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ESTIMATING ACCEPTANCE-SAMPLING PLANS FOR DEPENDENT PRODUCTION PROCESSES

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When acceptance sampling is used to judge the quality of an ongoing production process, the quality of successive items may exhibit statistical dependence that is not accounted for in standard acceptance-sampling plans. Computing the probabilities required to design sampling plans for general dependent processes is often complex, and sometimes intractable. This paper presents an efficient method for estimating single-sampling attribute plans for any production process model that can be simulated. Numerical illustrations are given.

Acceptance sampling is a widely used technique for economically assessing the quality of a "lot" of items. In this paper we consider single-sampling attribute plans of the form \((n, c)\) for "Type B" (continuous) production processes, where \(n\) is the number of items sampled and \(c < n\) is the acceptance number such that the lot is declared acceptable if no more than \(c\) defective items are discovered in the sample. The values of \(n\) and \(c\) are chosen to provide a prespecified "producer's risk" and "consumer's risk."

In standard acceptance-sampling plans the risk calculations are based on modeling the production process as a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables (e.g., \([1]\), \([2]\)). This model is plausible when the sample is drawn from a well-mixed lot of finished items. However, if items are sampled sequentially, either from an ongoing process or from a lot that retains its production order, then it is possible that the quality of successive items exhibits statistical dependence.

Bhat, Lal and Karunaratne \([3]\) and Sampathkumar \([9]\) extended the standard model by treating the quality of successive items as a two-state Markov chain. Their results facilitate derivation of acceptance-sampling plans for Markov-dependent processes. Sarkadi and Vinceze \([10]\) performed a similar analysis under the assumption that the quality of successive items can be modeled as a two-state Pólya process; they showed that standard plans applied to such a process lead to erroneous conclusions. Even for such simple process models the mathematical analysis is difficult.

In this paper we present a method for estimating single-sampling attribute plans for any production process that can be simulated. The method is computationally and statistically efficient, allows the user to specify the precision of the estimators in advance, and does not depend on the underlying process model. We include a numerical illustration that demonstrates the method and the consequences of using standard acceptance-sampling plans when the process is actually dependent.

Example

In this section we introduce an example to motivate the estimation of acceptance-sampling plans for dependent processes. The example is not intended to represent any particular production process or to be a recommended approach for process control, but similar models have been used in actual process control problems (e.g., \([1]\), \([2]\)) and in theoretical investigations (e.g., \([6]\)). In a later section some numerical results are given for this example.

Suppose that \(Z_1, Z_2, \ldots, Z_n\) are the measurements of some critical feature of successive items. The \(i\)-th item is within tolerance, and thus is acceptable, if \(\tau_L \leq Z_i \leq \tau_U\), where \(\tau_L\) and \(\tau_U\) are lower and upper tolerance limits, respectively. If we use the mapping

\[
X_i = \begin{cases} 
1 & \text{if } Z_i < \tau_L \text{ or } Z_i > \tau_U \\
0 & \text{otherwise}
\end{cases}
\]

then the process data are amenable to acceptance sampling. Notice that actual measurements are not required; only a "go, no-go" assessment is needed, which might make acceptance sampling economical.

Suppose that data collected from past measurements indicates that, when the process is in control, the critical measurements of successive items are well modeled by an ARMA(1,1) process; that is

\[
Z_i = \mu + \phi (Z_{i-1} - \mu) + \theta \epsilon_{i-1} + \epsilon_i \quad (1)
\]

where \(\mu\) is the desired nominal measurement, \(\epsilon_1, \epsilon_2, \ldots, \epsilon_n\) are i.i.d. \(N(0, \sigma^2)\) random variables, \(|\phi| < 1\) and \(|\theta| < 1\). If the process has been running long enough so that it is stationary, then (1) implies that the \(Z_i\) are identically distributed \(N(\mu, \sigma^2)\) random variables, where \(\sigma^2 = \sigma^2(1 + \theta^2 + 2\phi \theta)/(1 - \phi^2)\) \([7]\). However, the measurements are dependent, with lag correlations

\[
\rho_1 = \frac{(1 + \phi \theta)(\phi + \theta)}{1 + \theta^2 + 2\phi \theta} \\
\rho_h = \phi^{h-1} \rho_1, \quad h = 2, 3, \ldots
\]

In this example we assume that \(\phi\) and \(\theta\) are fixed characteristics of the production process, but deviations from the nominal process mean, \(\mu\), or the acceptable process variance, \(\sigma^2\), can occur. For instance, an unacceptable production process might be represented by

\[
Z_i' = \mu' + \phi (Z_{i-1}' - \mu') + \theta \epsilon'_{i-1} + \epsilon_i' \quad (2)
\]

where \(\mu' \neq \mu\) and/or \(\sigma' \neq \sigma\). Standard acceptance-sampling plans apply if \(\phi = \theta = 0\).
Process Model

Let the production process be represented by the stochastic process \( \{X_i, X_{i+1}, \ldots, X_N\} \), where \( N \) is the lot size. The possible outcomes for each \( X_i \) are 0, representing an acceptable item, or 1, representing a defective or unacceptable item; we make no other assumption about the joint or marginal distributions of the process. Items are inspected sequentially up to a maximum of \( n \leq N \) items.

Let \( C_i = \sum_{k=1}^{i} x_k \) be the cumulative number of defective items discovered through item \( i \), for \( i = 1, 2, \ldots, n \). To design single-sampling plans we need to know the probability that the lot is rejected under plan \((n, c)\), denoted \( \gamma_{n,c+1} = \Pr\{C_n \geq c + 1\} \). However, for the purpose of estimating \( \gamma_{n,c+1} \) an alternative representation is useful.

Define \( C_0 = 0 \) and let

\[
D_y = \begin{cases} 1 & \text{if } C_{i+1} = j - 1 \text{ and } X_i = 1 \\ 0 & \text{otherwise} \end{cases}
\]

for \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, i \). The random variable \( D_y \) indicates whether or not the \( i \)-th item inspected was the \( j \)-th defective item discovered. Then \( T_y = \sum_{i=1}^{N} D_y \) indicates whether or not the \( j \)-th defective item was discovered on or before inspecting item \( i \), for \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, i \). It follows immediately that

\[
\gamma_{n,c+1} = \Pr\{T_{n,c+1} = 1\} = E[T_{n,c+1}] = \sum_{n=1}^{\infty} E[D_{n+1}]. \tag{3}
\]

Thus, to estimate \( \gamma_{n,c+1} \) we can estimate \( E[T_{n,c+1}] \) or \( E[D_{n+1}] \) for \( \ell = c + 1, \ldots, n \). This turns out to be a useful representation, and it facilitates proving the following properties of the \{\( \gamma_y \)\}:

**Proposition 1:** The rejection probabilities \( \gamma_y \) satisfy:

(a) \( \gamma_{i+1,j} \geq \gamma_y \) for \( i = 1, 2, \ldots, N - 1 \) and \( j = 1, 2, \ldots, i \);

(b) \( \gamma_{i,j-1} \geq \gamma_y \) for \( i = 2, 3, \ldots, N \) and \( j = 2, 3, \ldots, i \).

**Proof:** Property (a) follows directly from (3) since

\[
\gamma_{i+1,j} = \sum_{\ell=i}^{\infty} E[D_{\ell+1}] \geq \sum_{\ell=1}^{i} E[D_{\ell+1}] = \gamma_y.
\]

To establish (b), recall that \( \gamma_y = \Pr\{T_y = 1\} \). But

\[
\{T_y = 1\} \iff \{\exists \ell : \ell \leq i \text{ and } D_y = 1\} \\
\iff \{C_{i+1} = j - 1, X_i = 1\} \\
\iff \{C_{i+1} = j - 1, X_i = 1\} \\
\iff \{D_{n+1} = 1\} \\
\iff \{T_{i+1} = 1\}.
\]

Thus, the event \( \{T_y = 1\} \subseteq \{T_{i+1} = 1\} \), which implies that \( \gamma_y = \Pr\{T_y = 1\} \leq \Pr\{T_{i+1} = 1\} = \gamma_{i+1,j} \).

These two properties are intuitively obvious: Property (a) states that the probability of rejecting the lot does not decrease if the sample size is increased while holding the acceptance number fixed; property (b) states that the probability of rejecting the lot does not decrease if the acceptance number is reduced while holding the sample size fixed. Ravindran et al. [8] proved an analogous result for the case of random sampling without replacement from a lot of size \( N \). In the next section we develop a method for estimating \( \gamma_y \) for all \( i \) and \( j \) such that the estimates have these same properties.

To identify an acceptance-sampling plan \((n, c)\) we specify a producer’s risk, \( 0 < \alpha < 1 \), which is associated with acceptable performance of the production process, and a consumer’s risk, \( 0 < \beta < 1 \), which is associated with unacceptable performance of the production process. In standard acceptance sampling plans the acceptable and unacceptable production processes are specified in terms of the parameter of the Bernoulli process, \( p = \Pr\{X_1 = 1\} \); the smaller value of \( p \) is called the Acceptable Quality Level (AQL) and the larger value of \( p \) is called the Lot Tolerance Percent Defective (LTPD). A more complex description will typically be required to specify dependent processes (e.g., the correlation between successive items is required in Bhat, Lal and Karunaratne’s [3] Markov chain model). Thus, we will refer to AQL and LTPD processes when estimating sampling plans.

When the production process is performing at the acceptable level we want the probability of rejecting or stopping the process under plan \((n, c)\), denoted \( \gamma_{n,c+1} \), to be no larger than \( \alpha \). On the other hand, when the process is performing at the unacceptable level we want the probability of rejecting the process, denoted \( \gamma_{n,c+1}^{\text{un}} \), to be at least \( \beta \). In the sections that follow we show how to estimate acceptance-sampling plans that attain these requirements.

**Estimating Sampling Plans**

We assume that \( m \) i.i.d. replications of the entire process \( \{X_1, X_2, \ldots, X_n\} \) can be simulated, and we attach an additional subscript \( k \) to \( X_i, C \) and \( D_y \) to denote observations on the \( k \)-th replication for \( k = 1, 2, \ldots, m \); e.g., \( X_{nk} \) denotes the quality of the \( i \)-th item produced on the \( k \)-th replication. Let

\[
Y_{ij} = \frac{\sum_{k=1}^{m} D_{ij}}{m}
\]

and

\[
S_{iy} = \frac{\sum_{k=1}^{m} Y_{ij}}{m}
\]

for \( i = 1, 2, \ldots, N \) and \( j = 1, 2, \ldots, i \), so that \( Y_{ij} \) is the number of replications on which the \( j \)-th defective item was the \( i \)-th item inspected and \( S_{iy} \) is the number of replications on which the \( j \)-th defective item occurred on or before the \( i \)-th item inspected.

Clearly, \( Y_{ij} = Y_{ij}/m \) is an unbiased estimator of \( E[D_{ij}] \), and \( S_{iy} = S_{iy}/m \) is an unbiased estimator of \( \gamma_y \) with variance \( \gamma_y(1 - \gamma_y)/m \). In addition, the estimators \{\( S_{iy} \)\} share the same properties as the \{\( \gamma_y \)\}.

**Theorem 1:** With probability 1, the estimated rejection probabilities \( \bar{S}_{iy} \) satisfy: (a) \( \bar{S}_{i+1,j} \geq \bar{S}_{iy} \) for \( i = 1, 2, \ldots, n \) if the acceptance number is reduced while holding the sample size fixed. Ravindran et al. [8] proved an analogous result for the case of random sampling without replacement from a lot of size \( N \). In the next section we develop a method for estimating \( \bar{S}_{iy} \) for all \( i \) and \( j \) such that the estimates have these same properties.
Let \( N_i \) be the number of items inspected until a decision is reached about the lot. For both semicurtailed and rectifying inspection,

\[
ASN = E[N_{n_i}] = \sum_{t=1}^{n} \ell \Pr\{\text{decision on item } \ell\}
\]

\[
= \sum_{t=1}^{n} \ell \Pr\{D_{t,c+1} = 1\} + n(1 - \gamma_{n,c+1}).
\]

Since \( Y_{t,c+1} \) is an unbiased estimator of \( \Pr\{D_{t,c+1} = 1\} \) and \( \bar{\gamma}_{n,c+1} \) is an unbiased estimator of \( \gamma_{n,c+1} \), an unbiased estimator of ASN is

\[
\hat{ASN} = \sum_{t=1}^{n} \ell Y_{t,c+1} + n(1 - \bar{\gamma}_{n,c+1})
\]

\[
= \frac{1}{m} \sum_{t=c+1}^{m} \ell Y_{t,c+1} + n \left( 1 - \frac{1}{m} \sum_{t=c+1}^{m} Y_{t,c+1} \right)
\]

\[
= n - \frac{1}{m} \sum_{t=c+1}^{m} (n - \ell) Y_{t,c+1}.
\]

Using similar reasoning and considerable algebra we can show that an estimator of the variance of the number sampled, \( \text{Var}[N_{n_i}] \), is

\[
\hat{\text{Var}}[\text{ASN}] = \frac{1}{m} \sum_{t=c+1}^{m} (n - \ell)^2 Y_{t,c+1}
\]

\[
- \left( \frac{1}{m} \sum_{t=c+1}^{m} (n - \ell) Y_{t,c+1} \right)^2.
\]

For rectifying inspection it is relevant to estimate the ATI. Let \( N_r \) be the total number of items inspected from a lot. Then

\[
ATI = E[N_r] = N\gamma_{n,c+1} + n(1 - \gamma_{n,c+1})
\]

for which the obvious estimator is

\[
\hat{ATI} = N\bar{\gamma}_{n,c+1} + n(1 - \bar{\gamma}_{n,c+1})
\]

\[
= n + (N - n)\bar{\gamma}_{n,c+1}.
\]

The corresponding estimator of \( \text{VAR}[N_r] \) is

\[
\hat{\text{VAR}}[\text{ATI}] = (N - n)^2\bar{\gamma}_{n,c+1}(1 - \bar{\gamma}_{n,c+1}).
\]

The AOQ is a measure of the quality of the items actually shipped. Let \( N_o \) be the number of defective items remaining in each lot after inspection, and let \( N_f \) be the number of items actually shipped, so that AOQ = \( E[N_o]\) / \( E[N_f]\). If \( R \) is the event that the lot is rejected and \( \overline{R} \) is the complementary event, then

\[
E[N_o] = E[N_o | R] \gamma_{n,c+1} + E[N_o | \overline{R}](1 - \gamma_{n,c+1})
\]

\[
= E[N_o | \overline{R}](1 - \gamma_{n,c+1})
\]

since there are no defectives in rejected lots under either strategy. We can estimate \( E[N_o | \overline{R}] \) by

\[
\hat{E}[N_o | \overline{R}] = \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} Y_{ji}}{m}
\]

which is the average number of defectives observed after item \( n \). Then

\[
\hat{E}[N_o] = \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} Y_{ji}}{m} \times (1 - \bar{\gamma}_{n,c+1})
\]
is an estimator of \( E[N_d] \).

In the case of rectifying inspection, \( E[N_d] = N \). On the other hand, for semicontinuous inspection

\[
E[N_d] = \sum_{\ell=0}^{n} (N-\ell) \Pr\{C_\ell = \ell\} + \sum_{i=0}^{n} (\ell-c-1) \Pr\{D_{\ell-c+1} = 1\}.
\]

Define \( S_d = m \) and \( S_{n+c+1} = 0 \). Then \( S_d - S_{n+c+1} \) is the number of replications with exactly \( j \) defective items in the first \( n \) items for \( j = 1, 2, \ldots, n \). Thus, an unbiased estimator of \( E[N_d] \) is

\[
\hat{E}[N_d] = \frac{\sum_{i=0}^{\ell}(N-\ell)(S_{d}-S_{n+c+1}) + \sum_{i=0}^{n}(\ell-c-1)Y_{\ell-c+1}}{m}
\]

Notice that, given the data \( Y_d \) and \( S_{n+c+1} \), the ASN, ATI and AOQ can be estimated for any sampling plan \((n,c)\).

However, to estimate the operating characteristic, AOQ, ATI or ASN curves as a function of the parameters of the underlying process a simulation experiment must be performed for each point desired on the curves.

**Simulation Design**

To estimate an acceptance-sampling plan with specified consumer’s risk and producer’s risk, two simulations are performed corresponding to the acceptable and unacceptable production processes, yielding two sets of estimators \( \{S_{d}^{AOQ}\} \) and \( \{S_{d}^{AOQ}\} \), respectively. Our goal, loosely stated, is to find \((n, c)\) such that \( E[S_{d}^{AOQ}] = \alpha \) and \( E[S_{d}^{AOQ}] = 1 - \beta \) (see a later section on searching for a plan for a precise definition). If the appropriate pair \((n, c)\) were known, then \( \text{Var}[S_{d}^{AOQ}] = (1 - \alpha)/m \) and \( \text{Var}[S_{d}^{AOQ}] = \beta(1 - \beta)/m \).

Since we specify \( \alpha \) and \( \beta \) in advance, we can use this knowledge to choose the number of replications, \( m \), required to achieve a prespecified precision.

Suppose \( E[S_{d}^{AOQ}] = \alpha \) exactly. Then an appropriate \((1 - \delta)\) 100% confidence interval for \( \alpha \), when \( m \) is large, is

\[
\hat{S}_{d}^{AOQ} = z_{\delta/2} \sqrt{(1 - \alpha)/m}
\]

where \( z_{\delta/2} \) is the \( 1 - \delta/2 \) quantile of the standard normal distribution. To be confident, at level \( 1 - \delta \), of having \( d \) digits of accuracy for estimating \( \alpha \) we need

\[
z_{\delta/2} \sqrt{(1 - \alpha)/m} < \frac{1}{2} \times 10^{-d}
\]

which implies \( m > z_{\delta/2}^2 \alpha(1 - \alpha)/(4 \times 10^{2d}) \). In most estimation problems such a calculation is not possible because \( \alpha \) is unknown. To account for both simulations we take

\[
m = \lfloor z_{\delta/2}^2(4 \times 10^{2d}) \times \max \{\alpha(1 - \alpha), \beta(1 - \beta)\} \rfloor + 1
\]

which guarantees that, for the appropriate \((n, c)\) combination, the estimators \( S_{d}^{AOQ} \) and \( S_{d}^{AOQ} \) of \( \gamma_{d}^{AOQ} \) and \( \gamma_{d}^{AOQ} \) achieve the desired precision. Of course, this is an indirect control on precision since we do not know the appropriate \((n, c)\), but the structure of the \( \{S_{d}\} \) established in Theorem 1 guarantees that the ordering of the \( \{S_{d}\} \) will be the same as the \( \{Y_d\} \). Thus, a search of the \( \{S_{d}\} \) should find the desired plan.

**Simulation Implementation**

The elements \( Y_d \) can be organized into a lower triangular array, \( Y \), as shown in Figure 1. The simulation begins with all cells in the array initialized to 0, and as the simulation progresses through \( m \) replications of \( N \) items each, a 1 is added to the \( i \)-th cell whenever the \( i \)-th item inspected is the \( j \)-th defective item. Thus, the array \( Y \) can be created directly without ever actually recording the \( D_{\ell} \). For a lot of size \( N \), \( N(N + 1)/2 \) cells are needed. Of course, the data can be stored in a single-dimension array with subscript \( \ell \) by using the mapping \( \ell = i(i - 1)/2 + j \).

The lower triangular array \( S \) with elements \( S_{n+c+1} \) is formed by summing the \( j \)-th column of \( Y \) from row \( j \) through row \( i \) (see Figure 1). There is no need to actually divide by \( m \) to form \( \hat{Y}_d \) and \( S_{n+c+1} \), since the search for \( S_{d}^{AOQ} \) and \( S_{d}^{AOQ} \) can be accomplished by comparing \( S_{d}^{AOQ} \) to \( m \alpha \) and \( S_{d}^{AOQ} \) to \( \beta \). This allows all of the data from the simulation to be stored as integers.

**Searching for a Plan**

Formally, the estimation problem is to determine \((n, c)\) such that

\[
\gamma_{d}^{AOQ} \leq \alpha \quad \text{and} \quad \gamma_{d}^{AOQ} \geq 1 - \beta \quad \text{(4)}
\]

Although there may be no plan that satisfies (4), more often there are several. In this section we discuss alternative criteria and describe how to search the arrays \( S_{d}^{AOQ} \) and \( S_{d}^{AOQ} \) to find the best plan.

If we are not concerned with strictly satisfying the inequalities in (4), then we may only require that \( E[S_{d}^{AOQ}] = \alpha \) and \( E[S_{d}^{AOQ}] = 1 - \beta \); thus choosing \((n, c)\) to minimize the loss function

\[
J = c + 1
\]

![Figure 1. Y = (Y_d). Organization of simulation data](image-url)
might be a reasonable criterion. Finding the combination that minimizes \( \ell(n,c) \) requires searching through all of the possible \( n \) and \( c \) combinations. We call the plan that minimizes \( \ell(n,c) \) the minimum-loss sampling plan.

Suppose that satisfying (4) is important. Since we must substitute estimates \( \hat{\gamma} \) for the true \( \gamma \)s, we should account for the precision of our estimators. One approach is to search for \( (n,c) \) combinations that satisfy

\[
\hat{S}_n^{\ell} = (1-\beta) \frac{\beta}{m} (1-\beta) \]

which guarantees, when \( m \) is large, that each inequality holds at confidence level \( 1-\delta \) (replacing \( \delta \) by \( 2/3 \) in (5) and (6) insures that the joint statement holds at level \( 1-\delta \) by the Bonferroni inequality). We say that a plan that satisfies (5) is \( \alpha \)-feasible, and a plan that satisfies (6) is \( \beta \)-feasible. For simplicity, we assume that \( \alpha \) and \( 1-\beta \) have been adjusted by the amounts specified above, but continue to refer to them as \( \alpha \) and \( 1-\beta \).

Define the \( \alpha \)-feasible set \( \mathcal{A} = \{(i,j) : \hat{S}_n^{\ell} \leq \alpha, i=1,2,...,N; j=1,2,...,i \} \). Some properties of \( \mathcal{A} \) follow immediately from Theorem 1:

**Corollary 1:**
1. There exists an \( i \) such that \( \hat{S}_n^{\ell} = \min_{i,j} \in \mathcal{A} \).
2. If \( (i,j) \in \mathcal{A} \) then \( (h,\ell) \in \mathcal{A} \) for all \( h = i, i-1,...,j \) and \( \ell = j, j-1,...,i \).
3. If \( (i,j) \in \mathcal{A} \) then \( (h,\ell) \in \mathcal{A} \) for all \( h = i, i+1,...,N \) and \( \ell = j, j-1,...,1 \).

Property 1 implies that the smallest \( \hat{S}_n^{\ell} \) is along the diagonal of \( S^{\ell} \); thus, if no diagonal element is \( \alpha \) feasible then no element is \( \alpha \) feasible. Properties 2 and 3 imply that \( \mathcal{A} \) is a region like the one in Figure 2.

Let \( \mathcal{F} = \{(i,j) : (i,j) \in \mathcal{A} \} \). We call \( \mathcal{F} \) the \( \alpha \)-feasible frontier, since \( (i,j) \in \mathcal{F} \) implies elements in \( S^{\ell} \) to the right and above element \( (i,j) \) are in \( S^{\ell} \), but elements to the left and below \( (i,j) \) are not in \( S^{\ell} \). Sampling plans that are optimal for many natural criteria are in \( \mathcal{F} \).

Consider \( \mathcal{F} \) and \( \mathcal{A} \). For all \( (i,j) \in \mathcal{A} \) there exists \( (i',j') \in \mathcal{F} \) such that \( \hat{S}_{n,i,i'}^{\ell} \geq \hat{S}_{n,i,j}^{\ell} \). Similarly, for all \( (i,j) \in \mathcal{F} \) there exists \( (i',j') \in \mathcal{F} \) such that \( \hat{S}_{n,i',j'}^{\ell} \geq \hat{S}_{n,i,j}^{\ell} \). These properties are immediate consequences of Theorem 1.

Thus, the \( \alpha \)-feasible plans with largest \( \hat{S}_n^{\ell} \) are in \( \mathcal{F} \). Also, the largest \( \hat{S}_n^{\ell} \) s corresponding to \( \alpha \)-feasible plans are in \( \mathcal{F} \), implying that if there are plans that are both \( \alpha \) and \( \beta \) feasible, then there will be at least one such plan in \( \mathcal{F} \).

Suppose that we are interested in minimizing the sample size \( n \) such that the sampling plan \( (n,c) \) is both \( \alpha \) and \( \beta \) feasible. Since the plans in \( \mathcal{F} \) have the largest

\[
\hat{S}_n^{\ell} \text{ such that the corresponding } \hat{S}_n^{\ell} \text{ is feasible, this plan can be found in } \mathcal{F}, \\
\text{if it exists. We call this plan the minimum-}n \text{ sampling plan. Notice that the plan may not be unique in } c, \\
\text{but it is the minimum-}n \text{ plan that also minimizes } c, \text{ which is sometimes considered desirable [4], [5].}
\]

Another possible criterion is to minimize \( \alpha - \hat{S}_n^{\ell} \) over all \( \alpha \) and \( \beta \)-feasible plans; that is, find the feasible plan that comes nearest to the specified producer's risk. Since the plans in \( \mathcal{F} \) have the largest feasible \( \hat{S}_n^{\ell} \), this plan can be found in \( \mathcal{F} \), if it exists. We call this plan the nearest-\( \alpha \) sampling plan.

To find either the minimum-\( n \) or nearest-\( \alpha \) plan we trace out \( \mathcal{F} \) in \( S^{\ell} \), checking \( \hat{S}_n^{\ell} \) for feasibility, and saving the best current plan along the way. The search requires at most \( 2N \) comparisons.

We have emphasized searching based on the matrix \( S^{\ell} \) and the producer's risk \( \alpha \). In a similar manner we could start with the \( \beta \)-feasible set and search for the nearest \( \alpha - \hat{S}_n^{\ell} \) plan. Other possible criteria might require looking at the intersection of the \( \alpha \) and \( \beta \)-feasible sets.

**Numerical Results**

In this section we present some numerical results for the ARMA(1,1) example introduced earlier. All programming was done in Fortran on a Vax 750 computer (the programs are available from the author on request). To verify the programs, experimental results were compared to results for Markov chain examples in [3].

We assume that the measurement process (1) is stationary (in 'steady state'). To simulate a stationary ARMA(1,1) process the initial vector \( (Z_0, e_0)' \) is sampled from the bivariate normal distribution with mean vector \( (\mu, 0)' \) and variance-covariance matrix

\[
\begin{pmatrix}
\sigma_Z^2 & \rho \sigma_Z \sigma_e \\
\rho \sigma_Z \sigma_e & \sigma_e^2
\end{pmatrix}
\]
The dependence in the process is a function of the parameters $\phi$ and $\theta$. The experiments used the cases shown in Table 1, which include: i.i.d. measurements, for which standard acceptance-sampling plans are appropriate (case 1); measurements that are correlated only with the preceding measurement (case 2); and measurements that exhibit diminishing correlation for items farther apart in sequence (cases 3, 4 and 5). In all cases the correlations are nonnegative—which seems intuitively to be the most natural situation—and the cases are ordered according to increasing dependence.

The long-run fraction of defective items, $p$, depends on the process mean, $\mu$, and the process variance, $\sigma^2$. We constructed cases in which the long-run fraction of defective items was 0.01 for the acceptable or AQL process and 0.10 for the unacceptable or LTPD process; see Table 2. However, in the first example the degradation in quality is due to a shift in the process mean, while in the second example an increase in the process variance degrades product quality.

For all of the experiments the producer's risk was fixed at $\alpha = 0.10$, the consumer's risk at $\beta = 0.10$, and the lot size at $N = 300$. The number of replications, $m$, was determined to provide 2 digits of accuracy with 99% confidence, implying $m = 23,889$ replications for these examples.

Results for the shifted mean and increased variance examples are given in Tables 3 and 4, respectively. For each case of process dependence in Table 1, the minimum-$n$, nearest-$\alpha$, and minimum-loss plans were estimated and they are displayed in that order. The estimated rejection probability for the AQL and LTPD processes are also displayed, as are the AOQ and ASN estimates for the AQL process assuming semicurtailed inspection.

The most outstanding feature of the results is that the value of $n$ increases with increasing process dependence for the minimum-$n$ and minimum-loss plans. The effect becomes even more pronounced when the dependence is increased further. For example, the parameter settings $\theta = 0.25$ and $\phi = 0.75$ or 0.9 yield minimum-$n$ plans (114,3) or (277,8), respectively, for the shifted mean example, and (98,3) or (199,6), respectively, for the increased variance example. For the nearest-$\alpha$ plans the AOQ tends to be smaller since they reject the AQL process.
process with probability as close as possible (but less than or equal to) $\alpha$.

To illustrate the consequences of using standard acceptance-sampling plans that assume independent item quality when in fact the process is dependent, we estimated the rejection probabilities for the case 1 plans if they were used with the case 2 through 5 processes. The results are given in Tables 5 and 6. For the minimum-$n$ and minimum-loss plans the rejection probabilities for the LTPD processes are often less than the desired 0.90, and significantly less for the highly dependent cases 4 and 5. This means that lots of unacceptable quality are too likely to be accepted. For the nearest-$\alpha$ plan, the rejection probability often exceeds the desired 0.10 for the AQL processes, meaning that lots of acceptable quality are too likely to be rejected. Again, the problem is more pronounced in the highly dependent cases.

### Discussion

This paper describes an efficient method for estimating single-sampling attribute plans for any production process that can be simulated. The estimation method could also be applied to actual data records collected from the production process of interest, if a large enough sample is available.

The type of sampling plan described here is probably most naturally used to judge the quality of an ongoing production process. In that context there may be no concept of a lot, and $N$ can be interpreted as the maximum sample size, $n$, that might be considered.

Extensions to other sampling strategies are possible. For example, modifying the method to estimate sampling plans that inspect every $T$-th item in sequence, rather than every item, is trivial. In addition, plans can be estimated that provide a specified AOQ or ASN since

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### Table 4. Results for Increased Variance Example (Minimum-$n$, Nearest-$\alpha$, and Minimum-Loss Plans)

<table>
<thead>
<tr>
<th>Case</th>
<th>$n$</th>
<th>$c$</th>
<th>$S_{ACL}$</th>
<th>$S_{LTPD}$</th>
<th>$S_{ACL}$</th>
<th>$S_{LTPD}$</th>
<th>AOQ</th>
<th>ASN</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>39</td>
<td>1</td>
<td>0.059</td>
<td>0.912</td>
<td>0.0088</td>
<td>38.24</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>107</td>
<td>2</td>
<td>0.095</td>
<td>0.999</td>
<td>0.0064</td>
<td>104.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>38</td>
<td>1</td>
<td>0.056</td>
<td>0.904</td>
<td>0.0088</td>
<td>37.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>1</td>
<td>0.066</td>
<td>0.911</td>
<td>0.0088</td>
<td>39.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>234</td>
<td>4</td>
<td>0.094</td>
<td>1.000</td>
<td>0.0021</td>
<td>229.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>39</td>
<td>1</td>
<td>0.064</td>
<td>0.904</td>
<td>0.0088</td>
<td>38.14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>1</td>
<td>0.067</td>
<td>0.908</td>
<td>0.0087</td>
<td>39.05</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>167</td>
<td>3</td>
<td>0.095</td>
<td>1.000</td>
<td>0.0043</td>
<td>163.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>39</td>
<td>1</td>
<td>0.064</td>
<td>0.901</td>
<td>0.0088</td>
<td>38.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>1</td>
<td>0.082</td>
<td>0.906</td>
<td>0.0086</td>
<td>40.67</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>294</td>
<td>5</td>
<td>0.095</td>
<td>1.000</td>
<td>0.0002</td>
<td>287.70</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>41</td>
<td>1</td>
<td>0.079</td>
<td>0.900</td>
<td>0.0087</td>
<td>39.75</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>63</td>
<td>2</td>
<td>0.052</td>
<td>0.907</td>
<td>0.0079</td>
<td>61.80</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>89</td>
<td>2</td>
<td>0.095</td>
<td>0.978</td>
<td>0.0070</td>
<td>85.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>47</td>
<td>1</td>
<td>0.110</td>
<td>0.901</td>
<td>0.0085</td>
<td>44.74</td>
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<td></td>
</tr>
</tbody>
</table>

### Table 5. Estimated Rejection Probabilities if the Standard Plans are Used for the Shifted Mean Example, Cases 2-5 (Minimum-$n$, Nearest-$\alpha$, and Minimum-Loss Plans)

<table>
<thead>
<tr>
<th>Plan</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$c$</td>
<td>$S_{ACL}$</td>
<td>$S_{LTPD}$</td>
<td>$S_{ACL}$</td>
</tr>
<tr>
<td>39</td>
<td>1</td>
<td>0.063</td>
<td>0.892</td>
<td>0.063</td>
</tr>
<tr>
<td>171</td>
<td>3</td>
<td>0.098</td>
<td>1.000</td>
<td>0.098</td>
</tr>
<tr>
<td>38</td>
<td>1</td>
<td>0.060</td>
<td>0.884</td>
<td>0.060</td>
</tr>
</tbody>
</table>

### Table 6. Estimated Rejection Probabilities if the Standard Plans are Used for the Increased Variance Example, Cases 2-5 (Minimum-$n$, Nearest-$\alpha$, and Minimum-Loss Plans)

<table>
<thead>
<tr>
<th>Plan</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$c$</td>
<td>$S_{ACL}$</td>
<td>$S_{LTPD}$</td>
<td>$S_{ACL}$</td>
</tr>
<tr>
<td>39</td>
<td>1</td>
<td>0.064</td>
<td>0.904</td>
<td>0.064</td>
</tr>
<tr>
<td>107</td>
<td>2</td>
<td>0.101</td>
<td>0.999</td>
<td>0.101</td>
</tr>
<tr>
<td>38</td>
<td>1</td>
<td>0.060</td>
<td>0.896</td>
<td>0.061</td>
</tr>
</tbody>
</table>

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these measures can be estimated for any plan. The estimation of multistage sampling plans is currently being investigated.

An inherent limitation of acceptance-sampling plans for dependent processes is that the plans cannot be tabulated. An application-specific process model must be developed for each situation in which the plans will be used. However, with computer collection of process data and appropriate statistical analysis software (to fit ARMA models, for example), the approach suggested here becomes practical: Given a process model, application-specific plans can be estimated quickly and automatically by a generic simulation program. The extra modeling effort is offset by the danger inherent in using tabulated plans in situations for which they were not designed.

This paper does not address the problem of developing and fitting process models to describe dependent production processes, however, which seems like an important area for further research. ARMA models are a rich class of noncausal models, and software for fitting such models is available in many commercial statistical analysis packages. When the physical mechanism causing defective items is understood (e.g., tool wear), process models that are based on the physical process might be candidates. And, since all process models are approximations, the sensitivity of the estimated sampling plans to errors in the model form or parameters (\( \phi \) and \( \theta \) in the ARMA(1, 1) process model) should be investigated as part of the simulation experimentation.

If a production process is known to exhibit statistical dependence, then it is legitimate to ask if acceptance-sampling plans that reduce the process data to counts make the most efficient use of process data. The answer is certainly no. However, given the extensive use of acceptance-sampling plans in practice, due partly to their simplicity, a direct extension to dependent processes is a practical first step toward a more general methodology for dependent production processes.

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REFERENCES


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