

## MODELING AND SIMULATING NON-POISSON ARRIVAL PROCESSES TO FACILITATE ANALYSIS

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### ABSTRACT

This paper introduces a method to model and simulate nonstationary, non-renewal arrival processes that depends only on the analyst setting intuitive and easily controllable parameters. Thus, it is suitable for assessing the impact of nonstationary, non-exponential and non-independent arrivals on simulated performance when they are suspected but no data are available.

### 1 INTRODUCTION

Arrival processes are the drivers of many stochastic simulation models, including, but not limited to, queueing and supply chain simulations. The stationary Poisson arrival process—implying interarrival times that are independent and identically distributed (i.i.d.) exponentials with a common mean—is well known, and often justified because it represents “arrivals from a large customer population making independent decisions about when to arrive.”

However, interarrival times are frequently more variable (e.g., telecommunications) or more regular (e.g., manufacturing orders) than Poisson. To handle this, general stationary renewal arrival processes are a feature of every commercial simulation product. Arrival processes with a time-varying arrival rate are also prevalent in practice, leading some of the same software products to include the capability to generate arrivals from a nonstationary Poisson process (NSPP). Fitting a renewal process or NSPP to data are problems that have been well studied (e.g., Law and Kelton 2000 and references therein).

Of course, stationary renewal processes and NSPPs do not address all of the possible departures from “Poissonness,” which lead Gerhardt and Nelson (2009) to consider nonstationary, non-Poisson (NSNP) arrival processes; NSNPs are generalizations of stationary renewal processes that allow a time-varying arrival rate. Their work provides methods for fitting and simulating NSNP processes.

The purpose of this paper is two-fold: From a basic theory perspective, we extend one of Gerhardt and Nelson’s results to facilitate generation of nonstationary, nonrenewal (NSNR) arrivals, which, in a sense, addresses the final remaining departure from Poisson arrivals (dependent interarrival times). However, rather than focusing on fitting NSNR processes to data, as Gerhardt and Nelson (2009) do, we provide a specific method designed to allow a user to easily and intuitively define NSNR processes without data. This facilitates assessing the impact of nonstationary, non-exponential, and dependent arrival processes on simulation results when no or only partial information on the arrival processes are available.

We believe that this situation is very common in practice: The modeler is aware that the arrivals are not well represented as Poisson, but has neither data nor enough information to fully specify the alternative. Therefore, the goal—if it is easy enough to do—is to see how much these non-Poisson features matter. *A central premise of this work is that modelers will analyze what they can readily model.* Thus, it is more important to be able to model non-Poisson features than it is to represent them perfectly.

### 2 THEORY

The goal is to define and simulate a sequence of interarrival times  $\{W_n, n \geq 1\}$  such that the arrival counting process  $I(t) = \max\{n \geq 0 : V_n \leq t\}$ , where  $V_n = \sum_{i=1}^n W_i$ , is nonstationary and non-renewal in easily controllable and understandable ways.

We begin with a set of nonnegative interevent times  $\{X_n, n \geq 1\}$ , and let  $S_n$  denote the time of the  $n^{\text{th}}$  event; that is,  $S_0 = 0$  and  $S_n = \sum_{i=1}^n X_i$ , for  $n = 1, 2, \dots$ . Let  $N(t)$  denote the number of events that have occurred on or before time  $t$ ; that is,  $N(t) = \max\{n \geq 0 : S_n \leq t\}$ , for  $t \geq 0$ . We assume that the process is initialized in equilibrium, so that, in particular,  $E\{N(t)\} = rt$ , for all  $t \geq 0$ , for some fixed arrival rate  $r > 0$ , and  $X_2, X_3, \dots$  are identically distributed.

The index of dispersion for counts for this process is

$$\text{IDC} = \lim_{t \rightarrow \infty} \frac{\text{Var}\{N(t)\}}{\text{E}\{N(t)\}} \quad (1)$$

which we assume exists. For a Poisson process  $\text{IDC} = 1$ ; for an equilibrium renewal process  $\text{IDC} = cv^2$ , the squared coefficient of variation of  $X$ ; but more generally the IDC captures both the variability and dependence in a stationary arrival process. Notice that (1) implies that for large  $t$ ,  $\text{Var}\{N(t)\} \approx \text{IDC} \cdot \text{E}\{N(t)\}$ . From here on we will assume  $r = 1$ .

Now suppose that  $r(t)$ ,  $t \geq 0$ , is the desired, integrable arrival rate for  $I(t)$ , and let  $R(t) = \int_0^t r(s) ds$ . For  $s \in \mathfrak{R}$ , define  $R^{-1}(s) \equiv \inf\{t : R(t) \geq s\}$ . Then we have the following algorithm for generating NSNR processes.

**Algorithm 1** *The Inversion Method for NSNR Processes*

1. Set  $V_0 = 0$ , index counter  $n = 1$ . Generate  $S_1$ . Set  $V_1 = R^{-1}(S_1)$ .
2. Return interarrival time  $W_n = V_n - V_{n-1}$ .
3. Set  $n = n + 1$ . Generate  $X_n$ . Set  $S_n = S_{n-1} + X_n$  and  $V_n = R^{-1}(S_n)$ .
4. Go to Step 2.

We have the following properties of  $I(t)$ .

**Theorem 1**  $\text{E}\{I(t)\} = R(t)$ , for all  $t \geq 0$ , and  $\text{Var}\{I(t)\} \approx \text{IDC} \cdot R(t)$ , for large  $t$ .

**Proof:** Since  $N$  is an equilibrium arrival process and  $r = 1$ , then  $\text{E}\{N(t)\} = t$ , for all  $t \geq 0$ , while  $\text{Var}\{N(t)\} \approx \text{IDC} \cdot t$ , for large  $t$ . Thus,

$$\begin{aligned} \text{E}\{I(t)\} &= \text{E}\{\text{E}[I(t)|N(R(t))]\} \\ &= \text{E}\{N(R(t))\} \\ &= R(t), \end{aligned}$$

for all  $t \geq 0$ , while

$$\begin{aligned} \text{Var}\{I(t)\} &= \text{E}\{\text{Var}[I(t)|N(R(t))]\} + \text{Var}\{\text{E}[I(t)|N(R(t))]\} \\ &= 0 + \text{Var}\{N(R(t))\} \\ &\approx \text{IDC} \cdot R(t), \end{aligned}$$

for large  $t$ .  $\square$

Thus,  $I(t)$  has the desired arrival rate, while preserving the IDC of the stationary base arrival process  $N(t)$  from which it was derived. When  $N(t)$  is a rate-1 Poisson process, this is the well-known inversion method for generating an NSNP. Gerhardt and Nelson (2009) extended this method (along with the so-called ‘‘thinning’’ method) to nonstationary, non-Poisson processes (but still renewal).

The IDC is not an intuitively understandable measure of variability and dependence. However, for many stationary

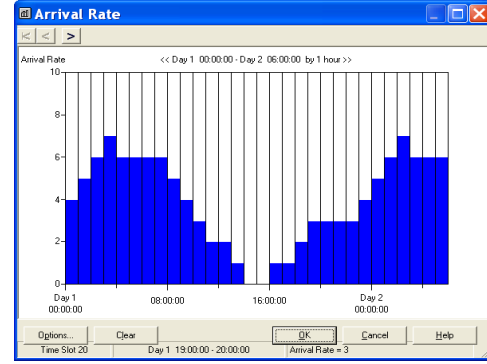


Figure 1: Graphical interface for specifying a piecewise constant arrival rate in Arena.

arrival processes it is equal to the index of dispersion for intervals (Gusella 1991),

$$\begin{aligned} \text{IDI} &= \lim_{n \rightarrow \infty} \frac{\text{Var}\{S_n\}}{n\text{E}^2\{X_2\}} \\ &= cv^2 \left( 1 + 2 \sum_{j=1}^{\infty} \rho_j \right) \end{aligned}$$

where  $\rho_j$  is the lag- $j$  autocorrelation of the stationary interarrival times (for  $\text{IDI} = \text{IDC}$  it is clear that the autocorrelation structure of the interarrival times must be summable, ruling out certain types of long-range dependence). The IDC summarizes the marginal variability of the interarrival times via  $cv^2$ , and the dependence among arrivals via  $\left( 1 + 2 \sum_{j=1}^{\infty} \rho_j \right)$ .

In summary, the inversion method attains the desired arrival rate while transferring the IDC of the base process to the NSNR arrival process.

### 3 MODELING ARRIVAL PROCESSES FOR ANALYSIS

The inversion method in Section 2 provides a basis for defining NSNR arrival processes with control over the arrival rate, marginal variability of the interarrival times, and dependence among the interarrival times. In this section we describe a specific implementation that is highly suitable for analysis.

#### 3.1 Arrival Rate

The desired arrival rate  $r(t)$  should be specified in an intuitive manner that also facilitates inversion of  $R(t)$ . A piecewise constant arrival rate fills this need, since  $R(t)$  is piecewise linear and therefore easily inverted. Figure 1 shows the

point-and-click graphical interface used in Arena to specify a piecewise constant arrival rate function hour by hour.

### 3.2 Base Process

For the base arrival process  $N(t)$ , we suggest the Markov MECO process of Johnson (1998). The Markov MECO is a particular case of a Markovian arrival process (MAP); MAPs represent interarrival times as the times to absorption of a continuous-time Markov chain (CTMC) where the initial state of the next interarrival time depends upon which absorbing state the previous interarrival time entered. The Markov MECO is based on the MECO (Mixture of Erlangs of Common Order) renewal process that can capture any feasible first three moments of the interrenewal time. The Markov MECO extends the MECO to nonrenewal arrivals by providing a parameter that controls the dependence between interarrival times (described more fully below).

As discussed in Gerhardt and Nelson (2009), a key benefit of using a MAP base processes is that it is easy to initialize in equilibrium, requiring only that the distribution of current state of the CTMC in equilibrium be computed; given the current state, the remaining time in that state is always exponentially distributed.

Since the arrival rate for the base Markov MECO must be 1, this leaves two additional parameters for the user:  $cv$ , the coefficient of variation of the interarrival times, and some measure of dependence (the Markov MECO also allows specification of a third moment of the interarrival time, but it is easy to build in defaults for that parameter). For a Markov MECO the dependence can be specified either as  $\rho_1$ , the lag-1 autocorrelation between arrivals, or as  $\left(1 + 2\sum_{j=1}^{\infty} \rho_j\right)$ ; these two are equivalent as the Markov MECO has geometrically decreasing autocorrelations.

Figure 2 shows a potential interface for allowing users to easily specify and modify an NSNR arrival process. Notice that a constant arrival rate with both slider bars at their midpoints is a Poisson arrival process, and this could be the default. Prespecified ranges for the variability and dependence can be embedded (e.g.,  $0 \leq cv \leq 10$  and  $\rho_1$  between the minimum and maximum feasible for the Markov MECO with that  $cv$ ); alternatively a different interface could all numerical values to be input for users with data or enough understanding of the parameters' meanings. The key point is that details of the inversion method or Markov MECO can be transparent to the user.

### ACKNOWLEDGMENTS

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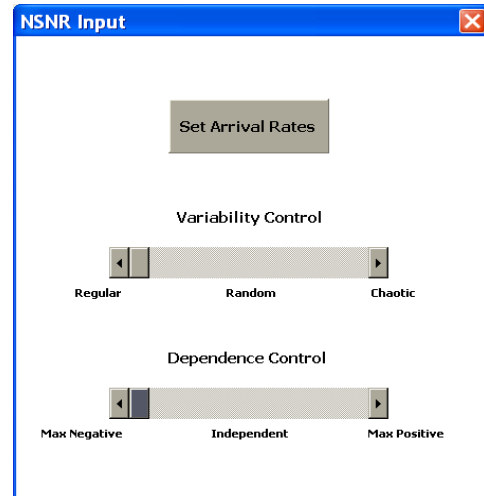


Figure 2: Mock up of interface for specifying NSNR arrival processes.

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