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Chance Constrained Selection of the Best

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Selecting the solution with the largest or smallest mean of a primary performance measure from a finite set of solutions while requiring secondary performance measures to satisfy certain constraints is called constrained selection of the best (CSB) in the simulation ranking and selection literature. In this paper, we consider CSB problems with secondary performance measures that must satisfy probabilistic constraints, and we call such problems chance constrained selection of the best (CCSB). We design procedures that first check the feasibility of all solutions and then select the best among all the sample feasible solutions. We prove the statistical validity of these procedures for variations of the CCSB problem under the indifference-zone formulation. Numerical results show that the proposed procedures can efficiently handle CCSB problems with up to 100 solutions, each with five chance constraints.

Keywords: simulation; ranking and selection; chance constraints; hypothesis test; statistical validity; multiple performance measures
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1. Introduction

Ranking and selection (R&S) procedures are often used to solve simulation optimization problems with a finite and small number of solutions (i.e., no more than 1,000 solutions). In these problems, the (random) performance of a solution may be observed by running a computer simulation experiment. The objective is often to find the solution with the best mean performance. Assuming we can allow the computational effort to simulate all solutions, then the search is exhaustive and the central problem is controlling statistical selection error (see, for instance, the review of Kim and Nelson 2006). The more efficiently we can control selection error, the larger the problem that can be handled in this way.

Most of the R&S procedures in the literature focus on only one performance measure and select the best solution based solely on it. In many practical situations, however, decision makers are interested in multiple performance measures. For instance, in inventory management, managers are concerned with expected cost but also the chance of a stock out, and in clinic scheduling, doctors are interested in their profits as well as the waiting times of their patients. A natural approach to handling multiple performance measures is to identify a primary one (e.g., cost and profit, respectively, in the examples) and maybe several secondary ones (e.g., probability of stock out and waiting times, respectively, in the examples) and then to optimize the expected value of the primary performance measure while requiring the secondary performance measures to satisfy one or more quality-of-service (QoS) constraint. This approach has been widely adopted in the stochastic programming literature (see, for instance, Birge and Louveaux 1997). In the context of R&S, we call this formulation *constrained selection of the best* (CSB), which has been studied only recently by Andradóttir and Kim (2010) and Healey et al. (2013). They formulate the problem as maximizing the expected value of the primary performance measure while requiring the expected values (means) of secondary performance measures to satisfy certain constraints. We call their formulation *expectation constrained selection of best* (ECSB). To solve the problem they assume that the primary and secondary outputs are jointly normally distributed with an unknown mean vector and covariance matrix. Recently, Hunter and Pasupathy (2013), Pasupathy et al. (2014), and Lee et al. (2012) also consider the ECSB problem. The goal of the first three papers is to allocate a simulation budget to all solutions to maximize the asymptotic rate of identifying the optimal feasible solution, whereas the last paper designs an easy-to-implement budget allocation rule under the optimal computing budget allocation (OCBA) formulation (see, for instance, Chen 1996 and Chen et al. 2000 for seminal work on the OCBA approach).

In this paper, we consider a special case of the ECSB problem that we call *chance constrained selection of the best* (CCSB): maximize (or minimize) the expected value of the primary performance measure while requiring the secondary performance measures to satisfy constraints with at least a given probability. For instance, in the inventory example, managers may choose to minimize the expected cost while requiring the probability of a stock out to be below 5%; and in the clinic scheduling example, doctors may choose to maximize the expected profit while requiring the probability of a patient waiting for more than 30 minutes to be below 10%. Because a probability can be written as the expectation of an indicator function, a CCSB is a form of ECSB. However, we know that the distribution of an indicator function is Bernoulli, so we can use this fact to solve a CCSB problem more efficiently than a generic ECSB problem and without any assumption as to the distribution of the secondary output measures that makes our procedure more robust. Exploiting this insight is a central contribution of the paper.

A CCSB representation of constraints is sometimes more reasonable than an ECSB formulation. For instance, in many problems in the service industry, the primary performance measures are financial outcomes, such as profit and cost; hence it makes sense to analyze their mean values. The secondary performance measures, contrarily, typically reflect QoS; therefore it makes more sense to analyze the probability of achieving a certain service standard, where the standard may be imposed internally by service commitments or externally by rules or regulations. In the stochastic programming literature, chance constrained programming was first formulated and considered by Charnes et al. (1958) and since then has been adopted as one of the most natural ways to handle stochastic constraints. For a recent review of the topic, refer to Prékopa (2003).

To design statistically valid procedures for CCSB problems, we take an indifference-zone approach initially proposed by Bechhofer (1954). We first consider the case where there is only one secondary performance measure and thus only a single chance constraint. We design a two-stage procedure: In the first stage, we check the feasibility of all solutions and calculate the sample variances of the primary performance measures of all sample feasible solutions using the available observations at the end of the stage. In the second stage, we select the best solution from all sample feasible solutions. A similar two-stage approach called procedure $\mathscr{AK}$ was proposed by Andradóttir and Kim (2010). The statistical validity of procedure $\mathscr{AK}$ cannot be proved, mainly because the sample sizes of all sample feasible solutions may be correlated with their second-stage sample means and the correlations are difficult to quantify in general (Andradóttir and Kim 2010). We solve this problem by designing feasibility tests that allocate a constant and fixed number of observations to all sample feasible solutions,[1] thus preserving the statistical validity of the second stage. This is possible because we take advantage of the structure of the chance constraint and formulate the feasibility checking as a hypothesis test on a probability with explicit control on both Type I and II errors. Once the feasibility tests are designed, we can use the fully sequential procedure $\mathscr{KN}$ of Kim and Nelson (2001) in the second stage to select the best feasible solution.

We next consider the case of multiple secondary performance measures. For instance, a hospital may care about the waiting times of various classes of patients (e.g., regular patients and critical patients), and a call center may care about waiting times of callers as well as the work loads of agents. One of the major difficulties in handling multiple secondary performance measures is the inefficiency caused by use of Bonferroni's inequality to ensure joint satisfaction of all constraints. When the constraints are on the expected values of the secondary performance measures, Batur and Kim (2010) allocate $\alpha/(mk)$ of the total error α to the Type I and II errors for feasibility checking, where m is the number of constraints and k is the number of solutions. We call this a *multiplicative rule* because it divides the total error α by $m \times k$. This makes feasibility checking terribly conservative even for problems of moderate size; e.g., $k = 100$ and $m = 10$.

In this paper, we develop two formulations to handle multiple secondary performance measures. In the first formulation, we group all secondary performance measures together into a joint chance constraint that requires all secondary performance measures to be above their corresponding standards simultaneously with a given probability. If the secondary performance measures reflect QoS of the solution, using a joint chance constraint is often a reasonable formulation of the problem. In the hospital example, for instance, the hospital authority may require that the probability that regular patients wait less than 60 minutes and critical patients wait less than 10 minutes to simultaneously be above 95%. Thus, the multiple secondary performance measures

[1] However, the numbers of observations for sample infeasible solutions may be constant or random, depending on whether a fixed-sample procedure or a sequential procedure is used for feasibility checking.

form only one chance constraint (avoiding the use of Bonferroni's inequality on the constraint), and the problem can be solved as a single secondary performance measure. In the stochastic programming literature, this formulation is also known as a joint chance constrained program; it was first proposed by Miller and Wagner (1965) and has been studied extensively since then (see, for instance, Hong et al. 2011, and references therein).

We also consider a formulation where there are multiple chance constraints for the secondary performance measures. Under this formulation, we carefully examine the multiplicative rule of Batur and Kim (2010) and find it unnecessary. Indeed, an *additive rule* that allocates an error of $\alpha/(m+k-1)$ to checking each constraint is sufficient, where α is the total error that includes both the feasibility checking and selection steps. When $m > 1$, the savings of simulation effort due to switching from the multiplicative rule to the additive rule is often significant. It is worthwhile observing that this additive rule can also be applied to the procedures of Andradóttir and Kim (2010) as well as Batur and Kim (2010) to make them more efficient.

Our procedures are related to Bernoulli selection problems because we exploit properties of Bernoulli random variables in the feasibility checking. In a Bernoulli selection problem, the goal is to choose the solution with the largest probability of success (see, for instance, Bechhofer et al. 1995, Chap. 7, for the related literature). The hypothesis-test formulation used in our feasibility checking is equivalent to the comparison-with-a-standard problem, as pointed out by Xu et al. (2010). Therefore, feasibility checking is also related to the literature on comparison with a standard (see, for instance, Nelson and Goldsman 2001 and Kim 2005).

Kim and Nelson (2001) proposed a fully sequential selection-of-the-best procedure known as procedure $\mathscr{KN}$ that allows unknown and unequal variances and the use of common random numbers (CRN). Procedure $\mathscr{KN}$ is particularly suitable for computer simulation experiments because simulation observations are often obtained sequentially on computers. In our chance constrained procedures, we use $\mathscr{KN}$ in the second stage to select the best solution from a group of sample feasible solutions. Although we focus only on two-stage procedures with a feasibility test followed by selection of the best in this paper, other types of procedures exist in the literature as well. For instance, procedure $\mathscr{AK}+$ in Andradóttir and Kim (2010) performs feasibility and optimality checking simultaneously to achieve higher efficiency. However, the statistical validity of procedure $\mathscr{AK}+$ cannot be guaranteed even under the normality assumption on both primary and secondary performance measures. Simultaneously running $\mathscr{AK}+$ was shown to be more efficient than two-stage $\mathscr{AK}$, but it is not always more efficient than our two-stage procedure when applied to CCSB problems, as shown in the numerical experiments in §6.1 and the online supplement EC.1 (available as supplemental material at http://dx.doi.org/10.1287/ijoc.2014.0628). In addition, there are practical problems for which it is useful to identify all feasible solutions. A byproduct of our procedure is a set of sample feasible solutions with a statistical guarantee. Simultaneously checking feasibility and optimality means that feasible solutions are likely to be eliminated.

Even though we consider only indifference-zone selection procedures in this paper, we believe that the CCSB problem is also an interesting and important problem for Bayesian R&S (see, for instance, Frazier 2010, §1 for a good overview). In addition to the CSB formulation for R&S problems with multiple performance measures, Butler et al. (2001) applied multiple attribute utility theory to address multiobjective problems, and Lee et al. (2010) incorporated the Pareto optimality concept into a R&S scheme to deliver a nondominated set of solutions.

The remainder of the paper is organized as follows. We formulate the CCSB problem in §2. In §3, we develop two feasibility checking procedures, one of fixed sample size and the other sequential. We combine the sequential feasibility checking procedure with procedure $\mathscr{KN}$ into a new two-stage sequential procedure for CCSB and discuss its error allocation and statistical validity in §4. In §5, we propose two formulations and procedures to handle multiple secondary performance measures, followed by numerical studies in §6. We conclude the paper in §7.

2. Problem Formulation and Solution Overview

Suppose that there are k solutions from which we need to select the best feasible solution. Let X_i and Y_i denote the primary and secondary performance measures, respectively, observed from running a simulation experiment at solution i, $i = 1, 2, \ldots, k$. Initially, we consider only one secondary performance measure, deferring the case of multiple secondary performance measures to §5. We formulate the CCSB problem as follows:

$$\max_{i=1,2,\ldots,k} \ \mathrm{E}(X_i) \quad \text{s.t.} \ \Pr\{Y_i \ge 0\} \ge 1-\gamma, \tag{1}$$

where $0 < \gamma < 1/2$ is the upper bound of the violation probability and is often set as 0.01, 0.05, or 0.1. If one is interested in $Y_i \ge b$ or $Y_i \le b$ instead of $Y_i \ge 0$, then Y_i can be redefined to fit into our formulation.

Suppose that we may run simulation experiments at solution i to observe independent observations of (X_i, Y_i), denoted as (X_{ij}, Y_{ij}), $j = 1, 2, \ldots$ for all $i = 1, 2, \ldots, k$. However, X_{ij} and Y_{ij} may be dependent because they are the outputs of a single simulation run. Moreover, we do not need (X_{ij}, Y_{ij}), $i = 1, 2, \ldots, k$, to be independent. Therefore, CRN may be used to induce positive correlations among (X_{ij}, Y_{ij}), $i = 1, 2, \ldots, k$, to make the comparisons sharper. Furthermore, for any $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots$ we assume that $X_{ij} \sim \mathrm{N}(\mu_i, \sigma_i^2)$ with unknown μ_i and σ_i^2, but we do not impose any specific distributional assumptions on Y_{ij}.

Andradóttir and Kim (2010) formulate the ECSB problem as follows:

$$\max_{i=1,2,\ldots,k} \ \mathrm{E}(X_i)$$
$$\text{s.t. } \ \mathrm{E}(Y_i) \geq 0,$$

and they assume that (X_i, Y_i) follows a bivariate normal distribution. Although $\Pr\{Y_i \geq 0\} = \mathrm{E}[1_{\{Y_i \geq 0\}}]$, where $1_{\{\cdot\}}$ is an indicator function, the formulation of CCSB is different from the formulation of ECSB because $1_{\{Y_i \geq 0\}}$ is a Bernoulli random variable instead of a normal random variable and it has many nice properties that facilitate feasibility checking.

Both ECSB and CCSB are reasonable formulations of constrained selection problems. Depending on practical considerations, one may be more suitable than the other. When Y_i is a performance measure related to quality of service, for instance, it may make more sense to use the CCSB formulation, where $Y_i \geq 0$ and $Y_i < 0$ define the events of satisfactory and unsatisfactory service, respectively. Then the chance constraint requires customer satisfaction with a probability at least $1 - \gamma$, which is a common approach to defining QoS requirements.

A high-level overview of our CCSB procedure follows, with the key contributions of our approach highlighted:

Initialization: Initialization includes selecting an overall statistical error allowance, a feasibility tolerance parameter for constraint checking, and an indifference-zone parameter for selection of the best. *How the overall error is allocated between feasibility checking and selection of the best is central to the validity and efficiency of the procedure.*

Feasibility Test: Each solution is simulated and declared either feasible or infeasible with a statistical guarantee of correctness up to the feasibility tolerance. *A key to our procedure is that the feasibility test simulates all solutions that are declared feasible for a fixed, prespecified number of observations; this is possible because of the Bernoulli distribution of chance constraints.* The feasibility test is introduced in §3.

Selecting the Best: An indifference-zone procedure is applied to select the best from among the solutions that are declared feasible, using the data from the feasibility test as the first stage of the procedure. *The fixed, prespecified feasibility-stage sample size avoids dependence between the first and subsequent stages that would invalidate the correct-selection guarantee.* Our procedures are described in §§4 and 5.

3. Feasibility Tests

The constraint in problem (1) is equivalent to $\Pr\{Y_i < 0\} \leq \gamma$. Let $p_i = \Pr\{Y_i < 0\}$ for all $i = 1, 2, \ldots, k$. Checking the feasibility of solution i is essentially a hypothesis test on a probability, which may be formulated as

$$H_0\colon p_i > \gamma \quad \text{vs.} \quad H_1\colon p_i \leq \gamma. \tag{2}$$

Therefore, rejecting H_0 indicates that solution i is feasible. Note that the presumption of any hypothesis test is that H_0 is true, and the goal of the hypothesis test is to collect enough evidence to reject H_0. Therefore, in our hypothesis, the presumption is that solution i is infeasible, and our goal is to use observations of Y_i to claim that solution i is feasible. This represents a conservative viewpoint toward feasibility, and it implies that claiming an infeasible solution feasible is more harmful than claiming a feasible solution infeasible. If one takes the opposite view, the hypothesis test may be formulated as $H_0\colon p_i \leq \gamma$ vs. $H_1\colon p_i > \gamma$.

Because Type I and II errors are both relevant, we want to determine an appropriate sample size n such that we achieve the following requirements on Type I and II errors:

$$\text{I: } \Pr\{\text{reject } H_0 \mid p_i > \gamma + \delta_{\gamma_1}\} \leq \beta_1, \tag{3}$$

$$\text{II: } \Pr\{\text{do not reject } H_0 \mid p_i \leq \gamma - \delta_{\gamma_2}\} \leq \beta_2, \tag{4}$$

where $\delta_{\gamma_1}, \delta_{\gamma_2} \geq 0$ may be viewed as tolerance levels on the constraints. Note that as long as $\beta_1 + \beta_2 < 1$ we cannot set $\delta_{\gamma_1} = 0$ and $\delta_{\gamma_2} = 0$ simultaneously and still control both Type I and II errors as desired. In fact, it is impossible to statistically guarantee identifying feasible solutions, even asymptotically, when stochastic constraints are tight (see Nelson 2013, Chap. 8). Objectives (3) and (4) solve this problem by employing an indifference-zone formulation.

Because we take a conservative point of view toward feasibility in this paper, hereafter we set $\delta_{\gamma_1} = 0$ and $\delta_{\gamma_2} = \delta_\gamma > 0$. Therefore, if solution i is infeasible (i.e., $p_i > \gamma$), it is declared a feasible solution with a probability less than β_1; if solution i is clearly feasible (i.e., $p_i \leq \gamma - \delta_\gamma$), it is declared an infeasible solution with a probability less than β_2; and if a solution is too close to the feasibility boundary (i.e., $\gamma - \delta_\gamma < p_i \leq \gamma$), then we do not have an explicit control of its Type II error, as shown in Figure 1. Note that the feasibility requirement is similar to that of Andradóttir and Kim (2010), except that we take a more restricted view on infeasibility. Once a solution

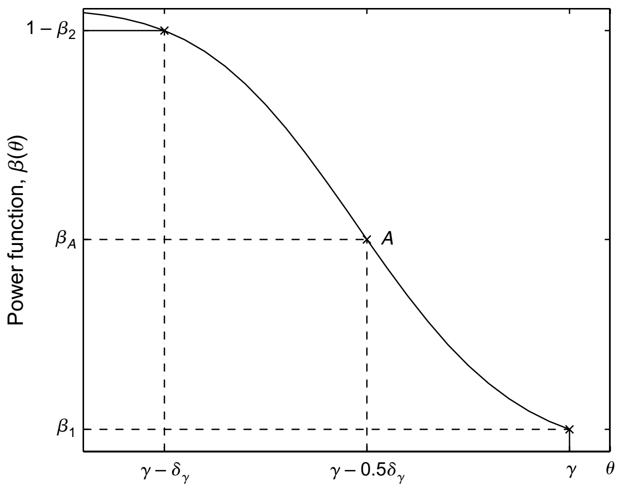

Figure 1 The Power Function $\beta(\theta) = \Pr\{\text{reject } H_0 \mid p_i = \theta\}$. Point A Is Used in §6.3

is infeasible, no matter how little it violates the constraint, we want to declare it infeasible with a controlled error. However, if a solution is feasible but with $\gamma - \delta_\gamma < p_i \le \gamma$, we may declare it infeasible with a probability that is larger than β_2, and this may cause the test to reject feasible solutions. To alleviate this problem, one can reduce the tolerance level δ_γ. However, reducing δ_γ may lead to an increase of the required sample size to make a decision.

As pointed out by Xu et al. (2010), if $\beta_1 = \beta_2$ in Equations (3) and (4), the hypothesis test (2) becomes a special case of comparisons with a standard (see, for instance, Nelson and Goldsman 2001 and Kim 2005). Indeed, Andradóttir and Kim (2010) also treat feasibility checking as a comparison with a standard and use the procedure of Kim (2005) to conduct the comparison. To follow this convention, we also set $\beta_1 = \beta_2 = \beta$, but the results can easily be extended to cases where $\beta_1 \ne \beta_2$.

3.1. Fixed-Sample-Size and Sequential Feasibility Tests

Suppose that we have $\{Y_{i1}, Y_{i2}, \ldots, Y_{in}\}$ for solution i. Let $Z_n = \sum_{j=1}^n 1_{\{Y_{ij}<0\}}$. (We do not make Z_n a function of solution i because the discussion that follows applies to any fixed solution.) An approach to testing the Hypothesis (2) is to determine an integer $m_\beta(n) \in \{0, 1, \ldots, n\}$ such that we reject H_0 if $Z_n \le m_\beta(n)$. To ensure Equation (3), we want

$$m_\beta(n) = \max\{m \in \{0, 1, \ldots, n\}\colon \Pr\{Z_n \le m \mid p_i = \gamma\} \le \beta\}.$$

It is easy to show that there will be such an $m_\beta(n)$ for all n large enough. To ensure Equation (4), we determine the sample size n, denoted as $n^*(\beta)$, such that

$$n^*(\beta) = \min\{n \in \{0, 1, \ldots\}\colon \Pr\{Z_n \ge m_\beta(n) + 1 \mid p_i = \gamma - \delta_\gamma\} \le \beta\}.$$

We can show that a solution $(m_\beta(n), n^*(\beta))$ exists provided $\beta < 1/2$ and $\delta_\gamma > 0$.

Note that Z_n is distributed according to a binomial distribution with parameters p_i and n. Let $F(x; n, p)$ denote the cumulative distribution function of a binomial distribution with parameters p and n. Then, for any $x \in \Re$,

$$F(x; n, p) = \sum_{i=0}^{\lfloor x \rfloor} \binom{n}{i} p^i (1-p)^{n-i},$$

where $\lfloor \cdot \rfloor$ is the floor function that rounds a number down to its nearest integer. We want the simultaneous solution to

$$m_\beta(n) = \max\{m \in \{0, 1, \ldots, n\}\colon F(m; n, \gamma) \le \beta\}, \quad (5)$$

$$n^*(\beta) = \min\{n \in \{0, 1, \ldots\}\colon F(m_\beta(n); n, \gamma - \delta_\gamma) \ge 1 - \beta\}. \quad (6)$$

Therefore, we can design the following procedure to test the feasibility of solution i:

Procedure 1 (Fixed-Sample-Size Feasibility Test)

Step 1. Given β, calculate $n = n^*(\beta)$ and $m_\beta(n)$.
Step 2. Run simulation experiments to observe $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ for solution i.
Step 3. Let $Z_n = \sum_{j=1}^n 1_{\{Y_{ij}<0\}}$. If $Z_n \ge m_\beta(n) + 1$, declare solution i infeasible; otherwise, declare it feasible.

Remark 1. When Procedure 1 is applied to all solutions $i = 1, 2, \ldots, k$, all solutions have the same fixed sample size $n^*(\beta)$ at the end of the test. Therefore, we call it a fixed-sample-size feasibility test. Furthermore, the fixed sample size $n^*(\beta)$ is a predetermined constant and it does not depend on the observations of Y_{ij}, $i = 1, 2, \ldots, k$. This property makes Procedure 1 different from the feasibility test procedures of Andradóttir and Kim (2010) and Batur and Kim (2010) where the sample sizes of all solutions are dependent on the observations of Y_{ij}, $i = 1, 2, \ldots, k$. In §4, we find this property particularly useful in designing statistically valid CCSB procedures.

We have the following theorem on the statistical validity of Procedure 1. The proof of the theorem is omitted as it is straightforward.

Theorem 1. *Suppose that Procedure* 1 *is used to test Hypothesis* (2). *Then*, $\Pr\{\text{reject } H_0\} \le \beta$ *if* $\Pr\{Y_i < 0\} \ge \gamma$ *and* $\Pr\{\text{do not reject } H_0\} \le \beta$ *if* $\Pr\{Y_i < 0\} \le \gamma - \delta_\gamma$.

Because a simulation model typically generates observations sequentially (a replication at a time), a minor modification may make the feasibility test more efficient. Let $Z_\tau = \sum_{j=1}^\tau 1_{\{Y_{ij}<0\}}$ for all $\tau \in \{0, 1, \ldots, n\}$. Clearly, $Z_\tau \le Z_n$ for any $\tau \in \{0, 1, \ldots, n\}$. Then

$Z_\tau \geq m_\beta(n)+1$ implies $Z_n \geq m_\beta(n)+1$. Therefore, in Procedure 1 we can stop the simulation of solution i and declare it infeasible at the sample size τ if $Z_\tau \geq m_\beta(n)+1$. This type of sequential test is known as a *curtailed test*, and the curtailed test on a probability (such as the one we use) is one of the few sequential tests that never take more samples, and may take fewer samples, than a fixed-sample-size version, thus "delivering positive benefit at no cost" (Siegmund 1985, p. 2). We call this the sequential feasibility test.

Procedure 2 (Sequential Feasibility Test)

Step 1. Given β, calculate $n = n^*(\beta)$ and $m_\beta(n)$. Let $\tau = 0$ and $Z_\tau = 0$.
Step 2. Let $\tau = \tau + 1$. Run a simulation experiment to observe $Y_{i\tau}$ and let $Z_\tau = Z_{\tau-1} + 1_{\{Y_{i\tau}<0\}}$.
Step 3. If $Z_\tau \geq m_\beta(n)+1$, declare solution i infeasible and end the procedure.
Step 4. If $\tau = n$, declare solution i feasible; otherwise, go to *Step* 2.

Remark 2. When Procedure 2 is applied to all solutions $i = 1, 2, \ldots, k$, all sample feasible solutions (i.e., the solutions that are claimed feasible by the test) have the same fixed sample size $n^*(\beta)$ at the end of the test. As noted in Remark 1, this is a useful property for designing statistically valid CCSB procedures.

It is interesting to see that both Procedures 1 and 2 can be done in a single stage under our formulation, thus leading to a constant and fixed sample size for all sample feasible solutions, while the feasibility test procedures of Andradóttir and Kim (2010) and Batur and Kim (2010) require two stages. This is because we use a chance constraint to handle the secondary performance measure. Thus, under Equations (3)–(4), the distribution of the Bernoulli random variable is known, and we may design a single-stage procedure to check the feasibility. Andradóttir and Kim (2010) and Batur and Kim (2010), however, use an expectation constraint to handle the secondary performance measure as in the formulation of ECSB. Under their indifference-zone formulation the means of the performance measures are known but the variances are not, so their procedures need a first stage to estimate the unknown variances of the secondary performance measure. Even though these sample variances are independent of the sample means of the secondary performance measures under the normality assumption, they may be dependent on the sample means of the primary performance measures. This makes it difficult to design statistically valid procedures to select the best solution from the set of sample feasible solutions (Andradóttir and Kim 2010).

3.2. Calculation of $m_\beta(n)$ and $n^*(\beta)$

Both Procedures 1 and 2 need to calculate the values of $m_\beta(n)$ and $n^*(\beta)$. Even though their values may be calculated using Equations (5) and (6), these calculations cannot be done efficiently because n is typically large when β and δ_γ are small. One solution is to use the normal approximation of a binomial distribution, which works well when n is large and γ is not too small (Casella and Berger 2002). Note that the normal approximation is also a common approach used in hypothesis tests on a probability (see, for instance, Tamhane and Dunlop 1999).

Under the normal approximation,

$$F(x; n, p) \approx \Phi\left(\frac{x - np}{\sqrt{np(1-p)}}\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Let $\tilde{m}_\beta(n)$ and $\tilde{n}^*(\beta)$ denote the continuous approximation of $m_\beta(n)$ and $n^*(\beta)$. Let $z_\alpha = \Phi^{-1}(\alpha)$ for any $0 < \alpha < 1$. Note that $z_\beta = -z_{1-\beta}$. Then, replacing $F(m; n, \cdot)$ in Equations (5) and (6) by the normal approximation above, we obtain two inequalities with two variables, m and n. After some algebra, we have

$$\tilde{n}^*(\beta) = \frac{z_{1-\beta}^2}{\delta_\gamma^2}\left(\sqrt{(\gamma-\delta_\gamma)(1-\gamma+\delta_\gamma)} + \sqrt{\gamma(1-\gamma)}\right)^2, \qquad (7)$$
$$\tilde{m}_\beta(n) = n\gamma - z_{1-\beta}\sqrt{n\gamma(1-\gamma)},$$

and we may set $m_\beta(n) = \lfloor \tilde{m}_\beta(n) \rfloor$ and $n^*(\beta) = \lceil \tilde{n}^*(\beta) \rceil$.

The normal approximation is accurate. We plot both the actual and approximated values in Figure 2 as functions of β and γ. If the exact values of $m_\beta(n)$ and $n^*(\beta)$ are required, one can search near the approximate solution to find the exact solutions $m_\beta(n)$ and $n^*(\beta)$.

4. Procedure CCSB

In this section, we propose a two-stage procedure. In the first stage, we test the feasibility of all solutions, and in the second stage, we conduct procedure $\mathcal{KN}$ to select the best from the sample feasible solutions. We first present the procedure in §4.1 and then discuss its error allocation and statistical validity in §4.2.

4.1. The Procedure

We propose a two-stage procedure for the CCSB problem. In the first stage, feasibility of all solutions are tested and sample infeasible solutions are eliminated. As the sequential procedure guarantees to reduce the total sample size in the feasibility test, we use it instead of the fixed-sample-size procedure in the first stage. In the second stage, procedure $\mathcal{KN}$ is conducted to select the best solution from the set of sample feasible solutions (with some changes in error allocation that are discussed in §4.2). Let α_1 be the error

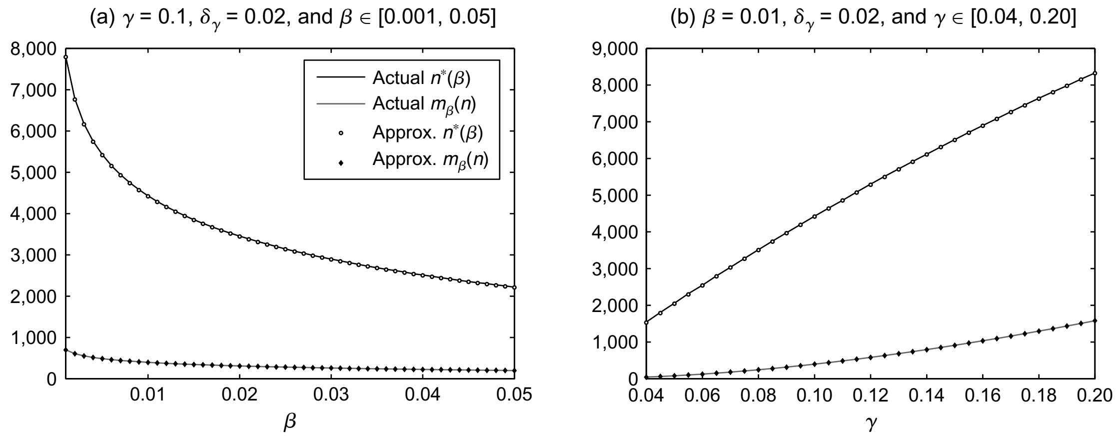

Figure 2 **The Actual Values and Normal Approximations of $m_\beta(n)$ and $n^*(\beta)$**

allocated to the feasibility test of each solution, and let α_2 be the error allocated to each pairwise elimination for selection of the best. The choices of α_1 and α_2 are important and will be described later.

Procedure 3 (Procedure CCSB)

Initialization. Select total error allowance $0 < \alpha < 1 - 1/k$, indifference-zone parameter δ, and feasibility tolerance parameter δ_γ. Choose α_1 and α_2. Solve for $n_0 = n^*(\alpha_1)$ and $m_{\alpha_1}(n_0)$. Let

$$h^2 = (n_0 - 1)[(2\alpha_2)^{-(2/(n_0-1))} - 1].$$

We discuss the choices of α_1 and α_2 in §4.2.

Feasibility Test. Let $\mathcal{F}$ denote the set of sample feasible solutions and set $\mathcal{F} = \varnothing$. Let $i = 1$.

Step 0. If $i > k$, where k is the number of solutions, terminate *Feasibility Test*; otherwise, let $\tau = 0$, $Z_\tau = 0$.

Step 1. Let $\tau = \tau + 1$ and $\mathcal{F}^{\text{old}} = \mathcal{F}$. Take an additional sample $(X_{i\tau}, Y_{i\tau})$ from solution i and let $Z_\tau = Z_{\tau-1} + 1_{\{Y_{i\tau} < 0\}}$.

Step 2. If $Z_\tau \geq m_{\alpha_1}(n_0) + 1$, declare solution i infeasible and go to *Step* 3; else if $\tau = n_0$, declare solution i feasible, let $\mathcal{F} = \mathcal{F}^{\text{old}} \cup \{i\}$ and go to *Step* 3; otherwise, go to *Step* 1.

Step 3. Let $i = i + 1$ and go to *Step* 0.

Selecting the Best. Let $\mathcal{I}$ denote the set of solutions still in contention and let $\mathcal{I} = \mathcal{F}$. For all $i \in \mathcal{I}$, calculate

$$\bar{X}_i(n_0) = \frac{1}{n_0} \sum_{l=1}^{n_0} X_{il}$$

and for all $i, j \in \mathcal{I}$ and $i \neq j$, calculate

$$S_{ij}^2 = \frac{1}{n_0 - 1} \sum_{l=1}^{n_0} (X_{il} - X_{jl} - [\bar{X}_i(n_0) - \bar{X}_j(n_0)])^2. \qquad (8)$$

Note that $\bar{X}_i(n_0)$ is the first-stage sample mean of the primary performance measure of solution i and S_{ij}^2 is the first-stage sample variance of the difference between the primary performance measures of solutions i and j.

Step 0. Set $r = n_0$.

Step 1. Set $\mathcal{I}^{\text{old}} = \mathcal{I}$ and let

$$\mathcal{I} = \{i \in \mathcal{I}^{\text{old}}\colon \bar{X}_i(r) \geq \bar{X}_j(r) - W_{ij}(r), \text{ for all } j \in \mathcal{I}^{\text{old}} \text{ and } j \neq i\},$$

where

$$W_{ij}(r) = \max\left\{0, \frac{\delta}{2r}\left(\frac{h^2 S_{ij}^2}{\delta^2} - r\right)\right\}.$$

Step 2. If $|\mathcal{I}| = 1$, then stop and let the solution with an index that is in $\mathcal{I}$ be the best; otherwise, let $r = r + 1$, take an additional sample (X_{ir}, Y_{ir}) from solution i for all $i \in \mathcal{I}$, and go to *Step* 1.

Note that we estimate the variance of the difference between the primary performance measures of any pair of sample feasible solutions using Equation (8). Therefore, our procedure allows the use of CRN to make comparisons between $\mathrm{E}(X_i)$ and $\mathrm{E}(X_j)$ sharper (Kim and Nelson 2001). The use of CRN may also introduce dependence between Y_i and Y_j. However, because the feasibility test in §3 is a marginal test for each solution individually, the statistical validity of the feasibility test is not affected when CRN are used. However, CRN was not considered by Andradóttir and Kim (2010) because they assume that all solutions are simulated independently to achieve a statistical guarantee for their proposed procedures.

4.2. Error Allocation and Statistical Validity

Let F and $\bar{F}$ denote the sets of feasible and infeasible solutions, respectively. We do not know which solutions are included in F and $\bar{F}$, but $F \cup \bar{F} = \{1, 2, \ldots, k\}$ and $F \cap \bar{F} = \varnothing$. Without loss of generality, assume that solution 1 (the identity of which is unknown) is the best feasible solution. Taking the typical indifference-zone approach in the literature (e.g., Kim and Nelson

2001 and Andradóttir and Kim 2010), we formulate the problem as follows:

$$\mathrm{E}(X_1) \ge \max_{i\in\{2,3,\ldots,k\}\cap F} \mathrm{E}(X_i) + \delta, \tag{9}$$

$$\Pr\{Y_1 \ge 0\} \ge 1-\gamma+\delta_\gamma. \tag{10}$$

Therefore, the best solution has an expected primary performance measure that is at least δ better than all other feasible solutions and has a secondary performance measure that is positive with a probability at least $1-\gamma+\delta_\gamma$.

Let $\mathscr{E}(i,j)$ denote the event that solution i eliminates solution j by the end of the second stage of Procedure CCSB if only those two solutions were considered in isolation. Note that the event of correct selection (CS) requires that solution 1 survives the feasibility test (i.e., $1\in\mathcal{F}$) and solution 1 survives the comparisons from all other solutions in $\mathcal{F}$. Then,

$$\begin{aligned}\Pr\{\mathrm{CS}\} &\ge \Pr\{1\in\mathcal{F} \text{ and } \mathscr{E}(1,i), \forall\, i\in\mathcal{F}, i\ne 1\}\\
&\ge \Pr\{1\in\mathcal{F} \text{ and } \bar F\cap\mathcal{F}=\varnothing \text{ and } \mathscr{E}(1,i),\\
&\qquad \forall\, i\in\mathcal{F}, i\ne 1\} \qquad (11)\\
&\ge \Pr\{1\in\mathcal{F} \text{ and } \bar F\cap\mathcal{F}=\varnothing \text{ and } \mathscr{E}(1,i),\\
&\qquad \forall\, i\in F, i\ne 1\} \qquad (12)\\
&\ge 1-\Pr\{1\notin\mathcal{F}\}-\Pr\{\exists\, i\in\bar F\colon i\in\mathcal{F}\}\\
&\qquad -\Pr\{\exists\, i\in F, i\ne 1\colon \mathscr{E}(i,1)\} \qquad (13)\\
&\ge 1-\Pr\{1\notin\mathcal{F}\}-\sum_{i\in\bar F}\Pr\{i\in\mathcal{F}\}\\
&\qquad -\sum_{i\in F, i\ne 1}\Pr\{\mathscr{E}(i,1)\} \qquad (14)\\
&\ge 1-\alpha_1-|\bar F|\alpha_1-(|F|-1)\alpha_2. \qquad (15)\end{aligned}$$

In this analysis, the critical step is Inequality (11), where we add the additional requirement $\bar F\cap\mathcal{F}=\varnothing$. Then, the event inside of $\Pr\{\cdot\}$ implies that

$$\{1\}\subseteq\mathcal{F}\subseteq F. \tag{16}$$

Because $\mathcal{F}\subseteq F$, we may enlarge $\mathcal{F}$ to F in Inequality (12). Inequalities (13) and (14) are the consequences of direct applications of Bonferroni's inequality, and Inequality (15) follows from Theorem 1 and Kim and Nelson (2001) under the assumption that Inequalities (9) and (10) are satisfied. Even though $|\bar F|$ and $|F|$ are unknown, we know that $|\bar F|+|F|=k$. Therefore, it is natural to choose $\alpha_1=\alpha_2=\alpha/k$. Then, $\Pr\{\mathrm{CS}\}\ge 1-\alpha$. We summarize the statistical validity of Procedure CCSB in the following theorem.

Theorem 2. *Suppose that Procedure CCSB is used to solve Problem* (1), *Equations* (9) *and* (10) *are satisfied, and* $\alpha_1=\alpha_2=\alpha/k$ *for any* $0<\alpha<1$. *Then,* $\Pr\{\mathrm{CS}\}\ge 1-\alpha$.

As pointed out by Remark 2, all sample feasible solutions have the same deterministic sample size n_0 at the end of the first stage. Therefore, we may use these samples to calculate the sample variances S_{ij}^2 without creating any unnecessary dependence that affects the statistical validity of procedure $\mathscr{KN}$. This is the major difference between Procedure CCSB and procedure $\mathscr{AK}$ of Andradóttir and Kim (2010) in terms of statistical validity.

Another difference between Procedures CCSB and $\mathscr{AK}$ is the error allocation scheme. Let

$$F_{\delta_\gamma}=\{i\in\{1,2,\ldots,k\}\colon \Pr\{Y_i\ge 0\}\ge 1-\gamma+\delta_\gamma\}. \tag{17}$$

Note that F_{δ_γ} denotes the set of solutions that are clearly feasible (i.e., $\Pr\{Y_i\ge 0\}$ is at least δ_γ larger than γ) and $\{1\}\subseteq F_{\delta_\gamma}$. When applying their error-allocation strategy to our formulation, Andradóttir and Kim (2010) and Batur and Kim (2010) essentially require

$$F_{\delta_\gamma}\subseteq\mathcal{F}\subseteq F, \tag{18}$$

which is a stronger requirement than ours (Equation (16)) and thus leads to more conservative allocations of the total error α.

In many practical situations, however, the indifference-zone assumptions, Equations (9) and (10), may not be satisfied. To allow for those situations, we follow the convention of the indifference-zone formulation and define the set of acceptable solutions. Without loss of generality, we now define solution 1 as the best *clearly feasible* solution—i.e., $\mathrm{E}(X_1)=\max_{i\in F_{\delta_\gamma}}\mathrm{E}(X_i)$. Then, we define the set of acceptable solutions as

$$A=\big\{i\in\{1,2,\ldots,k\}\colon \mathrm{E}(X_i)>\mathrm{E}(X_1)-\delta \text{ and } \Pr\{Y_i\ge 0\}\ge 1-\gamma\big\},$$

and call an event a good selection (GS) if the selected solution is in the set A. That is, the solutions in set A are feasible and within δ of the best clearly feasible solution. Figure 3 presents an example to illustrate the set A. In this example, while solution 1 is the best clearly feasible solution, solution 2 is the best feasible solution.

Corollary 1. *Suppose that Procedure CCSB is used to solve Problem* (1), *solution* 1 *is the best clearly feasible solution, and* $\alpha_1=\alpha_2=\alpha/k$ *for any* $0<\alpha<1$. *Then,* $\Pr\{\mathrm{GS}\}\ge 1-\alpha$.

Proof. The probability of a good selection can be bounded as follows:

$$\begin{aligned}&\Pr\{\mathrm{GS}\}\\
&\quad\ge \Pr\{1\in\mathcal{F} \text{ and } \mathscr{E}(1,i), \forall\, i\in\mathcal{F}\setminus A\}\\
&\quad\ge \Pr\{1\in\mathcal{F} \text{ and } \bar F\cap\mathcal{F}=\varnothing \text{ and } \mathscr{E}(1,i), \forall\, i\in\mathcal{F}\setminus A\}\end{aligned}$$

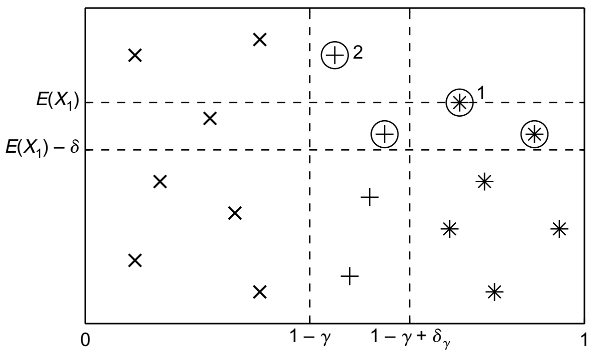

Figure 3 In This Example, the x-Axis Is $\Pr\{Y_i \geq 0\}$ and y-Axis Is $\mathrm{E}(X_i)$.

Note. × Represents Infeasible Solutions, + Represents Feasible But Not Clearly Feasible Solutions, ∗ Represents Clearly Feasible Solutions, and ○ Represents Acceptable Solutions

$$\begin{aligned}&\geq \Pr\{1\in\mathcal{F} \text{ and } \bar{F}\cap\mathcal{F}=\varnothing \text{ and } \mathscr{E}(1,i),\forall i\in F\backslash A\}\\&\geq 1-\Pr\{1\notin\mathcal{F}\}-\Pr\{\exists i\in\bar{F}\colon i\in\mathcal{F}\}\\&\quad-\Pr\{\exists i\in F\backslash A\colon \mathscr{E}(i,1)\}.\end{aligned}\tag{19}$$

Following the arguments in Kim and Nelson (2001), $\{1\}\subset A$, so $|A|\geq 1$. Then,

$$\begin{aligned}\Pr\{\exists i\in F\backslash A\colon \mathscr{E}(i,1)\}&\leq \sum_{i\in F/A}\Pr\{\mathscr{E}(i,1)\}\\&\leq (|F|-|A|)\alpha_2\leq(|F|-1)\alpha_2.\end{aligned}$$

Plugging this into Inequality (19), we have

$$\Pr\{\mathrm{GS}\}\geq 1-\alpha_1-|\bar{F}|\alpha_1-(|F|-1)\alpha_2=1-\alpha,$$

when $\alpha_1=\alpha_2=\alpha/k$.

5. Multiple Secondary Performance Measures

The formulation (1) is for a single secondary performance measure. In many practical situations, there will exist multiple secondary performance measures that the decision maker cares about. Let $\mathbf{Y}_i=(Y_i^{(1)},\ldots,Y_i^{(m)})'$ denote the vector of m secondary performance measures of solution i, $i=1,2,\ldots,k$. When simulating solution i at the jth replication, we observe $(X_{ij},Y_{ij}^{(1)},\ldots,Y_{ij}^{(m)})$, $i=1,2,\ldots,k$ and $j=1,2,\ldots$.

We consider two formulations to handle multiple secondary performance measures. In both formulations, for any solution i, $i=1,2,\ldots,k$, we consider m constraints corresponding to m secondary performance measures and treat $\{Y_i^{(s)}\geq 0\}$ as the event of satisfying the sth constraint, $s=1,2,\ldots,m$. However, the two formulations differ in how the probabilities of these events are considered.

5.1. Joint Chance Constraint

In the first formulation, the secondary performance measures are considered satisfactory if all of them satisfy their corresponding constraints simultaneously, i.e., only the event $\{Y_i^{(1)}\geq 0,\ldots,Y_i^{(m)}\geq 0\}$ is considered satisfactory, and a solution is feasible if its probability of being satisfactory is above a certain threshold. Therefore, we may formulate the CCSB problem with multiple secondary performance measures as

$$\begin{aligned}&\max_{i=1,2,\ldots,k}\ \mathrm{E}(X_i)\\&\text{s.t. } \Pr\{Y_i^{(1)}\geq 0,\ldots,Y_i^{(m)}\geq 0\}\geq 1-\gamma.\end{aligned}\tag{20}$$

In the stochastic programming literature, Problem (20) is also known as a joint chance constrained program.

Define

$$Y_i=\min\{Y_i^{(1)},\ldots,Y_i^{(m)}\}.\tag{21}$$

Then, the constraint in Problem (20) becomes

$$\Pr\{Y_i\geq 0\}\geq 1-\gamma,$$

which is the same as the constraint in Problem (1). Therefore, we convert a joint chance constraint into a single chance constraint. Note that this technique is also used by Hong et al. (2011). Procedure CCSB can be applied directly to Problem (20) to select the best feasible solution. We can define Y_i as in Equation (21) because our approach does not need a distributional assumption on Y_i.

An important benefit of formulation (20) is that we do not need to handle multiple constraints, thus avoiding the extra conservativeness introduced by using Bonferroni's inequality in error allocation (see, for instance, Batur and Kim 2010, and also §5.2 of this paper).

5.2. Multiple Individual Chance Constraints

In the alternative formulation, the secondary performance measures are considered satisfactory if each of them satisfies a separate chance constraint, and a solution is feasible if all chance constraints are satisfied simultaneously. Therefore, we may formulate the CCSB problem with multiple secondary performance measures as

$$\begin{aligned}&\max_{i=1,2,\ldots,k}\ \mathrm{E}(X_i)\\&\text{s.t. } \Pr\{Y_i^{(s)}\geq 0\}\geq 1-\gamma_s,\quad s=1,2,\ldots,m,\end{aligned}\tag{22}$$

and the tolerance level for each of the constraints is set as $\delta_{\gamma_s}>0$ for all $s=1,2,\ldots,m$. Without loss of generality, we denote solution 1 as the best feasible solution, and similar to Equations (9) and (10), we assume that

$$\mathrm{E}(X_1)\geq \max_{i\in\{2,3,\ldots,k\}\cap F}\mathrm{E}(X_i)+\delta,\tag{23}$$

$$\Pr\{Y_1^{(s)}\geq 0\}\geq 1-\gamma_s+\delta_{\gamma_s},\quad s=1,2,\ldots,m.\tag{24}$$

Problem (22) has m constraints. So the *Feasibility Test* step of Procedure CCSB cannot be applied directly, but the *Selecting the Best* step remains the same. We present the new feasibility test in §5.2.1 and discuss error allocation in §5.2.2.

5.2.1. Feasibility Test Procedure. Let $F_s = \{i \in \{1, 2, \ldots, k\}\colon \Pr\{Y_i^{(s)} \geq 0\} \geq 1 - \gamma_s\}$ denote the set of solutions that satisfy the sth constraint. Then it is clear that the set of feasible solutions is $F = \bigcap_{s=1}^m F_s$. Therefore, we may check the feasibility of each constraint of each solution. If any of the constraints appears to be violated, we claim the solution infeasible; otherwise, we claim it feasible.

Let

$$m_\beta^{(s)}(n) = \sup\{m \in \{0, 1, \ldots, n\}\colon F(m; n, \gamma_s) \leq \beta\},$$

$$n_s^*(\beta) = \inf\{n \in \{0, 1, \ldots\}\colon F(m_\beta^{(s)}(n); n, \gamma_s - \delta_{\gamma_s}) \geq 1 - \beta\}.$$

Let β_{1s} denote the Type I and II errors allocated to the sth constraint, $s = 1, 2, \ldots, m$. Note that, to test the sth constraint for solution i, we need $n_s^*(\beta_{1s})$ observations of $Y_i^{(s)}$. However, $Y_i^{(1)}, \ldots, Y_i^{(m)}$ are observed simultaneously. Therefore, to test all constraints for solutions i, we need

$$n_0 = \max_{s \in \{1,2,\ldots,m\}} n_s^*(\beta_{1s})$$

observations of $(Y_i^{(1)}, \ldots, Y_i^{(m)})$. One way to choose $\beta_{11}, \ldots, \beta_{1m}$ is to have them satisfy that

$$n_0 = n_1^*(\beta_{11}) = \cdots = n_m^*(\beta_{1m}). \tag{25}$$

By doing so, we can ensure that the feasibility tests of all individual constraints finish at the same sample size. Because we cannot start selection of the best until all constraint checking is complete, there is no benefit to having unequal sample sizes unless doing so would permit us to make the sample sizes for *all constraints* smaller; there is no reason to think we could achieve this. In §5.2.2 we discuss how to determine $\beta_{11}, \ldots, \beta_{1m}$ to satisfy Equation (25).

We replace the *Feasibility Test* step of Procedure CCSB by the following procedure, which tests all m constraints simultaneously.

Procedure 4 (Feasibility Test for Multiple Chance Constraints)

Feasibility Test. Let $\mathcal{F}$ denote the set of sample feasible solutions and set $\mathcal{F} = \varnothing$. Let $i = 1$.
- *Step* 0. If $i > k$, where k is the number of solutions, terminate the *Feasibility Test*; otherwise, let $\tau = 0$ and $Z_{\tau,s} = 0$.
- *Step* 1. Let $\tau = \tau + 1$ and $\mathcal{F}^{\text{old}} = \mathcal{F}$. Take an additional sample $(X_{i\tau}, Y_{i\tau}^{(1)}, \ldots, Y_{i\tau}^{(m)})$ from solution i and let $Z_{\tau,s} = Z_{\tau-1,s} + 1_{\{Y_{i\tau}^{(s)} < 0\}}$ for all $s = 1, 2, \ldots, m$.
- *Step* 2. If $Z_{\tau,s} \geq m_{\beta_{1s}}^{(s)}(n_0) + 1$ for any $s = 1, 2, \ldots, m$, declare solution i infeasible and go to *Step* 3; else if $\tau = n_s$, declare solution i feasible and let $\mathcal{F} = \mathcal{F}^{\text{old}} \cup \{i\}$ and go to *Step* 3; otherwise, go to *Step* 1.
- *Step* 3. Let $i = i + 1$ and go to *Step* 0.

5.2.2. Error Allocation. Now we discuss how to choose $\beta_{11}, \ldots, \beta_{1m}$ and α_2 so that the statistical validity of Theorem 2 can be extended to the case of multiple chance constraints. By Inequality (14), we have

$$\Pr\{\text{CS}\} \geq 1 - \Pr\{1 \notin \mathcal{F}\} - \sum_{i \in \bar{F}} \Pr\{i \in \mathcal{F}\} - \sum_{i \in F,\, i \neq 1} \Pr\{\mathcal{E}(i, 1)\}. \tag{26}$$

We analyze the three probability terms on the right-hand side of Inequality (26) one by one.

To analyze the first term, let $\mathcal{F}_s$ denote the set of solutions that are sample feasible for constraint s, $s = 1, 2, \ldots, m$. Note that $\mathcal{F}_s$, $s = 1, 2, \ldots, m$, are not available to us at the end of the feasibility-checking stage because if a solution violates a constraint the procedure stops and we do not know whether it satisfies other constraints. Nevertheless, we can conceptually define the sets assuming we use a fixed-sample-size feasibility test similar to Procedure 1, and it is clear that $\mathcal{F} \subseteq \mathcal{F}_s$ for all $s = 1, 2, \ldots, m$ and $\bigcap_{s=1}^m \mathcal{F}_s = \mathcal{F}$. Then,

$$\Pr\{1 \notin \mathcal{F}\} = \Pr\{1 \notin \mathcal{F}_s, \text{ for some } s = 1, 2, \ldots, m\} \leq \sum_{s=1}^m \Pr\{1 \notin \mathcal{F}_s\} \leq \sum_{s=1}^m \beta_{1s},$$

where the last inequality follows from Theorem 1 if Equation (24) is satisfied.

To analyze the second term on the right-hand side of Inequality (26), we notice that

$$\begin{aligned} \sum_{i \in \bar{F}} \Pr\{i \in \mathcal{F}\} &= \sum_{i \in \bar{F}} \Pr\{i \in \mathcal{F}_s, \text{ for all } s = 1, 2, \ldots, m\} \\ &\leq \sum_{i \in \bar{F}} \Pr\{i \in \mathcal{F}_s, \text{ for all } s = 1, 2, \ldots, m \text{ such that } i \notin F_s\} \\ &\leq \sum_{i \in \bar{F}} \max_{s \in \{t:\, i \notin F_t\}} \beta_{1s} \leq |\bar{F}| \max_{s \in \{1,2,\ldots,m\}} \beta_{1s}, \end{aligned} \tag{27}$$

where Inequality (27) holds because, if $i \in \bar{F}$, then there exists at least one $s \in \{1, 2, \ldots, m\}$ such that $i \notin F_s$, i.e., an infeasible solution violates at least one of the m constraints.

The third term on the right-hand side of Inequality (26) is clearly upper bounded by $(|F| - 1)\alpha_2$ by the property of procedure $\mathcal{KN}$. Let $\bar{\alpha}_1 = \max_{s=1,2,\ldots,m} \beta_{1s}$. Combining all three terms, we have

$$\Pr\{\text{CS}\} \geq 1 - \sum_{s=1}^m \beta_{1s} - |\bar{F}|\bar{\alpha}_1 - (|F| - 1)\alpha_2.$$

To ensure that $\Pr\{\text{CS}\} \geq 1 - \alpha$, we let $\alpha_2 = \bar{\alpha}_1$ and require that

$$\sum_{s=1}^m \beta_{1s} + (k - 1)\bar{\alpha}_1 = \alpha. \tag{28}$$

To better understand the implications of Equation (28), we first consider a special case where $\gamma_1 = \cdots = \gamma_m$ and $\delta_{\gamma_1} = \cdots = \delta_{\gamma_m}$. Then, by Equations (25) and (28), we have

$$\bar{\alpha}_1 = \beta_{11} = \cdots = \beta_{1m} = \frac{\alpha}{m+k-1}.$$

Therefore, the error allocated to the feasibility test of each individual constraint and the error allocated to the comparisons between each pair of solutions are $\alpha/(m+k-1)$, which we call an *additive rule* because we divide the total error α by the addition of the number of constraints m and the number of solutions k. Batur and Kim (2010) consider only the feasibility test (without the selection of the best) and the error allocated to the feasibility test of each individual constraint is $\alpha/(mk)$, which we call a *multiplicative rule*. The multiplicative rule is significantly more conservative than the additive rule when m is not equal to one. The difference between the two rules is because we use Equation (16) to define correct feasibility checking while Batur and Kim (2010) use Equation (18). If one's goal is to select the best feasible solution instead of identifying all feasible solutions, then Equation (16) is sufficient.

Now we consider how to determine $\beta_{11}, \ldots, \beta_{1m}$ based on Equations (25) and (28) for more general cases of γ_s and δ_{γ_s}, $s = 1, 2, \ldots, m$. We use the normal approximation approach of §3.2. Let

$$a_s = \frac{1}{\delta_{\gamma_s}}\left(\sqrt{(\gamma_s - \delta_{\gamma_s})(1-\gamma_s+\delta_{\gamma_s})} + \sqrt{\gamma_s(1-\gamma_s)}\right)$$

for all $s = 1, 2, \ldots, m$. To ensure Equation (25), we need

$$a_1 z_{1-\beta_{11}} = \cdots = a_m z_{1-\beta_{1m}} = \xi$$

for some $\xi > 0$. Then, we have

$$\beta_{1s} = \Phi^{-1}\left(-\frac{\xi}{a_s}\right), \quad s = 1, 2, \ldots, m. \qquad (29)$$

Therefore, we only need to find $\xi > 0$ such that Equation (28) is satisfied.

Let $\bar{a} = \max_{s\in\{1,2,\ldots,m\}} a_s$. Then, by Equation (28) and (29), ξ satisfies

$$\sum_{s=1}^{m} \Phi^{-1}\left(-\frac{\xi}{a_s}\right) + (k-1)\Phi^{-1}\left(-\frac{\xi}{\bar{a}}\right) = \alpha. \qquad (30)$$

Note that the left-hand side of Equation (30) is a decreasing function of ξ and α is in the range of the function. Therefore, Equation (30) has a unique root $\xi^* > 0$, which is

$$\beta_{1s} = \Phi^{-1}\left(-\frac{\xi^*}{a_s}\right), \quad s = 1, 2, \ldots, m.$$

6. Numerical Examples

In this section, we use several numerical examples to examine the performance of the proposed procedures in handling various CCSB problems.

6.1. Efficiency

We first illustrate the efficiency of Procedure CCSB proposed in §4.1 for a single-constraint problem by different numerical examples. These examples have also been formulated as ECSB problems to be solved by procedures $\mathscr{AK}$ and $\mathscr{AK}+$ in Andradóttir and Kim (2010). Section 6.1.1 describes the experimental configurations and §6.1.2 presents the main results, followed by a comparison with both $\mathscr{AK}$ and $\mathscr{AK}+$ in §6.1.3.

6.1.1. Configurations of Test Examples with a Single Constraint. We consider only one secondary performance measure for the set of test problems. Suppose that $X_i \sim N(\mu_i, \sigma_i^2)$ and $Y_i \sim N(\nu_i, 1)$ for all $i = 1, 2, \ldots, k$. We consider the following slippage configuration of means for the primary performance measure:

$$\mu_i = \begin{cases} \delta, & i = 1; \\ 0, & i = 2, 3, \ldots, b; \\ i\delta, & i = b+1, b+2, \ldots, k, \end{cases}$$

and various configurations of means for the secondary performance measure:

$$\nu_i = \begin{cases} -\Phi^{-1}(\gamma - \delta_\gamma), & i = 1; \\ -\Phi^{-1}(\gamma - c_1\delta_\gamma), & i = 2, 3, \ldots, b; \\ -\Phi^{-1}(\gamma + c_2\delta_\gamma), & i = b+1, b+2, \ldots, k, \end{cases} \qquad (31)$$

where $b = \lfloor (1+k)/2 \rfloor$, $\delta = 1/\sqrt{10}$ is the indifference-zone parameter and $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal distribution function. Setting $c_1 \geq 1$ and $c_2 \geq 0$ implies that solutions $1, 2, \ldots, b$ are feasible and the rest are infeasible. Note that, in this slippage configuration of means (where $c_1 = 1$ and $c_2 = 0$), we set $\mu_i = i\delta$ for all infeasible solutions $i = b+1, \ldots, k$ to make infeasible solutions particularly difficult to eliminate in the second stage if they are declared feasible.

Under the slippage configuration of means, we consider three variance configurations: all $\sigma_i^2 = 10^2$ in the equal-variance configuration, $\sigma_i^2 = 10^2[1+(i-1)\delta]$ in the increasing-variance configuration, and $\sigma_i^2 = 10^2/[1+(i-1)\delta]$ in the decreasing-variance configuration, respectively. For simplicity, we assume that (X_i, Y_i), $i = 1, 2, \ldots, k$, are mutually independent and X_i is also independent of Y_i for all $i = 1, 2, \ldots, k$.

Other parameters are specified as follows. Let the upper bound of the violation probability be $\gamma = 0.1$ and the tolerance level be $\delta_\gamma = 0.02$. Let the total error allowance be $\alpha = 0.05$ and the Type I and II errors defined in Equations (3) and (4) are chosen as

Table 1 Optimal $(m_\beta(n), n^*(\beta))$ with a Single Constraint

	$k=5$	$k=25$	$k=101$
$m_\beta(n)$	397	607	794
$n^*(\beta)$	4,434	6,776	8,862

$\alpha_1 = \alpha_2 = \alpha/k$. We set the number of solutions as $k = 5, 25, 101$. Using Equations (5) and (6), we compute the optimal $m_\beta(n)$ and $n^*(\beta)$ shown in Table 1, where $\beta = \alpha_1$ throughout this section.

6.1.2. Main Results for CCSB Formulation with a Single Constraint. In Table 2, we report detailed results for all solutions for the case where $k = 5$, including the average first-stage sample size (FSS), the observed feasibility probability (FP) and surviving probability[2] (SP) at the end of the first stage, the average total sample size (TSS), and the probability of the selection (PS) for each solution, over 1,000 independent macroreplications. In addition, the average total sample sizes for all solutions by the end of the first and second stages are reported in the last column of Table 2.

We have several findings from the results in Table 2. First, the sequential feasibility test can save about 4.1% of sampling effort even in the slippage configuration, where 4.1% is calculated by $(1 - \text{Total of FSS}/(kn_0)) \times 100\%$. Second, because feasibility checking in the first stage may require a significant amount of computational effort, it makes sense to use the first-stage samples for elimination rather than abandoning them, as in the restarting procedure in Andradóttir and Kim (2010). In fact, based on the information from the first-stage samples, there is a significant chance to make an elimination decision for feasible but inferior solutions (solutions 2 and 3 in this case) according to the difference between FP and SP. Third, in the configuration that we consider, it is often difficult to eliminate an infeasible solution once it passes the feasibility test because it has a larger primary performance measure than all feasible ones. Fourth, it is often harder (easier) to select the best for the increasing-variance (decreasing-variance) configuration than for the equal-variance configuration, which is consistent with the intuition that it is often difficult to eliminate inferior solutions with large variances.

Due to the space limitation, we summarize the numerical results with similar conclusions for $k = 25$ and $k = 101$ in Table EC.1 of the online supplement. In §6.1.3, we investigate the efficiency of Procedure CCSB compared with two existing procedures: $\mathcal{AK}$ and $\mathcal{AK}+$.

[2] Surviving probability of solution i denotes the probability that solution i has not been eliminated by $\mathcal{KN}$ procedure with just the first-stage n_0 samples.

6.1.3. Comparisons with Procedures $\mathcal{AK}$ and $\mathcal{AK}+$. Because CCSB problems can be formulated as ECSB problems, we also applied two competitors to Procedure CCSB, a two-stage procedure, $\mathcal{AK}$, and a simultaneously running procedure $\mathcal{AK}+$ from Andradóttir and Kim (2010).

To make a fair comparison, we set $\alpha_1 = \alpha_2 = \alpha/2 = 0.025$ for $\mathcal{AK}$ and $\alpha = 0.05$ for $\mathcal{AK}+$, $q = \gamma - \delta_\gamma/2 = 0.09$ and $\epsilon = \delta_\gamma/2 = 0.01$, and the initial-stage sample size $n_0 = 20$. All other parameters are the same as in §6.1.1. We only report detailed results for $k = 5$ with equal-variance configuration in Table 3. The numerical results for $k = 25$ and $k = 101$ are presented in the online supplement (EC.1.1). Note that $\mathcal{AK}+$ is a simultaneously running procedure that performs both feasibility checking and optimality checking at the same time, so it is not clear how to define a proper first stage for $\mathcal{AK}+$. In this paper, we only report the estimates (i.e., TSS and PS) when procedure $\mathcal{AK}+$ is completed.

From Table 3 we find that, without batching, procedure $\mathcal{AK}$ (and also $\mathcal{AK}+$, the results of which have been omitted here) may not be able to deliver the desired PCS (i.e., PS of solution 1) when constraints involve probabilities. Even though the probabilistic constraints can be written as expectations of Bernoulli random variables, the normality assumption for procedure $\mathcal{AK}$ and $\mathcal{AK}+$ fails. This has been observed in Andradóttir and Kim (2010, §6.3), and they suggest using batching to overcome this problem. We also report results for both $\mathcal{AK}$ and $\mathcal{AK}+$ when batching is used with a batch size of 10 replications. With batching, the desired PCS is nearly achieved by both procedures, which is consistent with the conclusion of Andradóttir and Kim (2010, §6.3). It is interesting to point out that, from the results in Table EC.2 in the online supplement, procedure $\mathcal{AK}$ achieves the desired PCS while $\mathcal{AK}+$ slightly fails, which implies the importance of selecting a proper batch size.

Comparing the other results for the equal-variance case with Table 2, we have additional observations: The total number of samples required for feasibility checking by procedure $\mathcal{AK}$ is less than that of Procedure CCSB (approximately 27%), but the total number of replications needed to select the best for procedure $\mathcal{AK}$ (and $\mathcal{AK}+$) is slightly more than that for Procedure CCSB (approximately 9% for $\mathcal{AK}$ and 3% for $\mathcal{AK}+$). Even though CCSB requires more samples for feasibility checking, these samples provide an accurate estimation of the sample variance of the primary performance measures (recall that CCSB uses $n_0 = n^*(\alpha/k)$ while $\mathcal{AK}$ and $\mathcal{AK}+$ use $n_0 = 20$ to estimate the sample variances), and they are still used at the second stage for selection of the best. It is also interesting to note that procedure $\mathcal{AK}$ allocates more replications to infeasible solutions (i.e., solutions 4 and 5)

Table 2 Summary of Single-Constraint CCSB Formulation When $k = 5$

Solution	1	2	3	4	5	Total
			Equal variance			
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.13×10^4
FP	0.992	0.992	0.988	0.015	0.008	—
SP	0.969	0.666	0.627	0.015	0.008	—
TSS	6.09×10^3	5.58×10^3	5.53×10^3	3.98×10^3	3.97×10^3	2.52×10^4
PS	0.950	0.013	0.014	0.015	0.008	—
			Increasing variance			
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.97×10^3	3.98×10^3	2.13×10^4
FP	0.985	0.989	0.989	0.011	0.009	—
SP	0.964	0.729	0.815	0.011	0.009	—
TSS	7.39×10^3	6.29×10^3	6.91×10^3	3.97×10^3	3.97×10^3	2.90×10^4
PS	0.948	0.020	0.012	0.011	0.009	—
			Decreasing variance			
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.97×10^3	3.98×10^3	2.13×10^4
FP	0.995	0.985	0.988	0.008	0.011	—
SP	0.973	0.523	0.443	0.008	0.011	—
TSS	5.53×10^3	5.25×10^3	4.98×10^3	3.97×10^3	3.98×10^3	2.37×10^4
PS	0.956	0.015	0.010	0.008	0.011	—

Table 3 Summary of Single-Constraint ECSB Formulation Using $\mathscr{AK}$ and $\mathscr{AK}+$ When $k = 5$

Solution	1	2	3	4	5	Total
			Procedure $\mathscr{AK}$, without batching			
FSS	2.83×10^3	2.88×10^3	2.83×10^3	3.50×10^3	3.54×10^3	1.56×10^4
FP	0.999	0.995	0.996	0.128	0.129	—
SP	0.998	0.977	0.985	0.128	0.129	—
TSS	7.22×10^3	6.07×10^3	6.23×10^3	4.00×10^3	3.96×10^3	2.75×10^4
PS	0.746	0.004	0.006	0.116	0.128	—
			Procedure $\mathscr{AK}$, with batching, batch size $= 10$			
FSS	2.96×10^3	2.83×10^3	2.87×10^3	3.51×10^3	3.60×10^3	1.58×10^4
FP	0.997	0.997	0.993	0.009	0.010	—
SP	0.996	0.972	0.973	0.009	0.010	—
TSS	8.22×10^3	6.77×10^3	6.89×10^3	3.53×10^3	3.61×10^3	2.90×10^4
PS	0.963	0.012	0.006	0.009	0.010	—
			Procedure $\mathscr{AK}+$, with batching, batch size $= 10$			
TSS	7.53×10^3	6.42×10^3	6.21×10^3	2.84×10^3	2.92×10^3	2.59×10^4
PS	0.946	0.014	0.011	0.012	0.017	—

Table 4 Optimal $(m_\beta(n), n^*(\beta))$ with Multiple Constraints

	$k = 5$	$k = 25$	$k = 101$
$m_\beta(n)$	472	626	799
$n^*(\beta)$	5,271	6,989	8,918

and requires more replications to select the best from surviving solutions.

The slippage configuration of means for the secondary performance measure (i.e., setting $c_1 = 1$ and $c_2 = 0$ in Equation (31)) indicates that the feasibility checking is difficult in general. However, in the nonslippage configuration, the probabilistic condition p_i may be far away from the violation probability bound γ. Therefore, we should expect $\mathscr{AK}$ to require fewer samples to complete feasibility checking because it estimates the sample variance of the secondary performance measure. However, as demonstrated by numerical examples in the online supplement (EC.1.3), we find that this benefit is usually offset by the inefficiency of $\mathscr{AK}$ for selection of the best because of its poor estimation of sample variance of the primary performance measure, unless the configuration is such that optimality checking is much easier than feasibility checking.

6.2. Examples with Multiple Secondary Performance Measures

We next consider the situation where there are multiple secondary performance measures in CCSB problems, which may be formulated as a joint chance-constrained problem or a multiple individual chance-constrained problem.

6.2.1. Configurations of Test Examples with Multiple Constraints and Main Results. For convenience, we assume that $Y_i^{(s)} \sim N(\nu_i^{(s)}, 1)$ are mutually independent and also independent of X_i, for $s = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, k$. For the joint-constraint formulation in Equation (20), we set

$$\nu_i^{(s)} = \begin{cases} -\Phi^{-1}(1-(1-\gamma+\delta_\gamma)^{1/m}), & i=1,2,\ldots,b; \\ -\Phi^{-1}(1-(1-\gamma)^{1/m}), & i=b+1,b+2,\ldots,k, \end{cases}$$

for all $s = 1, 2, \ldots, m$. For the multiple constraints formulation in Equation (22), we set $\gamma_s = \gamma = 0.1$, $\delta_{\gamma_s} = \delta_\gamma = 0.02$ and

$$\nu_i^{(s)} = \begin{cases} -\Phi^{-1}(\gamma-\delta_\gamma), & i=1,2,\ldots,b \\ & \quad \text{and } s=1,2,\ldots,m; \\ -\Phi^{-1}(\gamma-\delta_\gamma), & i=b+1,b+2,\ldots,\lfloor (b+k)/2 \rfloor \\ & \quad \text{and } s=1,2,\ldots,m-1; \\ -\Phi^{-1}(\gamma), & i=b+1,b+2,\ldots,\lfloor (b+k)/2 \rfloor \\ & \quad \text{and } s=m; \\ -\Phi^{-1}(\gamma), & i=\lfloor (b+k)/2 \rfloor+1, \lfloor (b+k)/2 \rfloor \\ & \quad +2,\ldots,k \text{ and } s=1,2,\ldots,m. \end{cases}$$

Under this setting, solutions $1, 2, \ldots, b$ are feasible, solutions $b+1, b+2, \ldots, \lfloor (b+k)/2 \rfloor$ violate only one constraint, and solutions $\lfloor (b+k)/2 \rfloor + 1, \lfloor (b+k)/2 \rfloor + 2, \ldots, k$ violate all constraints. For both formulations, we let $m = 5$, and all other parameter settings are the same as the problems reported in §6.1.1.

For the joint-constraint formulation, the optimal $(m_\beta(n), n^*(\beta))$ is the same as in Table 1. For the multiple constraints formulation, we choose $\alpha_1 = \alpha_2 = \alpha/(m+k-1)$ and provide the optimal $m_\beta(n)$ and $n^*(\beta)$ in Table 4.

The results for the joint-constraint and multiple-constraints formulations when $k = 5$ are reported in Tables 5 and 6, respectively. From Tables 5 and 6, we can draw similar conclusions as from Table 2. The procedures deliver the required probability of correct selection and the sequential feasibility test delivers a positive benefit at no cost: about 4.1% and 5.0% of savings for the joint-constraint and multiple-constraints formulations, respectively. We report the results of both formulations for the cases, where $k = 25$ and $k = 101$ in Table EC.3 in the online supplement, from which we find that the conclusions hold for these cases.

6.2.2. Comparison with Procedures $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$. To deal with problems having multiple secondary performance measures (i.e., the formulation in Equation (22)), we compare against procedure $\mathscr{AK}$ with feasibility-checking being replaced by either procedures $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ or $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ in Batur and Kim (2010) (for simplicity, we call procedures $\mathscr{AK} + \mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{AK} + \mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ as $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$, respectively, in this paper).

We now consider the example in §6.2.1 with $m = 5$ multiple secondary performance measures. We set $\alpha_1 = \alpha_2 = \alpha/2 = 0.025$, $q_s = q = \gamma - \delta_\gamma/2 = 0.09$, and $\epsilon_s = \epsilon = \delta_\gamma/2 = 0.01$ for $s = 1, 2, \ldots, m$, and the initial-stage sample size $n_0 = 20$. Because all solutions are simulated independently, we let $\beta = (1-(1-\alpha_1)^{1/k})/m$ in procedure $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and γ be the solution to $(1-\gamma)^k + (1-m\gamma)^k = 2-\alpha_1$ in procedure $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$. (Note that here β and γ are the notations used in describing the procedures in Batur and Kim 2010.) All other parameters are set the same as in §6.2.1. To make $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ yield the desired PCS, we only used batched output data with batch size equal to 10. Table 7 reports the detailed results for the case, where $k = 5$ under the equal-variance configuration.

Comparing these results with Table 6, we have the following findings: First, procedures $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ tend to be more conservative in terms of PCS than Procedure CCSB. This is because the "multiplicative rule" in $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ makes the feasibility checking more conservative than the "additive rule" in Procedure CCSB. Second, procedure $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ is more efficient than $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$, which is consistent with the conclusion in Batur and Kim (2010). Third, the total number of samples needed both for feasibility checking and selection of the best by $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$ are more than those needed by Procedure CCSB (approximately 17% for $\mathscr{F}_{\mathscr{B}}^{\mathscr{I}}$ and 8% for $\mathscr{F}_{\mathscr{A}}^{\mathscr{I}}$).

6.3. An Example Where the Best Solution Is Not Clearly Feasible

As mentioned in §3, our feasibility test does not have an explicit control of its power when the solution is feasible but not clearly feasible. In this section, we consider an example where the best feasible solution is not clearly feasible. Specifically, we consider the example for a single-constraint CCSB formulation and set the parameters the same as in §6.1.1 except that

$$\mu_i = \begin{cases} 2\delta, & i=1; \\ \delta, & i=2; \\ 0, & i=3,4,\ldots,b; \\ i\delta, & i=b+1,b+2,\ldots,k, \end{cases}$$

and

$$\nu_i = \begin{cases} -\Phi^{-1}(\gamma-0.5\delta_\gamma), & i=1; \\ -\Phi^{-1}(\gamma-\delta_\gamma), & i=2,3,\ldots,b; \\ -\Phi^{-1}(\gamma), & i=b+1,b+2,\ldots,k. \end{cases}$$

In this example, solution 1 is the best feasible solution but is not clearly feasible, solution 2 is the best solution among all *clearly feasible* solutions, and both are acceptable. We report the results for the case where $k = 5$ in Table 8, where PS in the last column represents the probability of good selection (PGS, i.e., the

Table 5 **Summary of Joint-Constraint Formulation When $k = 5$**

Solution	1	2	3	4	5	Total
Equal variance						
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.99×10^3	2.13×10^4
FP	0.991	0.991	0.986	0.006	0.009	—
SP	0.975	0.635	0.625	0.006	0.009	—
TSS	6.11×10^3	5.56×10^3	5.56×10^3	3.98×10^3	3.99×10^3	2.52×10^4
PS	0.961	0.010	0.014	0.006	0.009	—
Increasing variance						
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.13×10^4
FP	0.991	0.990	0.995	0.012	0.004	—
SP	0.977	0.759	0.848	0.012	0.004	—
TSS	7.57×10^3	6.30×10^3	7.12×10^3	3.98×10^3	3.98×10^3	2.90×10^4
PS	0.954	0.015	0.015	0.012	0.004	—
Decreasing variance						
FSS	4.43×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.13×10^4
FP	0.988	0.992	0.982	0.008	0.009	—
SP	0.969	0.517	0.413	0.008	0.009	—
TSS	5.41×10^3	5.18×10^3	4.90×10^3	3.97×10^3	3.97×10^3	2.34×10^4
PS	0.954	0.011	0.018	0.008	0.009	—

Table 6 **Summary of Multiple-Constraint Formulation When $k = 5$**

Solution	1	2	3	4	5	Total
Equal variance						
FSS	5.27×10^3	5.27×10^3	5.27×10^3	4.73×10^3	4.49×10^3	2.50×10^4
FP	0.970	0.972	0.971	0.008	0.000	—
SP	0.961	0.620	0.594	0.008	0.000	—
TSS	7.00×10^3	6.51×10^3	6.50×10^3	4.72×10^3	4.48×10^3	2.92×10^4
PS	0.953	0.017	0.022	0.008	0.000	—
Increasing variance						
FSS	5.27×10^3	5.27×10^3	5.27×10^3	4.72×10^3	4.49×10^3	2.50×10^4
FP	0.976	0.965	0.985	0.003	0.000	—
SP	0.973	0.753	0.838	0.003	0.000	—
TSS	8.58×10^3	7.40×10^3	8.09×10^3	4.72×10^3	4.49×10^3	3.33×10^4
PS	0.963	0.019	0.015	0.003	0.000	—
Decreasing variance						
FSS	5.27×10^3	5.27×10^3	5.27×10^3	4.74×10^3	4.49×10^3	2.50×10^4
FP	0.980	0.986	0.964	0.005	0.000	—
SP	0.976	0.473	0.392	0.005	0.000	—
TSS	6.28×10^3	6.03×10^3	5.85×10^3	4.74×10^3	4.49×10^3	2.74×10^4
PS	0.968	0.010	0.01	0.005	0.000	—

Table 7 **Summary of Multiple-Constraint ECSB Formulation Using $\mathcal{F}^{\mathcal{I}}_{\mathcal{B}}$ and $\mathcal{F}^{\mathcal{I}}_{\mathcal{A}}$ When $k = 5$**

Solution	1	2	3	4	5	Total
Procedure $\mathcal{F}^{\mathcal{I}}_{\mathcal{B}}$, with batching, batch size $= 10$						
FSS	7.00×10^3	6.97×10^3	6.92×10^3	5.27×10^3	3.00×10^3	2.92×10^4
FP	0.997	0.993	0.995	0.001	0.000	—
SP	0.995	0.649	0.654	0.001	0.000	—
TSS	9.21×10^3	8.48×10^3	8.36×10^3	5.27×10^3	3.00×10^3	3.43×10^4
PS	0.986	0.007	0.006	0.001	0.000	—
Procedure $\mathcal{F}^{\mathcal{I}}_{\mathcal{A}}$, with batching, batch size $= 10$						
FSS	7.30×10^3	7.24×10^3	7.24×10^3	5.43×10^3	1.14×10^3	2.83×10^4
FP	0.998	0.997	0.996	0.000	0.000	—
SP	0.997	0.536	0.555	0.000	0.000	—
TSS	8.67×10^3	8.11×10^3	8.22×10^3	5.43×10^3	1.14×10^3	3.16×10^4
PS	0.983	0.009	0.008	0.000	0.000	—

Table 8 Summary of CCSB Formulation When $k = 5$ with Multiple Acceptable Solutions

Solution	1	2	3	4	5	Total
			Equal variance			
FSS	4.34×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.12×10^4
FP	0.466	0.993	0.992	0.010	0.008	—
SP	0.456	0.813	0.397	0.010	0.008	—
TSS	4.80×10^3	5.54×10^3	5.09×10^3	3.96×10^3	3.95×10^3	2.33×10^4
PS	0.449	0.526	0.007	0.010	0.008	0.975
			Increasing variance			
FSS	4.34×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.12×10^4
FP	0.467	0.990	0.993	0.004	0.009	—
SP	0.459	0.820	0.388	0.004	0.009	—
TSS	4.86×10^3	5.55×10^3	5.05×10^3	3.95×10^3	3.95×10^3	2.34×10^4
PS	0.456	0.524	0.007	0.004	0.009	0.980
			Decreasing variance			
FSS	4.34×10^3	4.43×10^3	4.43×10^3	3.98×10^3	3.98×10^3	2.12×10^4
FP	0.440	0.989	0.988	0.005	0.012	—
SP	0.436	0.812	0.412	0.005	0.012	—
TSS	4.83×10^3	5.57×10^3	5.06×10^3	3.92×10^3	3.92×10^3	2.33×10^4
PS	0.431	0.542	0.010	0.005	0.012	0.973

probability of selecting either solution 1 or 2). In contrast to the results in Table 2, our procedure declares solution 1 infeasible with a probability that is larger than α_2 (which is illustrated as point A in Figure 1). However, the procedure selects an acceptable solution with a probability that is at least $1-\alpha$.

6.4. The Newsvendor Problem

The last example tested in this paper is a multiple-product newsvendor problem with correlated log-normal demands subject to some service level constraints. Suppose that there are three products with random demand $D = (D_1, D_2, D_3)^T$, where $\log(D) \sim N(\mu, \Sigma)$. The order quantity of the products is $x = (x_1, x_2, x_3)^T \in \mathbb{Z}^3_+$. Let c_{si} and c_{oi} denote the per unit shortage and overage costs for product i, $i = 1, 2, 3$. Then the expected total cost can be written as

$$\pi(x) = \mathrm{E}\left\{\sum_{i=1}^{3}[c_{si}(D_i - x_i)^+ + c_{oi}(x_i - D_i)^+]\right\},$$

where $y^+ = \max\{y, 0\}$. By Equations (20) and (22), we can formulate this problem as

$$\begin{aligned} \max_x \quad & \{-\pi(x)\} = -\sum_{i=1}^{3}[c_{si}\mathrm{E}(D_i - x_i)^+ + c_{oi}\mathrm{E}(x_i - D_i)^+] \\ \text{s.t.} \quad & \Pr\{D_i \le x_i,\ i = 1, 2, 3\} \ge 1 - \gamma, \end{aligned} \tag{32}$$

as a joint CCSB problem, and

$$\begin{aligned} \max_x \quad & \{-\pi(x)\} = -\sum_{i=1}^{3}[c_{si}\mathrm{E}(D_i - x_i)^+ + c_{oi}\mathrm{E}(x_i - D_i)^+] \\ \text{s.t.} \quad & \Pr\{D_i \le x_i\} \ge 1 - \gamma, \quad \text{for } i = 1, 2, 3, \end{aligned} \tag{33}$$

as a multiconstraint CCSB problem.

Let $\mu = (2.0, 2.5, 3.0)^T$ and

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix},$$

where $(\sigma_1, \sigma_2, \sigma_3) = (1.0, 1.1, 1.2)$, $c_{si} = 3$, and $c_{oi} = 1$ for $i = 1, 2, 3$. Suppose the order quantities for products 1, 2, 3 take values from the following sets: $x_1 \in \{15, 30, 50, 55\}$, $x_2 \in \{25, 75, 95, 100\}$, and $x_3 \in \{45, 115, 180, 210\}$ under the joint-constraint formulation while $x_1 \in \{15, 25, 30, 40\}$, $x_2 \in \{25, 55, 60, 75\}$, and $x_3 \in \{45, 90, 110, 145\}$ under the multiple constraints formulation. Then there are $k = 64$ solutions for each formulation.

We set the violation probability $\gamma = 0.1$, the tolerance level $\delta_\gamma = 0.02$, indifference-zone parameter $\delta = 1$, and total error allowance $\alpha = 0.05$. The optimal $m_\beta(n)$ and $n^*(\beta)$ are shown in Table 9.

Given the values of parameters and distribution of D, we can compute exact values of the objective functions and probability constraints in Formulations (32) and (33). For the joint constraint formulation, there are 12 feasible solutions, eight of which are clearly feasible, and the best feasible solution is $x_{b1} = (50, 75, 180)$ with the expected cost 263.0 and the benchmark solution (i.e., the best among all clearly feasible solutions) is $x_{g1} = (50, 95, 180)$ with the expected cost 279.9. For the multiple constraints case, there are 12 feasible solutions, four of which are clearly feasible, and the best feasible solution is $x_{b2} = (30, 55, 110)$ with the expected cost 176.6 and the benchmark solution is $x_{g2} = (40, 60, 110)$ with the expected cost 187.6. In both formulations, there are multiple acceptable solutions. In Tables EC.8 and EC.9 in the online supplement, we list the information for all solutions for both formulations.

Table 9 **Optimal $(m_\beta(n), n^*(\beta))$ for the Newsvendor Problem**

	Joint constraint	Multiple constraints
$\alpha_1 = \alpha_2$	α/k	$\alpha/(k+m-1)$
$m_\beta(n)$	732	737
$n^*(\beta)$	8,171	8,226

Table 10 **Summary for the Newsvendor Problem**

Configurations	Joint constraint	Multiple constraints
FSAS	2.81×10^5	3.09×10^5
Savings (%)	46.7	41.3
TSAS	3.88×10^5	6.58×10^5
PGS	0.999	0.999

In Table 10, we report the average first-stage sample size (FSS), the percentage of savings by using a sequential feasibility test (Saving%), the average total sample size (TSS), and the PGS, over 1,000 independent macroreplications. From the results we conclude that the procedures we propose can correctly select an acceptable solution. Furthermore, as the example is not in the slippage configuration, the sequential feasibility test demonstrates a substantial amount of savings (over 40% in both formulations) compared with the fixed-sample test.

7. Conclusions

In this paper we study CCSB problems where we select the best solution with the maximum or minimum expected value of the primary performance measure under the requirement that the secondary performance measures satisfy certain probabilistic constraints. We propose various two-stage procedures for CCSB problems. Specifically, in the first stage of the procedures, we design a fixed-sample feasibility test and a sequential feasibility test by transforming the probabilistic constraints to hypothesis tests on Bernoulli random variables. These tests not only select feasible solutions with the required Type I and II errors but also allow us to use the first-stage samples for selection in the second stage. In the second stage of the procedures, we use the $\mathcal{KN}$ procedure to sequentially select the best solution from the sample feasible solutions. We prove that our procedures can deliver the required PCS under the indifference-zone framework. To handle CCSB problems with multiple secondary performance measures, we propose two formulations: joint constraint and multiple constraints. We design two-stage procedures for both formulations and prove their statistical validities under the indifference-zone framework. We test our procedures using a number of examples, and the procedures can deliver the desired performances.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/ijoc.2014.0628.

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