t and the time-dependent probability of having an empty system at times greater than t for a system that has exactly one entity present at time t and has no previous or future arrivals.

6. Example

Next we present an example. We show both its network representation and equivalent single-node representation. We also calculate several time-dependent performance measures for this example using the MAPLE code that we have made available in downloadable form via the Online Supplement for this paper. The example is designed to illustrate our results, rather than to represent any particular real queueing system.

Consider the two-node network illustrated in Figure 1. A single-arrival process feeds the network, which includes immediate feedback at node 1 and feedback from node 2 to node 1. The time between arrivals is modeled by a time-dependent phase-type distribution with three phases, as illustrated in Figure 2. When we solve this problem numerically we use the specific values

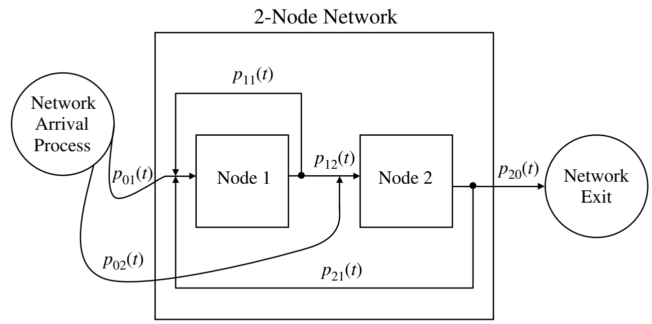

Figure 1 **Diagram of the Two-Node Network**

$$\boldsymbol{\mathcal{A}}(t) = \begin{pmatrix} 0 & \frac{1}{3} + \frac{1}{3}\cos\left(\frac{1}{4}\pi t\right) & \frac{2}{3} - \frac{1}{3}\cos\left(\frac{1}{4}\pi t\right) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.3 & 0 & 0.7 \\ 0.8 & 0.2 & 0 & 0 \end{pmatrix},$$

and

$$\boldsymbol{\lambda}(t) = \left(5, 8, 10 + 5\sin\left(\frac{1}{5}\pi t\right)\right).$$

The service processes at each node are also represented by three-phase, phase-type distributions, as shown in Figures 3 and 4, with specific values

$$\boldsymbol{\mathcal{B}}(t)^{[1]} = \begin{pmatrix} 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & \frac{1}{4} + \frac{1}{4}\sin\left(\frac{1}{6}\pi t\right) & \frac{3}{4} - \frac{1}{4}\sin\left(\frac{1}{6}\pi t\right) \\ 0 & 0 & 0.2 & 0.8 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix},$$

$$\boldsymbol{\mu}(t)^{[1]} = (3, 4, 7.5),$$

$$\boldsymbol{\mathcal{B}}(t)^{[2]} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}, \quad \text{and}$$

$$\boldsymbol{\mu}(t)^{[2]} = (2, 2, 2).$$

The Markov routing matrix among the nodes is

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\sin\left(\frac{1}{4}\pi t\right) & \frac{1}{2} - \frac{1}{2}\sin\left(\frac{1}{4}\pi t\right) & 0 \\ 0.15 & 0 & 0.85 \\ 0.7 & 0.3 & 0 \end{pmatrix}.$$

Thus, there are nonstationary components in the arrival, service, and routing processes. We illustrate

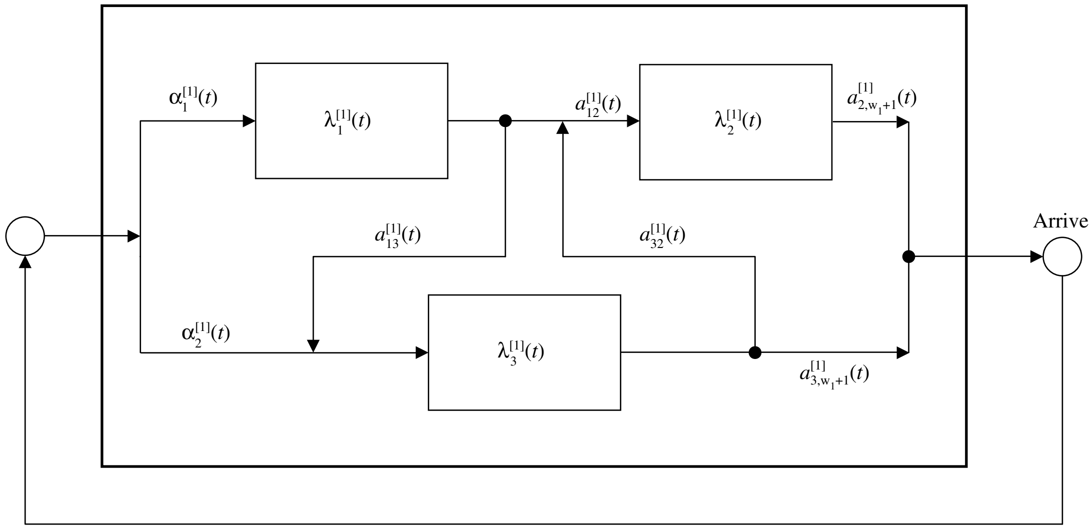

Figure 2 Diagram of the Time-Dependent Three-Phase Phase-Type Arrival Process

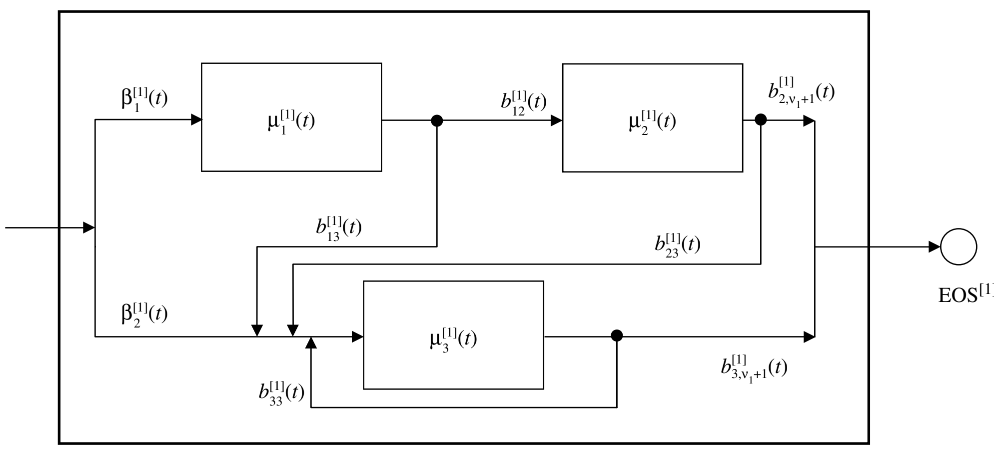

Figure 3 Diagram of the Time-Dependent Three-Phase Phase-Type Node-1 Service Process

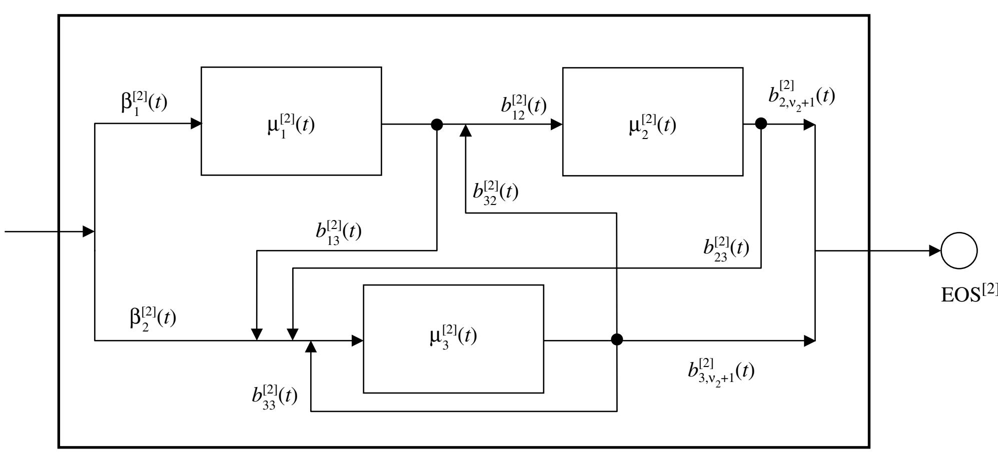

Figure 4 Diagram of the Time-Dependent Three-Phase Phase-Type Node-2 Service Process

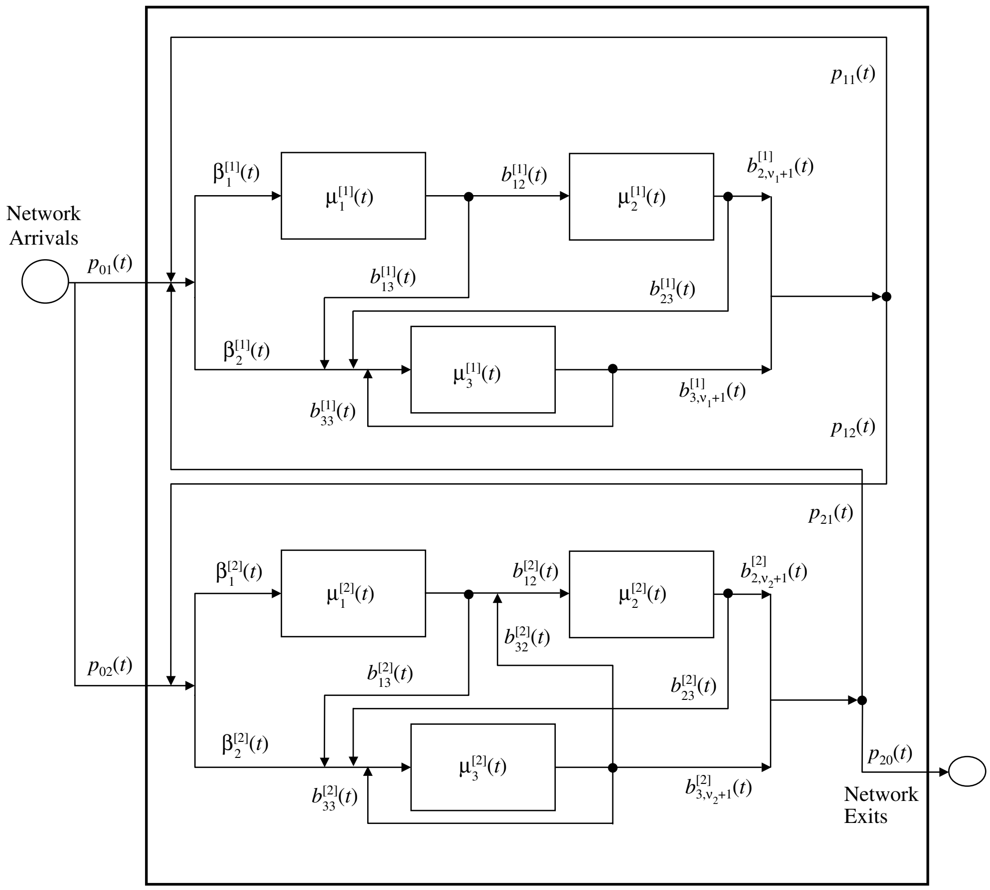

Figure 5 **The Single-Node Equivalent Representation of the Two-Node Network**

only a single entity class because having R classes is equivalent to R independent K-node networks and thus introduces no new complexity.

Figure 5 expands Figure 1 to show the phase-type service process within each node. A key result of this paper is that a network of $Ph_t/Ph_t/\infty$ queues is equivalent to a single $Ph_t/Ph_t/\infty$ node; our software constructs this equivalent representation automatically. The bold line in Figure 5 indicates the single node.

Using our MAPLE software, or any other software for numerically integrating systems of differential equations, we can compute time-dependent performance measures. Figure 6 displays the mean or expected number of entities in the network as a whole, and at each node in the network, from time $t = 0$ to $t = 10$, when the system begins empty and idle.

The variance of the number in the system and at each node for the same time period is shown in Figure 7. Notice that even though the curves in both figures have the same shape, they do not represent the same values. This establishes that the number in the system or at each node does not have a time-dependent Poisson distribution, because if they did, the mean and variance would be equal.

For a virtual entity arriving to the network, we can compute the mean, variance, and cdf of its sojourn time by applying the technique described in Nelson and Taaffe (2004) to the single-node representation.

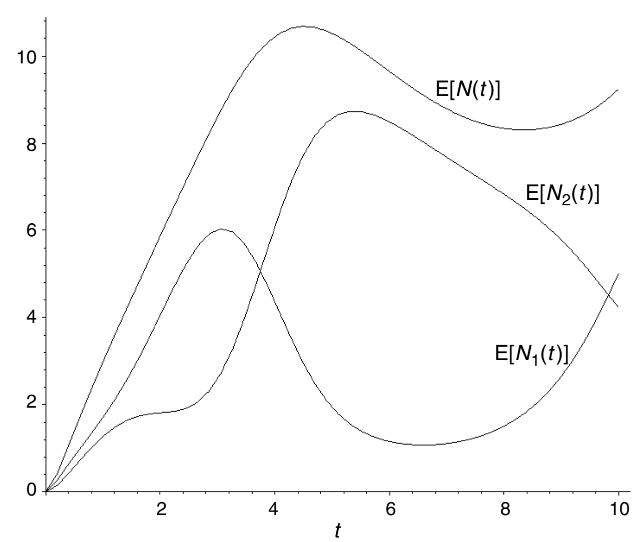

Figure 6 **Plot of the Time-Dependent Mean Number of Entities in the Network and at Each Node**

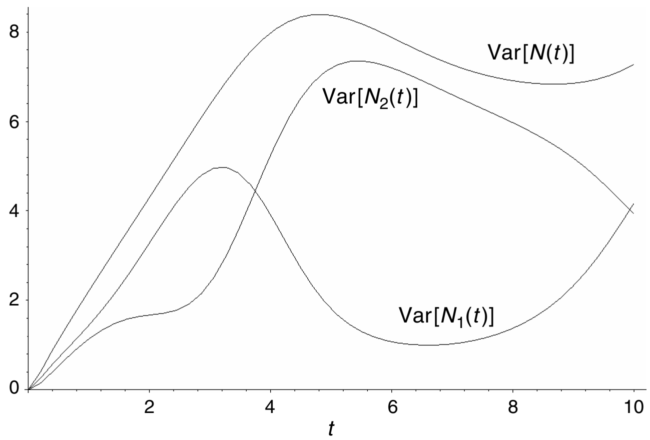

Figure 7 Plot of the Time-Dependent Variance of the Number of Entities in the Network and at Each Node

For example, a virtual entity arriving to our example network at time $t = 5$ would expect to spend 2.5 time units in the system, with a standard deviation of 2.2 time units. The cdf of the entity's sojourn time can be computed using our MAPLE software available via the Online Supplement.

7. Conclusions

In this paper we have shown the rather pleasing result that the $[Ph_t/Ph_t/\infty]^K$ queueing network can be viewed as a single $Ph_t/Ph_t/\infty$ queue. We have also demonstrated that such networks having multiple entity classes can be decomposed into separate single-node, single-class queueing systems. As a result the computational effort to analyze the time-dependent nodal and network behavior by entity class is a *linear* function of the number of nodes and the number of entity classes. All of the MAPLE code to evaluate such networks is in downloadable form via the Online Supplement.

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