IIE Transactions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713772245

ESTIMATING ACCEPTANCE-SAMPLING PLANS FOR DEPENDENT PRODUCTION PROCESSES

Barry L. Nelson[a]
[a] Department of Industrial and Systems Engineering, The Ohio State University, Columbus, Ohio

To cite this Article Nelson, Barry L.(1993) 'ESTIMATING ACCEPTANCE-SAMPLING PLANS FOR DEPENDENT PRODUCTION PROCESSES', IIE Transactions, 25: 6, 11 — 18
To link to this Article: DOI: 10.1080/07408179308964323
URL: http://dx.doi.org/10.1080/07408179308964323

ESTIMATING ACCEPTANCE-SAMPLING PLANS FOR DEPENDENT PRODUCTION PROCESSES

BARRY L. NELSON

Department of Industrial and Systems Engineering, The Ohio State University, Columbus, Ohio 43210

When acceptance sampling is used to judge the quality of an ongoing production process, the quality of successive items may exhibit statistical dependence that is not accounted for in standard acceptance-sampling plans. Computing the probabilities required to design sampling plans for general dependent processes is often complex, and sometimes intractable. This paper presents an efficient method for estimating single-sampling attribute plans for any production process model that can be simulated. Numerical illustrations are given.

■ Acceptance sampling is a widely used technique for economically assessing the quality of a "lot" of items. In this paper we consider single-sampling attribute plans of the form (n, c) for "Type B" (continuous) production processes, where n is the number of items sampled and $c < n$ is the acceptance number such that the lot is declared acceptable if no more than c defective items are discovered in the sample. The values of n and c are chosen to provide a prespecified "producer's risk" and "consumer's risk."

In standard acceptance-sampling plans the risk calculations are based on modeling the production process as a sequence of independent and identically distributed (i.i.d.) Bernoulli random variables (e.g., [11], [12]). This model is plausible when the sample is drawn from a well-mixed lot of finished items. However, if items are sampled sequentially, either from an ongoing process or from a lot that retains its production order, then it is possible that the quality of successive items exhibits statistical dependence.

Bhat, Lal and Karunaratne [3] and Sampathkumar [9] extended the standard model by treating the quality of successive items as a two-state Markov chain. Their results facilitate derivation of acceptance-sampling plans for Markov-dependent processes. Sarkadi and Vincze [10] performed a similar analysis under the assumption that the quality of successive items can be modeled as a two-state Pólya process; they showed that standard plans applied to such a process lead to erroneous conclusions. Even for such simple process models the mathematical analysis is difficult.

In this paper we present a method for *estimating* single-sampling attribute plans for *any* production process that can be simulated. The method is computationally and statistically efficient, allows the user to specify the precision of the estimators in advance, and does not depend on the underlying process model. We include a numerical illustration that demonstrates the method and the consequences of using standard acceptance-sampling plans when the process is actually dependent.

Example

In this section we introduce an example to motivate the estimation of acceptance-sampling plans for dependent processes. The example is not intended to represent any particular production process or to be a recommended approach for process control, but similar models have been used in actual process control problems (e.g., [1], [2]) and in theoretical investigations (e.g., [6]). In a later section some numerical results are given for this example.

Suppose that $Z_1, Z_2, \ldots, Z_N$ are the measurements of some critical feature of successive items. The i-th item is within tolerance, and thus is acceptable, if $\tau_L \leq Z_i \leq \tau_U$, where τ_L and τ_U are lower and upper tolerance limits, respectively. If we use the mapping

$$X_i = \begin{cases} 1 & \text{if } Z_i < \tau_L \text{ or } Z_i > \tau_U \\ 0 & \text{otherwise} \end{cases}$$

then the process data are amenable to acceptance sampling. Notice that actual measurements are not required; only a "go, no-go" assessment is needed, which might make acceptance sampling economical.

Suppose that data collected from past measurements indicates that, when the process is in control, the critical measurements of successive items are well modeled by an ARMA(1,1) process; that is

$$Z_i = \mu + \phi(Z_{i-1} - \mu) + \theta\epsilon_{i-1} + \epsilon_i \qquad (1)$$

where μ is the desired nominal measurement, $\epsilon_1, \epsilon_2, \ldots, \epsilon_N$ are i.i.d. $N(0, \sigma^2)$ random variables, $|\phi| < 1$ and $|\theta| < 1$. If the process has been running long enough so that it is stationary, then (1) implies that the Z_i are identically distributed $N(\mu, \sigma_Z^2)$ random variables, where $\sigma_Z^2 = \sigma^2(1 + \theta^2 + 2\phi\theta)/(1 - \phi^2)$ [7]. However, the measurements are dependent, with lag correlations

$$\varrho_1 = \frac{(1 + \phi\theta)(\phi + \theta)}{1 + \theta^2 + 2\phi\theta}$$

$$\varrho_h = \phi^{h-1}\varrho_1, \; h = 2, 3, \ldots$$

In this example we assume that ϕ and θ are fixed characteristics of the production process, but deviations from the nominal process mean, μ, or the acceptable process variance, σ_Z^2, can occur. For instance, an unacceptable production process might be represented by

$$Z_i' = \mu' + \phi(Z_{i-1}' - \mu') + \theta\epsilon_{i-1}' + \epsilon_i' \qquad (2)$$

where $\mu' \neq \mu$ and/or $\sigma' \neq \sigma$. Standard acceptance-sampling plans apply if $\phi = \theta = 0$.

 0740-817X/93/$3.00x.00

Process Model

Let the production process be represented by the stochastic process $\{X_1, X_2, \ldots, X_N\}$, where N is the lot size. The possible outcomes for each X_i are 0, representing an acceptable item, or 1, representing a defective or unacceptable item; we make no other assumption about the joint or marginal distributions of the process. Items are inspected sequentially up to a maximum of $n \leq N$ items.

Let $C_i = \sum_{\ell=1}^{i} X_\ell$ be the cumulative number of defective items discovered through item i, for $i=1,2,\ldots,n$. To design single-sampling plans we need to know the probability that the lot is rejected under plan (n, c), denoted $\gamma_{n,c+1} = \Pr\{C_n \geq c+1\}$. However, for the purpose of estimating $\gamma_{n,c+1}$ an alternative representation is useful.

Define $C_0 = 0$ and let

$$D_{ij} = \begin{cases} 1 & \text{if } C_{i-1} = j-1 \text{ and } X_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $i=1,2,\ldots,N$ and $j=1,2,\ldots,i$. The random variable D_{ij} indicates whether or not the i-th item inspected was the j-th defective item discovered. Then $T_{ij} = \sum_{\ell=j}^{i} D_{\ell j}$ indicates whether or not the j-th defective item was discovered on or before inspecting item i, for $i=1,2,\ldots,N$ and $j=1,2,\ldots,i$. It follows immediately that

$$\gamma_{n,c+1} = \Pr\{T_{n,c+1} = 1\} = \mathrm{E}[T_{n,c+1}] = \sum_{\ell=c+1}^{n} \mathrm{E}[D_{\ell,c+1}]. \quad (3)$$

Thus, to estimate $\gamma_{n,c+1}$ we can estimate $\mathrm{E}[T_{n,c+1}]$ or $\mathrm{E}[D_{\ell,c+1}]$ for $\ell = c+1,\ldots,n$. This turns out to be a useful representation, and it facilitates proving the following properties of the $\{\gamma_{ij}\}$:

Proposition 1: *The rejection probabilities γ_{ij} satisfy: (a) $\gamma_{i+1,j} \geq \gamma_{ij}$ for $i=1,2,\ldots,N-1$ and $j=1,2,\ldots,i$; and (b) $\gamma_{i,j-1} \geq \gamma_{ij}$ for $i=2,3,\ldots,N$ and $j=2,3,\ldots,i$.*

Proof: Property (a) follows directly from (3) since

$$\gamma_{i+1,j} = \sum_{\ell=j}^{i+1} \mathrm{E}[D_{\ell j}] \geq \sum_{\ell=j}^{i} \mathrm{E}[D_{\ell j}] = \gamma_{ij}.$$

To establish (b), recall that $\gamma_{ij} = \Pr\{T_{ij} = 1\}$. But

$$\begin{aligned} \{T_{ij} = 1\} &\Longleftrightarrow \{\exists \ell : \ell \leq i \text{ and } D_{\ell j} = 1\} \\ &\Longleftrightarrow \{C_{\ell-1} = j-1, X_\ell = 1\} \\ &\Longrightarrow \{\exists h : h < \ell \text{ and } C_{h-1} = j-2, X_h = 1\} \\ &\Longrightarrow \{D_{h,j-1} = 1\} \\ &\Longrightarrow \{T_{i,j-1} = 1\}. \end{aligned}$$

Thus, the event $\{T_{ij} = 1\} \subseteq \{T_{i,j-1} = 1\}$, which implies that $\gamma_{ij} = \Pr\{T_{ij} = 1\} \leq \Pr\{T_{i,j-1} = 1\} = \gamma_{i,j-1}$. □

These two properties are intuitively obvious: Property (a) states that the probability of rejecting the lot does not decrease if the sample size is increased while holding the acceptance number fixed; property (b) states that the probability of rejecting the lot does not decrease if the acceptance number is reduced while holding the sample size fixed. Ravindran *et al.*, [8] proved an analogous result for the case of random sampling without replacement from a lot of size N. In the next section we develop a method for estimating γ_{ij} for all i and j such that the estimates have these same properties.

To identify an acceptance-sampling plan (n, c) we specify a *producer's risk*, $0 < \alpha < 1$, which is associated with acceptable performance of the production process, and a *consumer's risk*, $0 < \beta < 1$, which is associated with unacceptable performance of the production process. In standard acceptance sampling plans the acceptable and unacceptable production processes are specified in terms of the parameter of the Bernoulli process, $p = \Pr\{X_i = 1\}$; the smaller value of p is called the *Acceptable Quality Level* (*AQL*) and the larger value of p is called the *Lot Tolerance Percent Defective* (*LTPD*). A more complex description will typically be required to specify dependent processes (e.g., the correlation between successive items is required in Bhat, Lal and Karunaratne's [3] Markov chain model). Thus, we will refer to AQL and LTPD *processes* when estimating sampling plans.

When the production process is performing at the acceptable level we want the probability of rejecting or stopping the process under plan (n, c), denoted $\gamma_{n,c+1}^{AQL}$, to be no larger than α. On the other hand, when the process is performing at the unacceptable level we want the probability of rejecting the process, denoted $\gamma_{n,c+1}^{LTPD}$, to be at least $1-\beta$. In the sections that follow we show how to estimate acceptance-sampling plans that attain these requirements.

Estimating Sampling Plans

We assume that m i.i.d. replications of the entire process $\{X_1, X_2, \ldots, X_N\}$ can be simulated, and we attach an additional subscript k to X_i, C_i and D_{ij} to denote observations on the k-th replication for $k=1,2,\ldots,m$; e.g., X_{ik} denotes the quality of the i-th item produced on the k-th replication. Let

$$Y_{ij} = \sum_{k=1}^{m} D_{ijk}$$

and

$$S_{ij} = \sum_{\ell=j}^{i} Y_{\ell j}$$

for $i=1,2,\ldots,N$ and $j=1,2,\ldots,i$, so that Y_{ij} is the number of replications on which the j-th defective item was the i-th item inspected and S_{ij} is the number of replications on which the j-th defective item occurred on or before the i-th item inspected.

Clearly, $\bar{Y}_{ij} = Y_{ij}/m$ is an unbiased estimator of $\mathrm{E}[D_{ij}]$, and $\bar{S}_{ij} = S_{ij}/m$ is an unbiased estimator of γ_{ij} with variance $\gamma_{ij}(1-\gamma_{ij})/m$. In addition, the estimators $\{\bar{S}_{ij}\}$ share the same properties as the $\{\gamma_{ij}\}$.

Theorem 1: *With probability* 1, *the estimated rejection probabilities $\bar{S}_{ij}$ satisfy: (a) $\bar{S}_{i+1,j} \geq \bar{S}_{ij}$ for $i=$*

$1,2,\ldots,N-1$ *and* $j=1,2,\ldots,i$; *and (b)* $\bar{S}_{i,j-1} \geq \bar{S}_{ij}$ *for* $i=2,3,\ldots,N$ *and* $j=2,\ldots,i$.

Proof: Property (a) follows directly from the definition of $\bar{S}_{ij}$ since

$$m\bar{S}_{i+1,j} = \sum_{t=j}^{i+1} Y_{tj} \geq \sum_{t=j}^{i} Y_{tj} = m\bar{S}_{ij}.$$

To establish (b), notice that $m\bar{S}_{ij} = \Sigma_{t=j}^{i} Y_{tj} = \Sigma_{t=j}^{i} \Sigma_{k=1}^{m} D_{tjk} = \Sigma_{k=1}^{m} \Sigma_{t=j}^{i} D_{tjk}$. For fixed replication k, at most one of $D_{jjk}, D_{j+1,jk},\ldots,D_{ijk}$ is equal to 1 (since the j-th defective can occur only once). Suppose $D_{tjk} = 1$. Then

$$\begin{aligned}\{D_{tjk} = 1\} &\Longrightarrow \{C_{t-1,k} = j-1, X_{tk} = 1\} \\ &\Longrightarrow \{\exists h : j-1 \leq h < t \text{ and } \\ &\qquad C_{h-1,k} = j-2, X_{hk} = 1\} \\ &\Longrightarrow \{D_{h,j-1,k} = 1\} \\ &\Longrightarrow \sum_{t=j-1}^{i} D_{t,j-1,k} \geq \sum_{t=j}^{i} D_{tjk}.\end{aligned}$$

Since this is true for all k, the result follows by summing both sides of the last inequality over k and dividing by m. □

The method of estimation suggested by these results is computationally efficient: For a given production process model, *all* of the $\{\gamma_{ij}\}$ can be estimated from one simulation experiment consisting of m replications. Direct estimation would require $N(N+1)/2$ simulation experiments.

In the following subsection we show that standard performance measures for acceptance-sampling plans can also be estimated from the same simulation data. The number of replications needed for a specified precision can be determined in advance, as will be shown in later sections, that also describe efficient ways to collect the simulation data and search for the (n, c) combination that provides the desired producer's risk and consumer's risk, respectively.

Performance of Sampling Plans

Three performance measures of a sampling plan are the *average sample number (ASN)*, the *average total inspection (ATI)* and the *average outgoing quality (AOQ)*. These three measures are easily estimated from the data Y_{ij} and S_{ij}. For illustration, we consider two strategies:

1. Sampling is terminated immediately if the $(c+1)$-st defective is discovered in the first n items, in which case the production process is stopped (no more items are produced). This is sometimes called *semicurtailed inspection*. We assume that all defective items are discarded.
2. Rejected lots are 100% inspected and all defective items, including those found in the first n, are replaced with acceptable items. This is sometimes called *rectifying inspection*.

Let N_N be the number of items inspected until a decision is reached about the lot. For both semicurtailed and rectifying inspection,

$$\begin{aligned}\text{ASN} &= \text{E}[N_N] = \sum_{\ell=c+1}^{n} \ell \Pr\{\text{decision on item } \ell\} \\ &= \sum_{\ell=c+1}^{n} \ell \Pr\{D_{\ell,c+1} = 1\} + n(1-\gamma_{n,c+1}).\end{aligned}$$

Since $\bar{Y}_{\ell,c+1}$ is an unbiased estimator of $\Pr\{D_{\ell,c+1} = 1\}$ and $\bar{S}_{n,c+1}$ is an unbiased estimator of $\gamma_{n,c+1}$, an unbiased estimator of ASN is

$$\begin{aligned}\widehat{\text{ASN}} &= \sum_{\ell=c+1}^{n} \ell \bar{Y}_{\ell,c+1} + n(1-\bar{S}_{n,c+1}) \\ &= \frac{1}{m}\sum_{\ell=c+1}^{n} \ell Y_{\ell,c+1} + n\left(1 - \frac{1}{m}\sum_{\ell=c+1}^{n} Y_{\ell,c+1}\right) \\ &= n - \frac{1}{m}\sum_{\ell=c+1}^{n} (n-\ell) Y_{\ell,c+1}.\end{aligned}$$

Using similar reasoning and considerable algebra we can show that an estimator of the variance of the number sampled, $\text{Var}[N_N]$, is

$$\widehat{\text{Var}}[\text{ASN}] = \frac{1}{m}\sum_{\ell=c+1}^{n} (n-\ell)^2 Y_{\ell,c+1} - \left(\frac{1}{m}\sum_{\ell=c+1}^{n} (n-\ell) Y_{\ell,c+1}\right)^2.$$

For rectifying inspection it is relevant to estimate the ATI. Let N_I be the total number of items inspected from a lot. Then

$$\text{ATI} = \text{E}[N_I] = N\gamma_{n,c+1} + n(1-\gamma_{n,c+1})$$

for which the obvious estimator is

$$\begin{aligned}\text{ATI} &= N\bar{S}_{n,c+1} + n(1-\bar{S}_{n,c+1}) \\ &= n + (N-n)\bar{S}_{n,c+1}.\end{aligned}$$

The corresponding estimator of $\text{VAR}[N_I]$ is

$$\widehat{\text{Var}}[\text{ATI}] = (N-n)^2 \bar{S}_{n,c+1}(1-\bar{S}_{n,c+1}).$$

The AOQ is a measure of the quality of the items actually shipped. Let N_D be the number of defective items remaining in each lot after inspection, and let N_S be the number of items actually shipped, so that AOQ $= \text{E}[N_D]/\text{E}[N_S]$. If $\mathscr{R}$ is the event that the lot is rejected and $\bar{\mathscr{R}}$ is the complementary event, then

$$\begin{aligned}\text{E}[N_D] &= \text{E}[N_D|\mathscr{R}]\gamma_{n,c+1} + \text{E}[N_D|\bar{\mathscr{R}}](1-\gamma_{n,c+1}) \\ &= \text{E}[N_D|\bar{\mathscr{R}}](1-\gamma_{n,c+1})\end{aligned}$$

since there are no defectives in rejected lots under either strategy. We can estimate $\text{E}[N_D|\bar{\mathscr{R}}]$ by

$$\hat{\text{E}}[N_D|\bar{\mathscr{R}}] = \frac{\Sigma_{i=n+1}^{N} \Sigma_{j=1}^{i} Y_{ij}}{m}$$

which is the average number of defectives observed after item n. Then

$$\hat{\text{E}}[N_D] = \frac{\Sigma_{i=n+1}^{N} \Sigma_{j=1}^{i} Y_{ij}}{m} \times (1-\bar{S}_{n,c+1})$$

is an estimator of $E[N_D]$.

In the case of rectifying inspection, $E[N_S] = N$. On the other hand, for semicurtailed inspection

$$E[N_S] = \sum_{\ell=0}^{c} (N-\ell)\Pr\{C_n = \ell\} + \sum_{\ell=c+1}^{n} (\ell - c - 1)\Pr\{D_{\ell,c+1} = 1\}.$$

Define $S_{n0} = m$ and $S_{n,n+1} = 0$. Then $S_{nj} - S_{n,j+1}$ is the number of replications with exactly j defective items in the first n items for $j = 1,2,\ldots,n$. Thus, an unbiased estimator of $E[N_S]$ is

$$\hat{E}[N_S] = \frac{\sum_{\ell=0}^{c}(N-\ell)(S_{n\ell} - S_{n,\ell+1}) + \sum_{\ell=c+1}^{n}(\ell - c - 1)Y_{\ell,c+1}}{m}.$$

Notice that, given the data Y_{ij} and S_{ij}, the ASN, ATI and AOQ can be estimated for *any* sampling plan (n, c). However, to estimate the operating characteristic, AOQ, ATI or ASN curves as a function of the parameters of the underlying process a simulation experiment must be performed for each point desired on the curves.

Simulation Design

To estimate an acceptance-sampling plan with specified consumer's risk and producer's risk, two simulations are performed corresponding to the acceptable and unacceptable production processes, yielding two sets of estimators $\{\bar{S}_{ij}^{AQL}\}$ and $\{\bar{S}_{ij}^{LTPD}\}$, respectively. Our goal, loosely stated, is to find (n, c) such that $E[\bar{S}_{n,c+1}^{AQL}] \approx \alpha$ and $E[\bar{S}_{n,c+1}^{LTPD}] \approx 1 - \beta$ (see a later section on searching for a plan for a precise definition). If the appropriate pair (n, c) were known, then $\mathrm{Var}[\bar{S}_{n,c+1}^{AQL}] \approx \alpha(1-\alpha)/m$ and $\mathrm{Var}[\bar{S}_{n,c+1}^{LTPD}] \approx \beta(1-\beta)/m$. Since we specify α and β in advance, we can use this knowledge to choose the number of replications, m, required to achieve a prespecified precision.

Suppose $E[\bar{S}_{n,c+1}^{AQL}] = \alpha$ exactly. Then an appropriate $(1-\delta)100\%$ confidence interval for α, when m is large, is

$$\bar{S}_{n,c+1}^{AQL} \pm z_{\delta/2}\sqrt{\alpha(1-\alpha)/m}$$

where $z_{\delta/2}$ is the $1 - \delta/2$ quantile of the standard normal distribution. To be confident, at level $1 - \delta$, of having d digits of accuracy for estimating α we need

$$z_{\delta/2}\sqrt{\alpha(1-\alpha)/m} < \frac{1}{2} \times 10^{-d}$$

which implies $m > z_{\delta/2}^2\alpha(1-\alpha)(4 \times 10^{2d})$. In most estimation problems such a calculation is not possible because α is unknown. To account for both simulations we take

$$m = \lfloor z_{\delta/2}^2(4 \times 10^{2d}) \times \max\{\alpha(1-\alpha), \beta(1-\beta)\}\rfloor + 1$$

which guarantees that, for the appropriate (n, c) combination, the estimators $\bar{S}_{n,c+1}^{AQL}$ and $\bar{S}_{n,c+1}^{LTPD}$ of $\gamma_{n,c+1}^{AQL}$ and $\gamma_{n,c+1}^{LTPD}$ achieve the desired precision. Of course, this is an indirect control on precision since we do not know the appropriate (n, c), but the structure of the $\{\bar{S}_{ij}\}$ established in Theorem 1 guarantees that the ordering of the $\{\bar{S}_{ij}\}$ will be the same as the $\{\gamma_{ij}\}$. Thus, a search of the $\{\bar{S}_{ij}\}$ should find the desired plan.

Simulation Implementation

The elements Y_{ij} can be organized into a lower triangular array, $\mathbf{Y}$, as shown in Figure 1. The simulation begins with all cells in the array initialized to 0, and as the simulation progresses through m replications of N items each, a 1 is added to the ij-th cell whenever the i-th item inspected is the j-th defective item. Thus, the array $\mathbf{Y}$ can be created directly without ever actually recording the D_{ijk}. For a lot of size N, $N(N+1)/2$ cells are needed. Of course, the data can be stored in a single-dimension array with subscript ℓ by using the mapping $\ell = i(i-1)/2 + j$.

The lower triangular array $\mathbf{S}$ with elements S_{ij} is formed by summing the j-th column of $\mathbf{Y}$ from row j through row i (see Figure 1). There is no need to actually divide by m to form $\bar{Y}_{ij}$ and $\bar{S}_{ij}$, since the search for $\bar{S}_{n,c+1}^{AQL}$ and $\bar{S}_{n,c+1}^{LTPD}$ can be accomplished by comparing S_{ij}^{AQL} to $m\alpha$ and S_{ij}^{LTPD} to $m(1-\beta)$. This allows all of the data from the simulation to be stored as integers.

Searching for a Plan

Formally, the estimation problem is to determine (n, c) such that

$$\gamma_{n,c+1}^{AQL} \le \alpha \text{ and } \gamma_{n,c+1}^{LTPD} \ge 1 - \beta. \qquad (4)$$

Although there may be no plan that satisfies (4), more often there are several. In this section we discuss alternative criteria and describe how to search the arrays $\mathbf{S}^{AQL} = \{S_{ij}^{AQL}\}$ and $\mathbf{S}^{LTPD} = \{S_{ij}^{LTPD}\}$ to find the best plan.

If we are not concerned with strictly satisfying the inequalities in (4), then we may only require that $E[\bar{S}_{n,c+1}^{AQL}] \approx \alpha$ and $E[\bar{S}_{n,c+1}^{LTPD}] \approx 1 - \beta$; thus choosing (n, c) to minimize the loss function

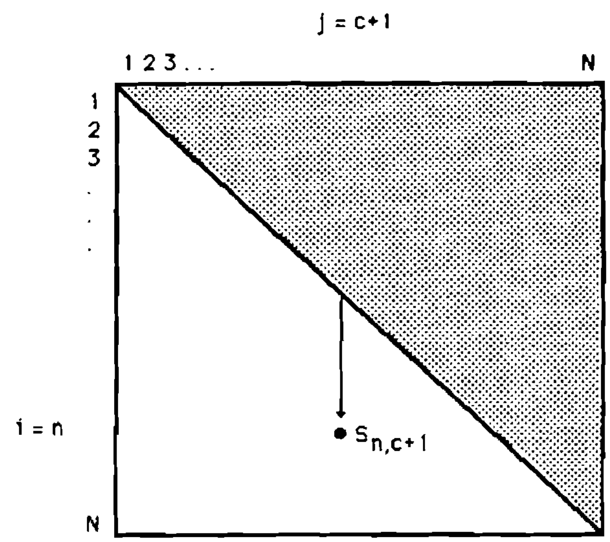

Figure 1. $Y = \{Y_{ij}\}$. Organization of simulation data

$$\ell(n,c) = \left|\bar{S}^{AQL}_{n,c+1} - \alpha\right| + \left|\bar{S}^{LTPD}_{n,c+1} - (1-\beta)\right|$$

might be a reasonable criterion. Finding the combination that minimizes $\ell(n,c)$ requires searching through all of the possible n and c combinations. We call the plan that minimizes $\ell(n,c)$ the *minimum-loss sampling plan*.

Suppose that satisfying (4) is important. Since we must substitute estimates $\bar{S}_{ij}$ for the true γ_{ij}, we should account for the precision of our estimators. One approach is to search for (n,c) combinations that satisfy

$$\bar{S}^{AQL}_{n,c+1} \leq \alpha - z_{\delta/2}\sqrt{\alpha(1-\alpha)/m} \tag{5}$$

$$\bar{S}^{LTPD}_{n,c+1} \geq 1 - \beta + z_{\delta/2}\sqrt{\beta(1-\beta)/m} \tag{6}$$

which guarantees, when m is large, that each inequality holds at confidence level $1-\delta$ (replacing δ by $\delta/2$ in (5) and (6) insures that the joint statement holds at level $1-\delta$ by the Bonferroni inequality). We say that a plan that satisfies (5) is *α feasible*, and a plan that satisfies (6) is *β feasible*. For simplicity, we assume that α and $1-\beta$ have been adjusted by the amounts specified above, but continue to refer to them as α and $1-\beta$.

Define the *α-feasible set* $\mathscr{A} = \{(i,j): \bar{S}^{AQL}_{ij} \leq \alpha, i = 1,2,\ldots,N; j=1,2,\ldots,i\}$. Some properties of $\mathscr{A}$ follow immediately from Theorem 1:

Corollary 1:

1. *There exists an ℓ such that $\bar{S}^{AQL}_{\ell\ell} = \min_{(i,j)\in\mathscr{A}} \bar{S}^{AQL}_{ij}$.*
2. *If $(i,j) \in \mathscr{A}$ then $(h,\ell) \in \mathscr{A}$ for $h = i, i-1,\ldots,j$ and $\ell = j, j+1,\ldots,i$.*
3. *If $(i,j) \notin \mathscr{A}$ then $(h,\ell) \notin \mathscr{A}$ for $h = i, i+1,\ldots,N$ and $\ell = j, j-1,\ldots,1$.*

Property 1 implies that the smallest $\bar{S}^{AQL}_{ij}$ is along the diagonal of $\mathbf{S}^{AQL}$; thus, if no diagonal element is α feasible then no element is α feasible. Properties 2 and 3 imply that $\mathscr{A}$ is a region like the one in Figure 2.

Let $\mathscr{F} = \{(i,j): (i,j) \in \mathscr{A}$ and $(i,j-1) \notin \mathscr{A}$ or $(i+1,j) \notin \mathscr{A}\}$. We call $\mathscr{F}$ the *α-feasible frontier*, since $(i,j) \in \mathscr{F}$ implies elements in $\mathbf{S}^{AQL}$ to the right and above element (i,j) are in $\mathscr{A}$, but elements to the left and below (i,j) are not in $\mathscr{A}$. Sampling plans that are optimal for many natural criteria are in $\mathscr{F}$.

Consider $\mathscr{F}$ and $\mathscr{A} \setminus \mathscr{F}$. For all $(i,j) \in \mathscr{A} \setminus \mathscr{F}$, there exists $(i',j') \in \mathscr{F}$ such that $\bar{S}^{AQL}_{i',j'} \geq \bar{S}^{AQL}_{ij}$. Similarly, for all $(i,j) \in \mathscr{A} \setminus \mathscr{F}$, there exists $(i',j') \in \mathscr{F}$ such that $\bar{S}^{LTPD}_{i',j'} \geq \bar{S}^{LTPD}_{i,j}$. These properties are immediate consequences of Theorem 1.

Thus, the α-feasible plans with largest $\bar{S}^{AQL}_{ij}$ are in $\mathscr{F}$. Also, the largest $\bar{S}^{LTPD}_{ij}$ s corresponding to α-feasible plans are in $\mathscr{F}$, implying that if there are plans that are both α and β feasible, then there will be at least one such plan in $\mathscr{F}$.

Suppose that we are interested in minimizing the sample size n such that the sampling plan (n,c) is both α and β feasible. Since the plans in $\mathscr{F}$ have the largest

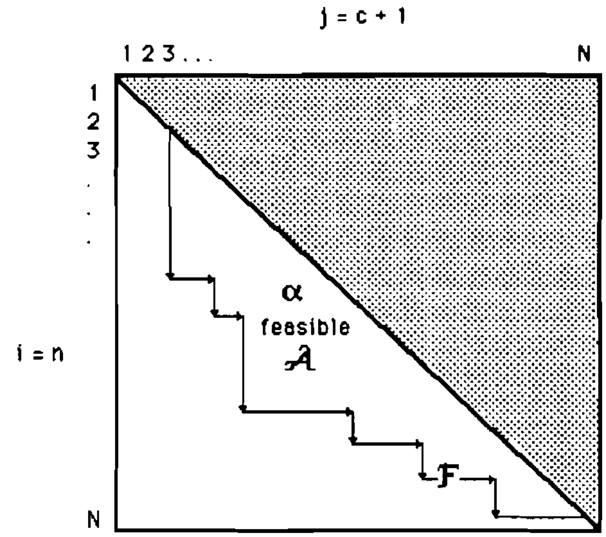

Figure 2. $S^{AQL} = \{S^{AQL}_{ij}\}$. Feasible set for estimator search

$\bar{S}^{LTPD}_{ij}$ s such that the corresponding $\bar{S}^{AQL}_{ij}$ is feasible, this plan can be found in $\mathscr{F}$, if it exists. We call this plan the *minimum-n sampling plan*. Notice that the plan may not be unique in c, but it is the minimum-n plan that also minimizes c, which is sometimes considered desirable [4], [5].

Another possible criterion is to minimize $\alpha - \bar{S}^{AQL}_{ij}$ over all α and β-feasible plans; that is, find the feasible plan that comes nearest to the specified producer's risk. Since the plans in $\mathscr{F}$ have the largest feasible $\bar{S}^{AQL}_{ij}$, this plan can be found in $\mathscr{F}$, if it exists. We call this plan the *nearest-α sampling plan*.

To find either the minimum-n or nearest-α plan we trace out $\mathscr{F}$ in $\mathbf{S}^{AQL}$, checking $\mathbf{S}^{LTPD}$ for feasibility, and saving the best current plan along the way. The search requires at most $2N$ comparisons.

We have emphasized searching based on the matrix $\mathbf{S}^{AQL}$ and the producer's risk α. In a similar manner we could start with the β-feasible set and search for the nearest $1-\beta$ plan. Other possible criteria might require looking at the intersection of the α and β-feasible sets.

Numerical Results

In this section we present some numerical results for the ARMA(1,1) example introduced earlier. All programming was done in Fortran on a Vax 750 computer (the programs are available from the author on request). To verify the programs, experimental results were compared to results for Markov chain examples in [3].

We assume that the measurement process (1) is stationary (in "steady state"). To simulate a stationary ARMA(1,1) process the initial vector $(Z_0, \epsilon_0)'$ is sampled from the bivariate normal distribution with mean vector $(\mu, 0)'$ and variance-covariance matrix

$$\begin{pmatrix} \sigma_Z^2 & \sigma^2 \\ \sigma^2 & \sigma^2 \end{pmatrix}.$$

The dependence in the process is a function of the parameters ϕ and θ. The experiments used the cases shown in Table 1, which include: i.i.d. measurements, for which standard acceptance-sampling plans are appropriate (case 1); measurements that are correlated only with the preceding measurement (case 2); and measurements that exhibit diminishing correlation for items farther apart in sequence (cases 3, 4 and 5). In all cases the correlations are nonnegative—which seems intuitively to be the most natural situation—and the cases are ordered according to increasing dependence.

The long-run fraction of defective items, p, depends on the process mean, μ, and the process variance, σ_Z^2. We constructed cases in which the long-run fraction of defective items was 0.01 for the acceptable or AQL process and 0.10 for the unacceptable or LTPD process; see Table 2. However, in the first example the degradation in quality is due to a shift in the process mean, while in the second example an increase in the process variance degrades product quality.

Table 1.
ARMA(1,1) Parameters and Dependence Structure

Case	ϕ	θ	ϱ_1	ϱ_h, h = 2,3,...
1	0	0	0	0
2	0	0.25	0.24	0
3	0.25	0	0.25	$0.25(0.25)^{h-1}$
4	0.25	0.25	0.45	$0.45(0.25)^{h-1}$
5	0.50	0.25	0.64	$0.64(0.50)^{h-1}$

For all of the experiments the producer's risk was fixed at $\alpha = 0.10$, the consumer's risk at $\beta = 0.10$, and the lot size at $N = 300$. The number of replications, m, was determined to provide 2 digits of accuracy with 99% confidence, implying $m = 23{,}889$ replications for these examples.

Results for the shifted mean and increased variance examples are given in Tables 3 and 4, respectively. For each case of process dependence in Table 1, the minimum-n, nearest-α, and minimum-loss plans were estimated and they are displayed in that order. The estimated rejection probability for the AQL and LTPD processes are also displayed, as are the AOQ and ASN estimates for the AQL process assuming semicurtailed inspection.

The most outstanding feature of the results is that the value of n increases with increasing process dependence for the minimum-n and minimum-loss plans. The effect becomes even more pronounced when the dependence is increased further. For example, the parameter settings $\theta = 0.25$ and $\phi = 0.75$ or 0.9 yield minimum-n plans (114,3) or (277,8), respectively, for the shifted mean example, and (98,3) or (199,6), respectively, for the increased variance example. For the nearest-α plans the AOQ tends to be smaller since they reject the AQL

Table 2. Process Parameters and Long-Run Fraction Defective

Quality	μ	σ_Z^2	τ_L	τ_U	p
AQL	10	1	7.4242	12.5758	0.01
LTPD	11.2940	1	7.4242	12.5758	0.10
AQL	10	0.03778	9.5	10.5	0.01
LTPD	10	0.09240	9.5	10.5	0.10

Table 3. Results for Shifted Mean Example (Minimum-n, Nearest-α, and Minimum-Loss Plans)

Case	n	c	$\bar{S}_{n,c+1}^{AQL}$	$\bar{S}_{n,c+1}^{LTPD}$	AOQ	ASN
1	39	1	0.058	0.914	0.0087	38.25
	171	3	0.095	1.000	0.0042	167.04
	38	1	0.055	0.905	0.0087	37.30
2	41	1	0.068	0.908	0.0086	40.02
	236	4	0.094	1.000	0.0021	231.08
	40	1	0.065	0.900	0.0087	39.08
3	42	1	0.072	0.907	0.0086	40.93
	50	1	0.095	0.948	0.0083	48.27
	41	1	0.068	0.900	0.0086	40.00
4	46	1	0.091	0.910	0.0086	44.35
	296	5	0.095	1.000	0.0001	289.70
	45	1	0.089	0.904	0.0084	43.44
5	71	2	0.063	0.907	0.0076	69.38
	209	4	0.095	0.999	0.0029	203.27
	54	1	0.131	0.899	0.0081	50.94

process with probability as close as possible (but less than or equal to) α.

To illustrate the consequences of using standard acceptance-sampling plans that assume independent item quality when in fact the process is dependent, we estimated the rejection probabilities for the case 1 plans if they were used with the case 2 through 5 processes. The results are given in Tables 5 and 6. For the minimum-n and minimum-loss plans the rejection probabilities for the LTPD processes are often less than the desired 0.90, and significantly less for the highly dependent cases 4 and 5. This means that lots of unacceptable quality are too likely to be accepted. For the nearest-α plan, the rejection probability often exceeds the desired 0.10 for the AQL processes, meaning that lots of acceptable quality are too likely to be rejected. Again, the problem is more pronounced in the highly dependent cases.

Discussion

This paper describes an efficient method for estimating single-sampling attribute plans for any production process that can be simulated. The estimation method could also be applied to actual data records collected from the production process of interest, if a large enough sample is available.

The type of sampling plan described here is probably most naturally used to judge the quality of an ongoing production process. In that context there may be no concept of a lot, and N can be interpreted as the maximum sample size, n, that might be considered.

Extensions to other sampling strategies are possible. For example, modifying the method to estimate sampling plans that inspect every T-th item in sequence, rather than every item, is trivial. In addition, plans can be estimated that provide a specified AOQ or ASN since

Table 4. Results for Increased Variance Example (Minimum-n, Nearest-α, and Minimum-Loss Plans)

Case	n	c	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	AOQ	ASN
1	39	1	0.059	0.912	0.0088	38.24
	107	2	0.095	0.999	0.0064	104.04
	38	1	0.056	0.904	0.0088	37.29
2	40	1	0.066	0.911	0.0088	39.07
	234	4	0.094	1.000	0.0021	229.10
	39	1	0.064	0.904	0.0088	38.14
3	40	1	0.067	0.908	0.0087	39.05
	167	3	0.095	1.000	0.0043	163.00
	39	1	0.064	0.901	0.0088	38.11
4	42	1	0.082	0.906	0.0086	40.67
	294	5	0.095	1.000	0.0002	287.70
	41	1	0.079	0.900	0.0087	39.75
5	63	2	0.052	0.907	0.0079	61.80
	89	2	0.095	0.978	0.0070	85.95
	47	1	0.110	0.901	0.0085	44.74

Table 5. Estimated Rejection Probabilities if the Standard Plans are Used for the Shifted Mean Example, Cases 2-5 (Minimum-n, Nearest-α, and Minimum-Loss Plans)

Plan		Case 2		Case 3		Case 4		Case 5	
n	c	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$
39	1	0.063	0.892	0.063	0.884	0.072	0.859	0.087	0.795
171	3	0.098	1.000	0.098	1.000	0.109	1.000	0.131	0.997
38	1	0.060	0.884	0.060	0.875	0.069	0.849	0.084	0.785

Table 6. Estimated Rejection Probabilities if the Standard Plans are Used for the Increased Variance Example, Cases 2-5 (Minimum-n, Nearest-α, and Minimum-Loss Plans)

Plan		Case 2		Case 3		Case 4		Case 5	
n	c	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$	$\bar{S}^{AQL}_{n,c+1}$	$\bar{S}^{LTPD}_{n,c+1}$
39	1	0.064	0.904	0.064	0.901	0.073	0.884	0.088	0.842
107	2	0.101	0.999	0.101	0.999	0.110	0.997	0.128	0.993
38	1	0.060	0.896	0.061	0.894	0.070	0.876	0.085	0.832

these measures can be estimated for any plan. The estimation of multistage sampling plans is currently being investigated.

An inherent limitation of acceptance-sampling plans for dependent processes is that the plans cannot be tabled. An application-specific process model must be developed for each situation in which the plans will be used. However, with computer collection of process data and appropriate statistical analysis software (to fit ARMA models, for example), the approach suggested here becomes practical: Given a process model, application-specific plans can be estimated quickly and automatically by a generic simulation program. The extra modeling effort is offset by the danger inherent in using tabled plans in situations for which they were not designed.

This paper does not address the problem of developing and fitting process models to describe dependent production processes, however, which seems like an important area for further research. ARMA models are a rich class of noncausal models, and software for fitting such models is available in many commercial statistical analysis packages. When the physical mechanism causing defective items is understood (e.g., tool wear), process models that are based on the physical process might be candidates. And, since all process models are approximations, the sensitivity of the estimated sampling plans to errors in the model form or parameters (ϕ and θ in the ARMA(1,1) process model) should be investigated as part of the simulation experimentation.

If a production process is known to exhibit statistical dependence, then it is legitimate to ask if acceptance-sampling plans that reduce the process data to counts make the most efficient use of process data. The answer is certainly no. However, given the extensive use of acceptance-sampling plans in practice, due partly to their simplicity, a direct extension to dependent processes is a practical first step toward a more general methodology for dependent production processes.

Acknowledgements

The author acknowledges the help of Donald Warsing on an initial investigation into this topic, and the helpful comments of three anonymous referees. This research was partially supported by National Science Foundation Grant No. ECS-8707634.

REFERENCES

[1] Alwan, L. C., and Roberts, H. V., "Time Series Modeling for Statistical Process Control," *Journal of Business and Economic Statistics*, 6, 87-95 (1988).

[2] Berthouex, P. M., Hunter, W. G., and Pallesen, L., "Monitoring Sewage Treatment Plants: Some Quality Control Aspects," *Journal of Quality Technology*, 10, 139-149 (1978).

[3] Bhat, U. N., Lal, R., and Karunaratne, M., "A Sequential Inspection Plan for Markov Dependent Production Processes," *IIE Transactions*, 22, 56-64 (1990).

[4] Hald, A., "The Determination of Single Sampling Attribute Plans with Given Producer's and Consumer's Risk," *Technometrics*, 9, 401-415 (1967).

[5] Jaech, J. L., "Determination of Acceptance Numbers and Sample Sizes for Attribute Sampling Plans," *Journal of Quality Technology*, 12, 187-190 (1980).

[6] Johnson, R. A., and Bagshaw, M., "The Effect of Serial Correlation on the Performance of CUSUM Tests," *Technometrics*, 16, 103-112 (1974).

[7] Nelson, C. R., *Applied Time Series Analysis for Managerial Forecasting*, Holden-Day, San Francisco (1973).

[8] Ravindran, A., Shin, W. S., Arthur, J. L., and Moskowitz, H., "Nonlinear Integer Programming Models for Acceptance Sampling," *Computers & Operations Research*, 13, 611-622 (1986).

[9] Sampathkumar, V. S., "A Tightened *m*-Level Continuous Sampling Plan for Markov Dependent Production Processes," *IIE Transactions*, 16, 257-261 (1984).

[10] Sarkadi, K., and Vincze, I., *Mathematical Methods of Statistical Quality Control*, Academic Press, N.Y. (1974).

[11] Schilling, E. G., *Acceptance Sampling in Quality Control*, Dekker, N.Y. (1982).

[12] Wadsworth, H. M., Stephens, K. S., and Godfrey, A. B., *Modern Methods for Quality Control and Improvement*, Wiley, N.Y. (1986).

Barry L. Nelson is an associate professor in the Department of Industrial and Systems Engineering at The Ohio State University. His Ph.D. is from the School of Industrial Engineering at Purdue University. His research interests are design and analysis of simulation experiments, particularly methods for statistically efficient simulation. He teaches courses in simulation and stochastic processes. Continuing memberships include ASA, IIE, ORSA, and TIMS. Dr. Nelson is an Associate Editor for *Naval Research Logistics* and *Operations Research*, and is Secretary-Treasurer of the TIMS College on Simulation.

Received May 1990; revised September 1990. Handled by the Department of Engineering Statistics and Applied Probability.