# SOME PROPERTIES OF SIMULATION INTERVAL ESTIMATORS UNDER DEPENDENCE INDUCTION

Barry L. NELSON

*Department of Industrial and Systems Engineering, The Ohio State University, Columbus, OH 43210, USA*

Received March 1987
Revised July 1987

Some basic, but surprising properties of confidence intervals formed under antithetic-variates and common-random-numbers variance reduction techniques are derived. It is shown that improved point estimator performance, the primary goal of variance reduction techniques, does not necessarily imply improved interval estimator performance.

simulation * statistical analysis * variance reduction

## 1. Introduction

Variance reduction techniques (VRTs) represent a choice between a crude experiment and a more statistically efficient experiment for estimating the parameters of a stochastic process. For up-to-date surveys of VRTs, see Nelson [11] and Wilson [17]. The most widely used VRTs change the joint distribution of the simulation output random variables upon which the point estimators are based. When this change is achieved by inducing dependence among input random variables we call them *dependence induction* (DI) VRTs. For the purposes of this paper the DI problem is as follows [11]:

Let $Z$ be an estimator of an unknown scalar parameter $\theta$, where

$$Z = k^{-1} \sum_{j=1}^{k} g_j(\boldsymbol{X}_j). \tag{1}$$

The simulation input is $\boldsymbol{X}_j = \{X_{1j}, X_{2j}, \ldots, X_{lj}\}$ with cumulative distribution function (cdf) $F_j$, and the simulation output is $Y_j = g_j(\boldsymbol{X}_j)$, where $g_j$ is a scalar-valued function. In the crude experiment the $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k\}$ are independent. The DI problem is to select a joint distribution for $\{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k\}$ that reduces var$[Z]$ but preserves the marginal distributions $\{F_1, \ldots, F_k\}$. Dependence is induced among the input random variables $\{\boldsymbol{X}_j\}$ in hopes of realizing a favorable covariance structure among the outputs $\{Y_j\}$. Ideally, the joint distribution of the outputs would be specified directly, but the distribution of $Y_j$ is usually unknown because $g_j$ is implicit in the simulation code.

Frequently, a distinction is made between DI problems in which $\theta$ is an absolute parameter of a single stochastic process and problems in which $\theta$ is the difference between the parameters of two stochastic processes. In terms of (1), when $g_j = g$ for all $j$ and the $\{\boldsymbol{X}_j\}$ are identically distributed, then $Z$ estimates an absolute parameter. On the other hand, when $g_{2j-1}$ is the output of one process and the $\{\boldsymbol{X}_{2j-1}\}$ are identically distributed, while $g_{2j}$ is the negative of the output of a second process and the $\{\boldsymbol{X}_{2j}\}$ are identically distributed, then $Z$ estimates a difference.

We consider only these two cases of (1). Additionally, we assume that dependence is induced only between pairs $(\boldsymbol{X}_{2j-1}, \boldsymbol{X}_{2j})$, which includes the two most widely used DI VRTs: *antithetic variates* (AV) and *common random numbers* (CRN). Thus, dependence is induced between pairs of simulation outputs, $(Y_{2j-1}, Y_{2j})$ for $j =$

0167-6377/87/$3.50  

$1, 2, \ldots, k/2$, with disjoint pairs independent.

Although AV and CRN can cause an increase in point estimator variance (e.g., Wright and Ramsey [18]), as a practical matter the chances of counterproductive results are small; more likely is that no dependence is induced between outputs and the variance of the point estimator is unchanged. However, if any correlation in the correct direction is achieved then point estimator performance, as measured by its variance, is improved. The same cannot be said of interval estimators for $\theta$. This paper derives some basic properties of confidence intervals under DI, results that should be as well known as the antithetic-variates and common-random-numbers VRTs.

For completeness, we mention that techniques for generating bivariate input random variables with extremal joint distributions (maximum or minimum achievable covariance) and given marginal distributions are usually based on the inverse cdf method of variate generation (Whitt [16]). Variate generation methods for DI are outside the scope of this paper; see Bratley, Fox and Schrage [1] for a general reference, and Cheng [2], Fishman and Moore [3], and Schmeiser and Kachitvichyanukul [13] for variate generation algorithms that facilitate DI.

Section 2 below describes the basic properties of standard interval estimators under DI. Section 3 extends these results to output analysis methods used in steady-state simulation. Section 4 discusses the assumptions on which the results are based, and the Appendix outlines how the results are derived.

## 2. Basic properties of interval estimators

From a practical perspective, it is useful to think of AV as inducing compensating (negative) correlation for estimating an absolute parameter, and CRN as guaranteeing homogeneous experimental conditions (positive correlation) for estimating the difference between two parameters. We consider the antithetic-variate estimator first.

The problem is to estimate an absolute parameter $\theta = \mathrm{E}[Y_j]$. In the crude experiment, the $\{Y_j; j = 1, 2, \ldots, k\}$ are assumed to be independent and identically distributed (i.i.d.) normal random variables with marginal mean $\theta$ and common unknown variance $\sigma^2 < \infty$, denoted $\mathrm{N}(\theta, \sigma^2)$. The point estimator for the crude experiment is the sample mean, denoted $\bar{Y}$, which has variance $\sigma^2/k$. A $(1-\alpha)100\%$ confidence interval (c.i.) for $\theta$ is $\bar{Y} \pm H$, where the half width $H = t_{\alpha/2}(k-1)S/\sqrt{k}$, $t_{\alpha/2}(k-1)$ is the $1-\alpha/2$ quantile of the $t$ distribution with $k-1$ degrees of freedom, and $S^2$ is the sample variance of the $\{Y_j\}$.

In the AV experiment, the $\{Y_j\}$ are still marginally $\mathrm{N}(\theta, \sigma^2)$, but $\mathrm{corr}[Y_{2j-1}, Y_{2j}] = \rho \le 0$, for $j = 1, 2, \ldots, k/2$. The correlation $\rho$ is the *achieved* dependence induction, and is usually not known. The point estimator is again the sample mean, denoted $\hat{Y}$ for AV, which has variance $(1+\rho)\sigma^2/k$. Let $\hat{Y}_j = (Y_{2j-1} + Y_{2j})/2$. A $(1-\alpha)100\%$ c.i. for $\theta$ is $\hat{Y} \pm H_a$ where $H_a = t_{\alpha/2}(k/2-1)S_a\sqrt{2/k}$, and

$$S_a^2 = \left(\frac{k}{2} - 1\right)^{-1} \sum_{j=1}^{k/2} \left(\hat{Y}_j - \hat{Y}\right)^2.$$

While $\mathrm{var}[\hat{Y}] \le \mathrm{var}[\bar{Y}]$, it is not certain that the confidence interval associated with the AV experiment is superior. Under the given assumptions both c.i.s have the correct probability of coverage, $1-\alpha$. Thus, we consider four other performance measures: $\mathrm{E}[Q]$, $\mathrm{var}[Q]$ and $\mathrm{CV}[Q]$ (coefficient of variation), where $Q = H$ or $H_a$, and also

$$\beta(\theta + \Delta; Z) = \Pr\{\, |Z - (\theta + \Delta)| \le Q \,\},$$

where $Q = H$ when $Z = \bar{Y}$, and $Q = H_a$ when $Z = \hat{Y}$. The function $\beta(\cdot;\cdot)$ gives the probability that the c.i. includes values $\Delta$ units from $\theta$. For all four performance measures the lower their values the better. Evaluation of c.i. procedures by these and other measures is discussed in Kang and Schmeiser [7].

For fixed $\sigma^2$, the properties of the half width depend on the number of replications, $k$, the achieved correlation induction, $\rho$, and the confidence level $\alpha$. Figures 1 and 2 show the function $\gamma(\alpha, k)$ with the property that $\rho \le \gamma(\alpha, k)$ implies $\mathrm{E}[H_a] \le \mathrm{E}[H]$ or $\mathrm{var}[H_a] \le \mathrm{var}[H]$, respectively, as a function of $\alpha$ and $k$. There are two immediate conclusions: (1) For $k \ge 30$, the induced negative correlation required for both reduced point estimator variance and shorter expected c.i. half width is modest for all values of $\alpha$, but smaller values of $k$ impose increasingly more difficult requirements; and (2) $\rho \le -1/2$ is required for all values of $k$ to achieve improved c.i. stability as measured by the variance of the half width. In practical problems

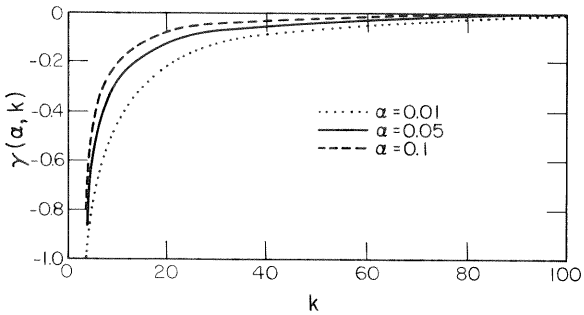


Fig. 1. The function $\gamma(\alpha, k)$ such that $\rho \le \gamma(\alpha, k)$ implies $E[H_a] \le E[H]$ for confidence level $\alpha$ and $k$ replications.

$\rho \le -1/2$ may be optimistic. The coefficient of variation of the half width depends only on $k$, and $CV[H_a]$ is always greater than $CV[H]$. The ratio $CV[H_a]/CV[H] \to \sqrt{2}$ as $k \to \infty$.

The function $\beta$ also depends on $k$, $\rho$ and $\alpha$. For $k = 10$ and $\alpha = 0.05$, Figure 3 shows $\beta$ for the crude experiment versus the AV experiment with $\rho$ ranging from 0 (no induced correlation) to $-0.08$. As expected, smaller $\rho$ leads to better performance for AV, and $\rho = -0.5$ is needed to compete with the crude c.i. at 10 replications. When $k$ is small, considerable dependence must be achieved, while increasing $k$ allows the AV c.i. to perform as well or better than the crude c.i. for $\rho$ closer to 0.

We consider the common-random-numbers estimator next. The problem is to estimate $\theta = E[Y_{2j-1} - Y_{2j}] = \mu_1 - \mu_2$, the difference between two parameters. Suppose that the pairs $(Y_{2j-1}, Y_{2j})$ are marginally $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, for $j = 1, 2, \ldots, k/2$. Let $D_j = Y_{2j-1} - Y_{2j}$, and assume initially that $\sigma_1^2 \ne \sigma_2^2$ (both finite and unknown). In the crude experiment all the $\{Y_j\}$ are independent. The point estimator for $\theta$ is the sample mean of the $\{D_j\}$, denoted $\bar{D}$, which has variance $2(\sigma_1^2 + \sigma_2^2)/k$. A $(1-\alpha)100\%$ c.i. for $\theta$ is $\bar{D} \pm H_d$, where $H_d = t_{\alpha/2}(k/2 - 1) S_d \sqrt{2/k}$, and $S_d^2$ is the sample variance of the $\{D_j\}$.

In the CRN experiment, the marginal distributions of the $\{Y_j\}$ are unchanged, but $\text{corr}[Y_{2j-1}, Y_{2j}] = \rho \ge 0$, for $j = 1, 2, \ldots, k/2$. The point estimator is still the sample mean, denoted $\tilde{D}$ for CRN, which has variance $2(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)/k$. The c.i. half width, denoted $H_c$, is functionally the same as $H_d$, and it is easy to show that it has no larger expectation and variance, identical coefficient of variation, and no larger $\beta$ function versus

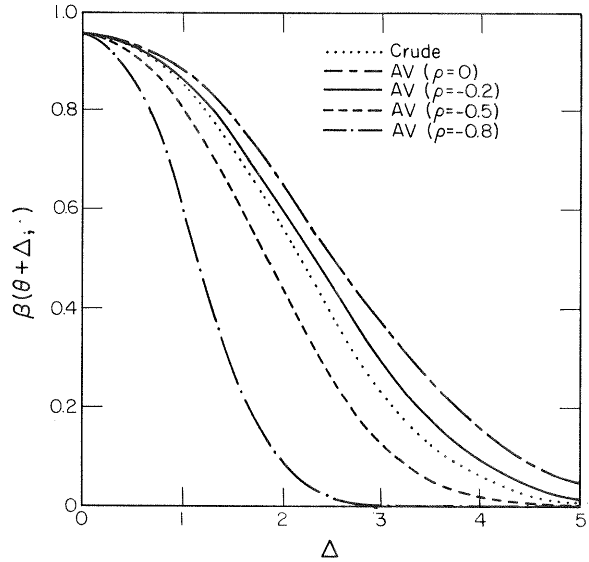


Fig. 3. Probability of covering values $\Delta\sigma/\sqrt{k}$ from $\theta$ for confidence level $\alpha = 0.05$ and $k = 10$ replications.

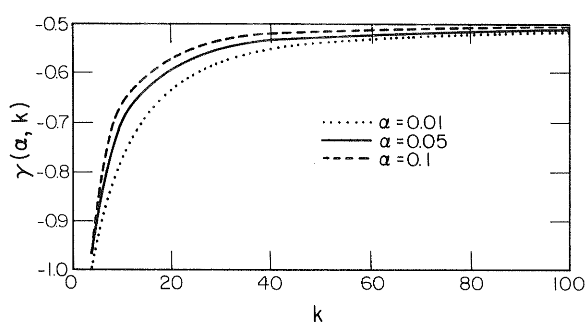


Fig. 2. The function $\gamma(\alpha, k)$ such that $\rho \le \gamma(\alpha, k)$ implies $\text{var}[H_a] \le \text{var}[H]$ for confidence level $\alpha$ and $k$ replications.

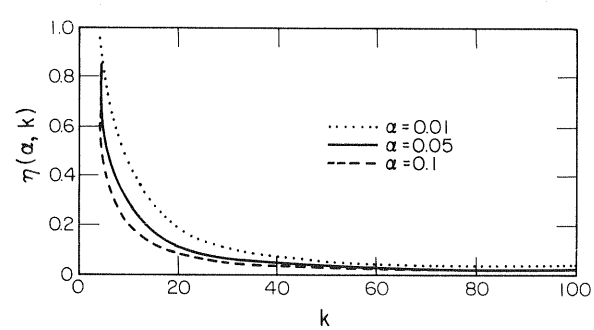


Fig. 4. The function $\eta(\alpha, k)$ such that $\rho \ge \eta(\alpha, k)$ implies $E[H_c] \le E[H_p]$ for confidence level $\alpha$ and $k$ replications.

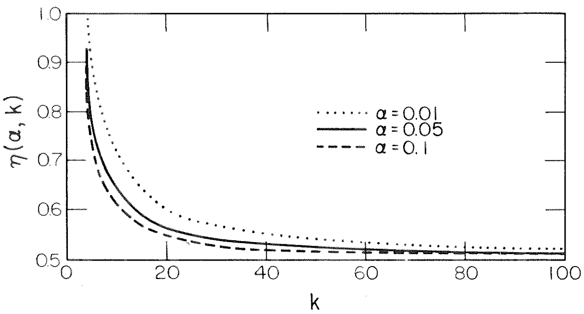


Fig. 5. The function $\eta(\alpha, k)$ such that $\rho \geq \eta(\alpha, k)$ implies $\text{var}[H_c] \leq \text{var}[H_p]$ for confidence level $\alpha$ and $k$ replications.

the crude experiment for all values of $k$, $\alpha$ and $\rho$.

However, if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ then CRN may not dominate. In this case a $(1-\alpha)100\%$ c.i. for $\theta$ from the crude experiment is $\bar{D} \pm H_p$, where $H_p = t_{\alpha/2}(k-2)S_p\sqrt{4/k}$, and $S_p^2$ is the pooled variance estimator from $\{Y_{2j-1}\}$ and $\{Y_{2j}\}$. The CRN point and interval estimators are as before. Figures 4 and 5 show the function $\eta(\alpha, k)$ with the property that $\rho \geq \eta(\alpha, k)$ implies $\text{E}[H_c] \leq \text{E}[H_p]$ or $\text{var}[H_c] \leq \text{var}[H_p]$, respectively, as a function of $\alpha$ and $k$. The conclusions are similar to AV. Again, the ratio $\text{CV}[H_c]/\text{CV}[H_p] \to \sqrt{2}$ as $k \to \infty$. Figure 6 shows $\beta$ for the crude and CRN c.i.s with $k = 20$, which is similar to Figure 3 except for a larger number of replications.

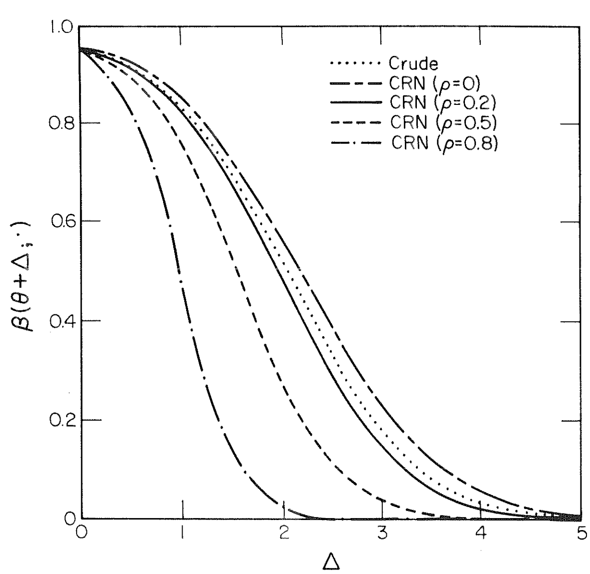


Fig. 6. Probability of covering values $\Delta\sigma\sqrt{4/k}$ from $\theta$ for confidence level $\alpha = 0.05$ and $k = 20$ replications.

## 3. Steady-state simulations

Provided the normality assumption is approximately true, the results in the previous section apply to both terminating (finite-horizon) simulations employing $k$ replications, and steady-state (infinite-horizon) simulations when replication methods are employed and the length of each replication (in units of simulated time or number of observations) is fixed. A common characteristic of these situations is that the distribution of $Y_j$ is not a function of $k$. If the experimenter makes as many replications as the budget allows, then the results above indicate how the DI interval estimator is expected to perform relative to the crude interval estimator as a function of the number of replications, the achieved dependence induction, and the confidence level.

However, replication methods are not always employed in steady-state simulation, and when they are the length of the replications may be a decision variable. In this section we consider the performance of interval estimators when the total simulation budget is fixed at $n$ observations of the output variable of interest, first when the budget is allocated among $k$ replications of length $m = n/k$, and second when one replication of length $n$ is divided into $k$ non-overlapping batches of size $m = n/k$. The distributions of the replication means and batch means are functions of $k$ for these methods. Because of initial condition bias, $d$ outputs may be discarded from the beginning of each replication, but we assume that $d = 0$. The conclusion is that the results above are still valid.

The steady-state simulation problem and associated output analysis methods are summarized in [1] and Law and Kelton [9], which should be supplemented by Schruben [14] for the method of standardized time series. Heidelberger and Iglehart [6] and Glynn [4] discuss dependence induction in conjunction with the regenerative method of output analysis, which we do not consider.

### 3.1. Replications with a fixed budget

Consider estimating an absolute parameter using $k$ replications of length $m$ when $n = km$ is fixed. Changing notation slightly, let $Y_{ij}$ be the $i$th output from the $j$th replication, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$. For fixed $j$, the output process $\{Y_{ij};\ i = 1, 2, \ldots, m\}$ may (and in general will) be dependent; we assume that the process is covariance stationary, and in particular that $E[Y_{ij}] = \theta$ for all $i$ and $j$. Let $\bar{Y}_j(k)$ be the sample mean of the $j$th replication, so that $\bar{Y}_j(k)$ takes the place of $Y_j$ as the $j$th output in (1). Finally, let $\bar{Y}(k)$ be the grand mean of all outputs, which has expectation $\theta$ and variance that depends on $k$ (Kelton [8]).

In the crude experiment the $\{\bar{Y}_j(k);\ j = 1, 2, \ldots, k\}$ are i.i.d. and a c.i. for $\theta$ is formed as in the previous section with $\{\bar{Y}_j(k)\}$ as the basic data. In the AV experiment let the basic data be $\hat{Y}_j(k) = (\bar{Y}_{2j-1}(k) + \bar{Y}_{2j}(k))/2$, for $j = 1, 2, \ldots, k/2$, and let $\hat{Y}(k)$ denote the grand mean. The fundamental assumption is that

$$\mathrm{var}\left[\hat{Y}_j(k)\right] = \frac{(1+\rho)}{2}\mathrm{var}\left[\bar{Y}_j(k)\right], \qquad (2)$$

with $\rho \le 0$, and that $\bar{Y}_j(k)$ and $\hat{Y}_j(k)$ are normally distributed. It is important to notice that $\mathrm{var}\{\bar{Y}_j(k)]$ changes as a function of $k$, but not necessarily according to a simple relationship such as $\sigma^2/k$.

Under these assumptions, the results in Figures 1, 2 and 3 are still valid, with the only change being that the units on $\Delta$ in Figure 3 are $\sqrt{\mathrm{var}[\bar{Y}(k)]}$.

Similarly, the earlier results for estimating a difference using the CRN experiment are still valid. Let $\theta = E[\bar{Y}_{2j-1}(k) - \bar{Y}_{2j}(k)] = \mu_1 - \mu_2$, and $\bar{D}_j(k) = \tilde{D}_j(k) = \bar{Y}_{2j-1}(k) - \bar{Y}_{2j}(k)$ for $j = 1, 2, \ldots, k/2$, where $\bar{Y}_j(k)$ is again the sample mean of the $j$th replication of length $m$. The $\{\bar{D}_j(k)\}$ represent the crude experiment in which $\bar{Y}_{2j-1}(k)$ and $\bar{Y}_{2j}(k)$ are independent, and the $\{\tilde{D}_j(k)\}$ represent the CRN experiment. We assume that the $\{\bar{Y}_j(k)\}$ are normally distributed, that $\mathrm{var}[\bar{Y}_{2j-1}(k)] = \mathrm{var}[\bar{Y}_{2j}(k)]$ for all $j$, and that

$$\mathrm{var}\left[\tilde{D}_j(k)\right] = (1-\rho)\ \mathrm{var}\left[\bar{D}_j(k)\right], \qquad (3)$$

with $\rho \ge 0$ (in the case of unequal variances CRN dominates the crude experiment, as before). Using the $\{\bar{Y}_j(k)\}$ as the basic data, the results in Figures 4, 5 and 6 are still valid, with the only change being that the units on $\Delta$ in Figure 6 are $\sqrt{\mathrm{var}[\bar{D}(k)]}$, where $\bar{D}(k)$ is the sample mean of the $\{\bar{D}_j(k)\}$.

### 3.2. *Batch means with a fixed budget*

Consider estimating an absolute parameter from a single covariance stationary replication of length $n$. Partition the replication into $k$ disjoint 'batches' of $m$ consecutive outputs, where $n = km$ is fixed. Let $Y_{ij}$ be the $i$th output in the $j$th batch, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$, and let $\bar{Y}_j(k)$ be the sample mean of the $j$th batch. The fundamental assumption of batch means analysis is that there exists a number of batches $k^* \le n$ such that for $k \le k^*$ the batch means $\{\bar{Y}_j(k)\}$ are i.i.d. normal random variables (Schmeiser [12]). Thus, when $k \le k^*$ a c.i. can be formed as in Section 3.1. with the batch means playing the role of the replications means. It is important to remember that $k^*$ is seldom if ever known.

In the AV experiment there are two replications of length $n/2$, each divided into $k/2$ batches of size $m$. Letting $\bar{Y}_{2j-1}(k)$ and $\bar{Y}_{2j}(k)$, for $j = 1, 2, \ldots, k/2$, correspond to batch means from the first and second replication, respectively, then $\hat{Y}_j(k)$ can be defined as in Section 3.1 and a c.i. formed. Assuming that (2) holds and that $k \le k^*$, all the previous results are still valid, with the only change being that the units on $\Delta$ in Figure 3 are $\sqrt{\mathrm{var}[\bar{Y}]}$, where $\bar{Y}$ is the mean of the $\{\bar{Y}_j(k)\}$ which does not change with $k$.

For estimating a difference, suppose there is a single replication of length $n/2$ from each process. Letting $\bar{Y}_{2j-1}(k)$ and $\bar{Y}_{2j}(k)$, for $j = 1, 2, \ldots, k/2$, correspond to batch means of length $m$ from the first and second replication, respectively, then $\bar{D}_j(k)$ and $\tilde{D}_j(k)$ can be defined as in Section 3.1 and c.i.s formed. Assuming that (3) holds and that $k \le k^*$, all the previous results are still valid, with the only change being that the units on $\Delta$ in Figure 6 are $\sqrt{\mathrm{var}[\bar{D}]}$, where $\bar{D}$ is the sample mean of the $\{\bar{D}_j(k)\}$.

## 4. Comments and disclaimers

The results in this paper show that interval estimator performance under DI is much more sensitive than point estimator performance to the dependence induction actually achieved. In particular, comparable c.i. stability for the crude and DI experiments always requires considerable induced dependence. The achieved dependence is usually unknown, but rules of thumb can be derived from our analysis.

The analysis depends on assumptions about the joint distribution of simulation output processes. Our philosophy was to adopt the assumptions that are typically made when applying the methods of replications and batch means and to understand the performance of the c.i. procedures when they are valid. This minimum level of understanding seems essential if the procedures are to be used effectively in practice. However, it is important to understand the limitations of the results.

In the case of a fixed total budget, the joint distribution of the replication or batch means may depend on the replication length or batch size, $m$, which changes with the number of replications or batches, $k$. For instance, the smaller $k$ is the better the approximation of independence for the batch means, and the better the approximation of normality for the replication and batch means. Thus, the appropriateness of the c.i. procedure may depend on a small value of $k$. However, our results indicate that if $k$ is too small and the induced dependence is modest, then the crude experiment will yield a superior c.i. procedure.

Because of issues related to synchronization of random number streams and monotonicity of the $\{g_j\}$, the replication length or batch size may also affect the dependence induction actually achieved [1]. For example, if synchronization is incomplete then the induced dependence between pairs of replications or batches may be weaker for larger $m$, counter to (2) and (3). This phenomenon further supports the claim that $k$ should not be too small.

In the fixed budget case we assumed that the number of outputs discarded from the beginning of each replication, $d$, was 0. This assumption is not restrictive in the case of batch means since we discard $d$ outputs from the beginning of at most two replications, and the c.i. is based on a fixed amount of remaining data. However, in the case of multiple replications the amount of data discarded is $kd$, so that the usable budget decreases as $k$ increases. Assuming $d=0$ avoids the need to explicitly characterize the joint distribution of the simulation output processes, but $d>0$ forces us to characterize it. Nelson [10] addresses this issue by using a surrogate process to represent the simulation output process. Preliminary analysis indicates that c.i. performance is not necessarily monotonically improved as $k$ increases when $d>0$.

For estimating a parameter that is a difference, we only considered procedures that employ equal sample sizes from each process. Since we are simulating it is always possible to guarantee this condition. However, point estimator variance can sometimes be reduced by intelligently allocating sampling effort unequally between the two processes. The analysis in this paper could be extended to procedures for unequal samples sizes such as the one proposed by Welch [15].

## Appendix

We derive the results in Section 2 for replications first, then show that they apply to replications and batch means with a fixed budget (Section 3).

*Properties of the confidence interval half widths*

Suppose $V$ is a random variable, $b$ a positive integer, and $\lambda$ a positive constant such that

$$\frac{(b-1)V^2}{\lambda^2} \sim \chi^2(b-1), \tag{A.1}$$

where $\chi^2(b-1)$ denotes a random variable having the chi-squared distribution with $b-1$ degrees of freedom, and '$\sim$' denotes equivalent in distribution. Let $I=\varepsilon V/\sqrt{b}$, where $\varepsilon$ is a positive constant. Then from Schmeiser [12, Appendix A] we have the following results:

$$\mathrm{E}[I]=\varepsilon\sqrt{\frac{2}{b-1}}\,\frac{\Gamma(b/2)}{\Gamma((b-1)/2)}\,\frac{\lambda}{\sqrt{b}}, \tag{A.2}$$

and

$$\mathrm{var}[I]=\varepsilon^2\left(1-\frac{2}{b-1}\,\frac{\Gamma^2(b/2)}{\Gamma^2((b-1)/2)}\right)\frac{\lambda^2}{b}. \tag{A.3}$$

By definition,

$$\mathrm{CV}[I]=\frac{\sqrt{\mathrm{var}[I]}}{\mathrm{E}[I]}. \tag{A.4}$$

Letting $Z$ denote a point estimator for $\theta$ and $I$ the associated confidence interval half width, the point and interval estimators in Section 2 can be summarized in Table 1, where $\delta=\sigma_1^2+\sigma_2^2$.

Under the assumptions in Section 2, (A.1) holds for each $(b-1)V^2/\lambda^2$ combination. Thus, substituting into (A.2), (A.3) and (A.4) yields the expectation, variance, and coefficient of variation of each c.i. half width. The relationships between crude and DI c.i.s are determined by comparing

Table 1

| $Z$ | $I$ | $V^2$ | $\lambda^2$ | $b$ | $\varepsilon$ |
|---|---|---|---|---|---|
| $\bar{Y}$ | $H$ | $S^2$ | $\sigma^2$ | $k$ | $t_{\alpha/2}(k-1)$ |
| $\hat{Y}$ | $H_a$ | $S_a^2$ | $(1-\rho)\sigma^2/2$ | $k/2$ | $t_{\alpha/2}(k/2-1)$ |
| $\bar{D}$ | $H_d$ | $S_d^2$ | $\delta$ | $k/2$ | $t_{\alpha/2}(k/2-1)$ |
| $\bar{D}$ | $H_p$ | $S_p^2$ | $\sigma^2$ | $k-1$ | $\sqrt{4(k-1)/k}\, t_{\alpha/2}(k-2)$ |
| $\tilde{D}$ | $H_c$ | $S_d^2$ | $\delta-2\rho\sigma_1\sigma_2$ | $k/2$ | $t_{\alpha/2}(k/2-1)$ |

the appropriate results. To determine the behavior of $\gamma(\alpha, k)$ and $\eta(\alpha, k)$ as $k \to \infty$ we need two results from Goldsman and Schruben [5, Appendix]: As $b \to \infty$,

$$\frac{t_{\alpha/2}(cb-e)\sqrt{\dfrac{2}{cb-e}}\,\dfrac{\Gamma((cb-e+1)/2)}{\Gamma((cb-e)/2)}}{t_{\alpha/2}(b-1)\sqrt{\dfrac{2}{b-1}}\,\dfrac{\Gamma(b/2)}{\Gamma((b-1)/2)}} \to 1, \tag{A.5}$$

and

$$\frac{t_{\alpha/2}^2(cb-e)\left(1-\dfrac{2}{cb-e}\,\dfrac{\Gamma^2((cb-e+1)/2)}{\Gamma^2((cb-e)/2)}\right)}{t_{\alpha/2}^2(b-1)\left(1-\dfrac{2}{b-1}\,\dfrac{\Gamma^2(b/2)}{\Gamma^2((b-1)/2)}\right)} \to c^{-1} \tag{A.6}$$

for $c$ a positive integer and $e$ a non-negative integer.

First we determine the function $\gamma(\alpha, k)$ such that $\rho \le \gamma(\alpha, k)$ implies $E[H_a] \le E[H]$. Satisfying the inequality requires

$$\rho \le \left(\frac{t_{\alpha/2}(k-1)\sqrt{\dfrac{2}{k-1}}\,\dfrac{\Gamma(k/2)}{\Gamma((k-1)/2)}}{t_{\alpha/2}(k/2-1)\sqrt{\dfrac{4}{k-2}}\,\dfrac{\Gamma(k/4)}{\Gamma((k-2)/4)}}\right)^2 -1, \tag{A.7}$$

so that $\gamma(\alpha, k)$ is equal to the right-hand side of (A.7). Using (A.5) with $b = k/2$, $c = 2$ and $e = 1$ shows that $\gamma(\alpha, k) \to 0$ as $k \to \infty$.

Determining $\gamma(\alpha, k)$ such that $\rho \le \gamma(\alpha, k)$ implies $\text{var}[H_a] \le \text{var}[H]$ requires satisfying the inequality

$$\rho \le \frac{t_{\alpha/2}^2(k-1)\left(1-\dfrac{2}{k-1}\,\dfrac{\Gamma^2(k/2)}{\Gamma^2((k-1)/2)}\right)}{t_{\alpha/2}^2(k/2-1)\left(1-\dfrac{4}{k-2}\,\dfrac{\Gamma^2(k/4)}{\Gamma^2((k-2)/4)}\right)} -1, \tag{A.8}$$

so that $\gamma(\alpha, k)$ is equal to the right-hand side of A.8). Using (A.6) with $b = k/2$, $c = 2$ and $e = 1$ shows that $\gamma(\alpha, k) \to -1/2$ as $k \to \infty$.

Next we determine the function $\eta(\alpha, k)$ such that $\rho \ge \eta(\alpha, k)$ implies $E[H_c] \le E[H_p]$. Satisfying the inequality requires

$$\rho \ge 1 - \left(\frac{t_{\alpha/2}(k-2)\sqrt{\dfrac{2}{k-2}}\,\dfrac{\Gamma((k-1)/2)}{\Gamma((k-2)/2)}}{t_{\alpha/2}(k/2-1)\sqrt{\dfrac{4}{k-2}}\,\dfrac{\Gamma(k/4)}{\Gamma((k-2)/4)}}\right)^2, \tag{A.9}$$

so that $\eta(\alpha, k)$ is equal to the right-hand side of (A.9). Using (A.5) with $b = k/2$, $c = 2$ and $e = 2$ shows that $\eta(\alpha, k) \to 0$ as $k \to \infty$.

Determining $\gamma(\alpha, k)$ such that $\rho \ge \eta(\alpha, k)$ implies $\text{var}[H_c] \le \text{var}[H_p]$ requires satisfying the inequality

$$\rho \ge 1 - \frac{t_{\alpha/2}^2(k-2)\left(1-\dfrac{2}{k-2}\,\dfrac{\Gamma^2((k-1)/2)}{\Gamma^2((k-2)/2)}\right)}{t_{\alpha/2}^2(k/2-1)\left(1-\dfrac{4}{k-2}\,\dfrac{\Gamma^2(k/4)}{\Gamma^2((k-2)/4)}\right)}, \tag{A.10}$$

so that $\eta(\alpha, k)$ is equal to the right-hand side of (A.10). Using (A.6) with $b = k/2$, $c = 2$ and $e = 2$ shows that $\eta(\alpha, k) \to 1/2$ as $k \to \infty$.

Finally, the inequalities $E[H_c] \le E[H_d]$ and $\text{var}[H_c] \le \text{var}[H_d]$ for all values of $\rho \ge 0$ (unequal variance case) are immediate: The same c.i. procedure is used in both the crude and CRN experiments. However, the population variance in the CRN experiment is $\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, versus $\sigma_1^2 + \sigma_2^2$ for the crude experiment. Thus, $H_c$ has no larger expectation and variance when $\rho \ge 0$.

The results for coefficient of variation of the half width follow from the results for expectation and variance by substitution into (A.4).

To obtain the fixed-budget results, we replace the $Z$ and $\lambda^2$ columns in table 1 with those in

Table 2

| $Z$ | $\lambda^2$ |
|---|---|
| $\bar{Y}(k)$ | $\operatorname{var}[\bar{Y}_j(k)]$ |
| $\hat{Y}(k)$ | $(1+\rho)\operatorname{var}\{\bar{Y}_j(k)]/2$ |
| $\bar{D}(k)$ | $\delta$ |
| $\bar{D}(k)$ | $\operatorname{var}[\bar{Y}_j(k)]$ |
| $\tilde{D}(k)$ | $\delta - 2\rho\sqrt{\operatorname{var}\left[\bar{Y}_{2j-1}(k)\right]}\sqrt{\operatorname{var}\left[\bar{Y}_{2j}(k)\right]}$ |

table 2, where $\delta = \operatorname{var}[\bar{Y}_{2j-1}(k)] + \operatorname{var}[\bar{y}_{2j}(k)]$.

Under the assumptions in Section 3, (A.1) still applies, so that all of the results follow directly.

*Properties of the function $\beta$*

Suppose that $Z$ is a random variable having an $N(\theta, \lambda^2/b)$ distribution. Then if $V$ is independent of $Z$, the random variable

$$T = \frac{Z - (\theta + \Delta)k}{V/\sqrt{b}} \tag{A.11}$$

has a non-central $t$ distribution with $b-1$ degrees of freedom and non-centrality parameter $-\Delta/(\lambda/\sqrt{b})$. For all the c.i. procedures, except when $V = S_p$ (pooled variance case), substituting for $Z$, $V$ and $b$ in (A.11) from the tables above allows the function $\beta(\cdot;\cdot)$ to be derived from properties of the non-central $t$ distribution as in [12].

For the pooled variance case when $Z = \bar{D}$ or $\bar{D}(k)$, the appropriate non-central $t$ random variable is

$$T = \frac{Z - (\theta + \Delta)}{S_p\sqrt{4/k}},$$

which has $k-2$ degrees of freedom and non-centrality parameter $-\Delta/\lambda\sqrt{4/k}$.

## References


[1] P. Bratley, B.L. Fox and L.E. Schrage, *A Guide to Simulation*, Springer, New York, 1983.

[2] R.C.H. Cheng. "Generation of multivariate normal samples with given sample mean and covariance matrix", *J. Statist. Comput. Simul.* **21**, 39–49 (1985).

[3] G.S. Fishman and L.R. Moore, III, "Sampling from a discrete distribution while preserving monotonicity", *Amer. Statistician* **38**, 219–223 (1984).

[4] P.W. Glynn, "Regenerative structure of Markov chains simulated via common random numbers", *Opns. Res. Lett.* **4**, 49–53 (1985).

[5] D. Goldsman and L. Schruben, "Asymptotic properties of some confidence interval estimators for simulation output", *Mgmt. Sci.* **30**, 1217–1225 (1984).

[6] P. Heidelberger and D.L. Iglehart, "Comparing stochastic systems using regenerative simulation with common random numbers", *Adv. Appl. Prob.* **11**, 804–819 (1979).

[7] K. Kang and B. Schmeiser, "Methods for evaluating and comparing confidence-interval procedures", Technical Report, Department of Industrial Engineering, University of Miami, Coral Gables, FL, 1983.

[8] W.D. Kelton, "Replication splitting and variance for simulating discrete-parameter stochastic processes", *Opns. Res. Lett.* **4**, 275–279 (1985).

[9] A.M. Law and W.D. Kelton, *Simulation Modeling and Analysis*, McGraw-Hill, New York, 1982.

[10] B.L. Nelson, "Antithetic-variate splitting for steady-state simulations", Working Paper Series no. 1986-013, Department of Industrial and Systems Engineering, Ohio State University, Columbus, OH, 1986.

[11] B.L. Nelson, "A perspective on variance reduction in dynamic simulation experiments", *Commun. Statist.-Simul. Comput.* **B16** (1987).

[12] B. Schmeiser, "Batch size effects in the analysis of simulation output", *Opns. Res.* **30**, 556–568 (1982).

[13] B. Schmeiser and V. Kachitvichyanukul, "Correlation induction without the inverse transform", *1986 Winter Simulation Conference Proceedings*, 266–274 (1986).

[14] L. Schruben, "Confidence interval estimation using standardized time series", *Opns. Res.* **31**, 1090–1108 (1983).

[15] B.L. Welch, "The generalization of 'Student's' problem when several different population variances are involved", *Biometrika* **34**, 28–35 (1947).

[16] W. Whitt, "Bivariate distributions with given marginals", *Ann. Statist.* **4**, 494–501 (1980).

[17] J.R. Wilson, "Variance reduction techniques for digital simulation", *Am. J. Math. Mgmt. Sci.* **4**, 277–312 (1984).

[18] R. Wright and T. Ramsey, "On the effectiveness of common random numbers", *Mgmt. Sci* **25**, 649–656 (1979).

# SOME PROPERTIES OF SIMULATION INTERVAL ESTIMATORS UNDER DEPENDENCE INDUCTION: ERRATUM

Barry L. NELSON

*Department of Industrial and Systems Engineering, The Ohio State University, Columbus, OH 43210, USA*

Received December 1987

Typesetting errors in B.L. Nelson (1987), 'Some Properties of Simulation Interval Estimators Under Dependence Induction', *Operations Research Letters* 6, no. 4, pp. 169–176:

Page 171, column 1, line 9: $-0.08$ should be $-0.8$.
Page 173, column 1, paragraph 4, line 4: $\overline{D}_j(k) = \overline{D}_j(k) = \overline{Y}_{2j-1}(k) - \hat{Y}_{2j}(k)$ should be $\overline{D}_j(k) = \tilde{D}_j(k) = \overline{Y}_{2j-1}(k) - \overline{Y}_{2j}(k)$.
Page 173, column 1, paragraph 4, line 8: $\{\overline{D}_j(k)\}$ should be $\{\tilde{D}_j(\mathrm{k})\}$.
Page 176, eq. (A.11):

$$T = \frac{Z - (\theta + \Delta)k}{V/\sqrt{b}}$$

should be

$$T = \frac{Z - (\theta + \Delta)}{V/\sqrt{b}}.$$

 