Online Supplement

This support material is intended for an online supplement. It primarily contains a summary the MDE/DDE approach used to approximate time-dependent nodal-size moments and departure-count moments for non-stationary Markovian queues, and the specific results used in this paper. As such, much of the material simply documents an application of known results. The exceptions are Appendices B and D. In Appendix B we make some modifications to known approximations for the $Ph_t/Ph_t/s/c$ queue to approximate a $Ph_t/Ph_t/s/\infty$ node; these modifications are important for our work, but not foundational contributions. In Appendix D we list the network parameters of seven networks examined in Section 4.

A Calculating Departure Count Moments for the $MAP_t/Ph/\infty$ Node

Recall that to develop our traffic flow approximations we used numerically exact results for two-node $MAP_t/Ph/\infty \to \cdot/Ph/\infty$ networks. A closed system of infinite-server MDEs is provided in [28] and recounted here. For our work we need the DDEs for the $MAP_t/Ph/\infty$ node, which we derive here. We use the notation in Section 2.2.

Utilizing the techniques of Taaffe and co-authors (e.g., see [23, 27, 31] and related papers), we can derive the p^{th} DDE, namely

$$\begin{aligned}\frac{d}{d\tau}\mathbb{E}\{D_t^p(t+\tau)\} \;=\; & \sum_{i=1}^{m_b}\mu_i f_i\left[\sum_{j=1}^{m_a}\left(\mathbb{E}\left\{N_i(t+\tau), J(t+\tau)=j\right\}\right.\right.\\ &\left.\left.+\sum_{q=1}^{p-1}\binom{p}{q}\mathbb{E}\left\{D_t^q(t+\tau)N_i(t+\tau), J(t+\tau)=j\right\}\right)\right], \quad \text{(A.1)}\end{aligned}$$

for $p = 1, 2, \ldots$, where the moments on the right-hand side of (A.1) are the *partial moments*[1] of the random variables with respect to arrival phase j, for $j = 1, 2, \ldots, m_a$. Notice the

[1] A partial moment is evaluated only over those elements that are in a particular subset of the sample space [36]. That is, for random variable X, $\mathbb{E}\{X, X \in U\} = \sum_{x\in U} x \Pr\{X = x, X \in U\}$, for sample space subset U.

derivatives in the DDEs are with respect to the interval length τ, while the terms on the right-hand side of the DDEs are evaluated at time $t+\tau$.

Since $\mathrm{Var}\{D_t(t+\tau)\} = \mathbb{E}\{D_t^2(t+\tau)\} - (\mathbb{E}\{D_t(t+\tau)\})^2$, we need only numerically integrate (A.1) over $[t, t+\tau)$ for $p = 1, 2$ to evaluate the variance. However, this requires knowing values for $\mathbb{E}\{N_i(t+\tau), J(t+\tau)=j\}$ and $\mathbb{E}\{D_t(t+\tau)N_i(t+\tau), J(t+\tau)=j\}$, for $j = 1, 2, \ldots, m_a$ and $i = 1, 2, \ldots, m_b$; we present differential equations for each now.

First we express the p^{th} partial-MDEs (PMDEs) for the upstream node, introduced in [28] and recounted here:

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left\{(N_i(t))^p, J(t)=j\right\} = \\
&\quad -\upsilon_j(t)\mathbb{E}\left\{(N_i(t))^p, J(t)=j\right\} \\
&\quad + \sum_{h=1}^{m_a} \upsilon_h(t)\Bigg[\Big(a_{hj}(t) + d_h(t)\alpha_{hj}(t)\Big)\mathbb{E}\left\{(N_i(t))^p, J(t)=h\right\} \\
&\qquad + \beta_i d_h(t)\alpha_{hj}(t)\sum_{q=1}^{p-1}\binom{p}{q}\mathbb{E}\left\{(N_i(t))^q, J(t)=h\right\}\Bigg] \\
&\quad + \mu_i \sum_{q=0}^{p-1}\binom{p}{q}(-1)^{p-q}\,\mathbb{E}\left\{(N_i(t))^{q+1}, J(t)=j\right\} \\
&\quad + \sum_{k=1,k\neq i}^{m_b} \mu_k b_{ki}\sum_{q=0}^{p-1}\binom{p}{q}\mathbb{E}\left\{(N_i(t))^q N_k(t), J(t)=j\right\},
\end{aligned} \tag{A.2}$$

for $p = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, m_a$, and $i = 1, 2, \ldots, m_b$. Unlike the DDEs, the derivatives in the MDEs are with respect to the current time t, and the right-hand terms in the MDEs are evaluated at t as well.

We also can derive the partial cross-system-departure differential equations (PCSDDEs) needed to close (A.1), namely

$$\begin{aligned}
&\frac{d}{d\tau}\mathbb{E}\left\{D_t(t+\tau)N_i(t+\tau), J(t+\tau)=j\right\} = \\
&\quad -\Big(\upsilon_j(t+\tau)+\mu_i\Big)\mathbb{E}\left\{D_t(t+\tau)N_i(t+\tau), J(t+\tau)=j\right\} \\
&\quad +\sum_{h=1}^{m_a}\upsilon_h(t+\tau)\Bigg[\beta_i d_h(t+\tau)\alpha_{hj}(t+\tau)\mathbb{E}\left\{D_t(t+\tau), J(t+\tau)=h\right\} \\
&\qquad +\Big(a_{hj}(t+\tau)+d_h(t+\tau)\alpha_{hj}(t+\tau)\Big)\mathbb{E}\left\{D_t(t+\tau)N_i(t+\tau), J(t+\tau)=h\right\}\Bigg] \\
&\quad +\mu_i f_i\Big(\mathbb{E}\left\{(N_i(t+\tau))^2, J(t+\tau)=j\right\}-\mathbb{E}\left\{N_i(t+\tau), J(t+\tau)=j\right\}\Big) \\
&\quad +\sum_{k=1,k\neq i}^{m_b}\mu_k\Bigg(b_{ki}\mathbb{E}\left\{D_t(t+\tau)N_k(t+\tau), J(t+\tau)=j\right\} \\
&\qquad +f_k\mathbb{E}\left\{N_i(t+\tau)N_k(t+\tau), J(t+\tau)=j\right\}\Bigg),
\end{aligned} \tag{A.3}$$

for $j = 1, 2, \ldots, m_a$ and $i = 1, 2, \ldots, m_b$. Notice the right-hand sides of (A.2) and (A.3) include values for $\mathbb{E}\{N_i(t+\tau)N_k(t+\tau), J(t+\tau) = j\}$ and $\mathbb{E}\{D_t(t+\tau), J(t+\tau) = j\}$, for $j = 1, 2, \ldots, m_a$ and i, $k = 1, 2, \ldots, m_b$ ($i \neq k$). We find the partial product-moment differential equations (PPMDEs) in [28]:

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left\{N_i(t)N_k(t), J(t)=j\right\} = \\
&\quad -\Big(\upsilon_j(t)+\mu_i+\mu_k\Big)\mathbb{E}\left\{N_i(t)N_k(t), J(t)=j\right\} \\
&\quad +\sum_{h=1}^{m_a}\upsilon_h(t)\Bigg[\Big(a_{hj}(t)+d_h(t)\alpha_{hj}(t)\Big)\mathbb{E}\left\{N_i(t)N_k(t), J(t)=h\right\} \\
&\qquad +d_h(t)\alpha_{hj}(t)\Big(\beta_i\mathbb{E}\left\{N_k(t), J(t)=h\right\}+\beta_k\mathbb{E}\left\{N_i(t), J(t)=h\right\}\Big)\Bigg] \\
&\quad +\mu_k b_{ki}\Big(\mathbb{E}\left\{(N_k(t))^2, J(t)=j\right\}-\mathbb{E}\left\{N_k(t), J(t)=j\right\}\Big) \\
&\quad +\mu_i b_{ik}\Big(\mathbb{E}\left\{(N_i(t))^2, J(t)=j\right\}-\mathbb{E}\left\{N_i(t), J(t)=j\right\}\Big) \\
&\quad +\sum_{r=1,r\neq i,k}^{m_b}\mu_r\Big(b_{ri}\mathbb{E}\left\{N_r(t)N_k(t), J(t)=j\right\}+b_{rk}\mathbb{E}\left\{N_r(t)N_i(t), J(t)=j\right\}\Big),
\end{aligned} \tag{A.4}$$

for $j = 1, 2, \ldots, m_a$ and i, $k = 1, 2, \ldots, m_b$ ($i \neq k$).

Finally, we close the system of MDEs and DDEs with the first partial departure-moment differential equations (PDDEs):

$$\begin{aligned}
&\frac{d}{d\tau}\mathbb{E}\left\{D_t(t+\tau), J(t+\tau)=j\right\} = \\
&\quad -\upsilon_j(t+\tau)\mathbb{E}\left\{D_t(t+\tau), J(t+\tau)=j\right\} \\
&\quad +\sum_{h=1}^{m_a}\upsilon_h(t+\tau)\Big[a_{hj}(t+\tau)+d_h(t+\tau)\alpha_{hj}(t+\tau)\Big]\mathbb{E}\left\{D_t(t+\tau), J(t+\tau)=h\right\} \\
&\quad +\sum_{i=1}^{m_b}\mu_i f_i \mathbb{E}\left\{N_i(t+\tau), J(t+\tau)=j\right\},
\end{aligned} \tag{A.5}$$

for $j = 1, 2, \ldots, m_a$. Summing (A.5) across arrival phases $j = 1, 2, \ldots, m_a$ yields (A.1), for $p = 1$.

Thus, numerically integrating (A.2)–(A.5) over interval $[t, t+\tau)$ yields values for $\mathbb{E}\{D_t(t+\tau)\}$ and $\mathrm{Var}\{D_t(t+\tau)\}$, for $t \geq 0$ and $\tau > 0$. Additionally, numerically integrating these equations over $[t, t+2\tau)$ and $[t+\tau, t+2\tau)$ yields values for $\mathrm{Cov}\{D_t(t+\tau), D_{t+\tau}(t+2\tau)\} = 1/2[\mathrm{Var}\{D_t(t+2\tau)\} - \mathrm{Var}\{D_{t+\tau}(t+2\tau)\} - \mathrm{Var}\{D_t(t+\tau)\}]$, and

$$\mathrm{Corr}\{D_t(t+\tau), D_{t+\tau}(t+2\tau)\} = \frac{\mathrm{Cov}\{D_t(t+\tau), D_{t+\tau}(t+2\tau)\}}{\sqrt{\mathrm{Var}\{D_t(t+\tau)\}\mathrm{Var}\{D_{t+\tau}(t+2\tau)\}}}.$$

B The $Ph_t/Ph/s/\infty$ Node: MDEs, DDEs, and Closure Techniques

In this section we introduce the system of MDEs and DDEs for the $Ph_t/Ph/s/\infty$ node, where $s < \infty$ (described as "finite-server, infinite-buffer"), and propose techniques for closing this system using approximations for unknown terms. We recount the simpler case of finite-buffer nodes (i.e., $Ph_t/Ph/s/c$ nodes, for $c < \infty$), using this case to introduce the Pólya-Eggenberger (PE) distribution in Appendix B.1. In Appendix B.2 we derive the MDEs and DDEs for the finite-server, infinite-buffer node, and define our technique for employing the PE as surrogate in the infinite-buffer case in Appendix B.3.

B.1 Employing the PE Distribution in a Finite-Buffer Model

B.1.1 Closure Techniques for Finite-Server, Finite-Buffer MDEs

In calculating moments for the expected number of entities in the nonstationary, finite-buffer $Ph_t/M_t/s/c$ queueing node (i.e., $s \leq c < \infty$), Ong and Taaffe [31] derive partial moment differential equations (PMDEs), which give the instantaneous rate of change of the joint expectation of powers of the number of entities in the system and the current phase of the next entity arrival. The left-hand side of these equations is $d/dt\,[\mathbb{E}\{N^p(t),\ J_a(t)=j\}]$, where $N^p(t)$ represents the p^{th} power of the node size ($p = 0, 1, 2, \ldots$), while $J_a(t)$ represents the current phase of the next entity arrival ($j = 1, 2, \ldots, m_a$, where m_a is the order of the Ph interarrival distribution), at time $t \geq 0$. Notice that M_t service process indicates $m_b = 1$.

The right-hand sides of the Ong and Taaffe PMDEs include both the partial moments mentioned above and the following $3m_a$ joint state probabilities: $\Pr\{N(t) = s-1,\ J_a(t) = j\}$, $\Pr\{N(t) = s,\ J_a(t) = j\}$, $\Pr\{N(t) = c,\ J_a(t) = j\}$, for arrival phases $j = 1, 2, \ldots, m_a$. Since approximations are needed to evaluate these joint state probabilities, the PMDEs are *pseudo-closed* by a surrogate distribution to approximate these values [27]. The authors employ the following steps for $s \leq c < \infty$ nodes; we utilize this same algorithm when $s < c = \infty$ in our work.

1. At time t, we have system moments from numerical integration of the finite-server MDEs and DDEs; see Appendix B.2.

2. Choose parameters of the surrogate system-size distribution to match the moments in Step 1 (e.g., (B.2) and (B.3) below); see Appendix B.1.2.

3. Plug specified parameters into the probability mass function for the surrogate (e.g., (B.1)) to calculate approximations for the state probabilities necessary to close the MDEs and DDEs.

4. Numerically integrate MDEs and DDEs to time $t + \tau$. Go to Step 1.

B.1.2 Surrogate Distributions: The Pólya-Eggenberger

Clark [2] was the first to propose using the Pólya-Eggenberger (PE) as a surrogate; the PE distribution is described by the number of times in a sequence of $n \geq 1$ trials that a blue ball is selected from an urn that initially has b blue balls and r red balls in it, such that after every pull, the chosen ball is replaced and u balls of the same color as the most recently pulled ball are added to the urn. We reparameterize the distribution, defining $p \equiv b/(b+r)$ and $\alpha \equiv u/(b+r)$. Letting X denote the random variable representing the number of pulls that were blue, we have

$$\Pr\{X = k\} = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(p + i\alpha) \prod_{j=0}^{n-k-1}((1-p) + j\alpha)}{\prod_{m=0}^{n-1}(1 + m\alpha)}, \tag{B.1}$$

for $k = 0, 1, \ldots, n$.

By deriving equations for $\mathbb{E}\{X\}$ and $\mathbb{E}\{X^2\}$ in terms of p and α, Clark [2] describes a technique in which the PE distribution serves as a surrogate for state probabilities in a nonstationary $M_t/M_t/s$ system. He shows that

$$p = \mathbb{E}\{X\}/n. \tag{B.2}$$

Solving further for the incremental probability α, Clark defines the quantity α^*:

$$\alpha^* \equiv \frac{\mathbb{E}\{X\}^2 + \mathbb{E}\{X\}(1-p) - \mathbb{E}\{X^2\}}{\mathbb{E}\{X^2\} - n\mathbb{E}\{X\}}. \tag{B.3}$$

If $\alpha^* \leq -\min\{p,\ 1-p\}/(n-1)$, then the PE parameter $\alpha = -\min\{p,\ 1-p\}/(n-1) + 10^{-4}$. Otherwise, $\alpha = \alpha^*$.

Ong and Taaffe [31] use a partitioning technique to solve the PMDEs. For each p^{th} partial system moment, they define two joint partial moments,

$$E_{1,j}^{(p)} \equiv \mathbb{E}\left\{N^p(t),\ N(t) \leq s-1,\ J_a(t) = j\right\},\ E_{2,j}^{(p)} \equiv \mathbb{E}\left\{N^p(t),\ s \leq N(t) \leq c,\ J_a(t) = j\right\}, \tag{B.4}$$

for $p = 0, 1, 2, \ldots$, such that $E^{(0)}_{1,j} \equiv \Pr\{N(t) \le s-1,\ J_a(t) = j\}$, $E^{(0)}_{2,j} \equiv \Pr\{s \le N(t) \le c,\ J_a(t) = j\}$, for $j = 1, 2, \ldots, m_a$. To complete the approximations, the authors match the respective pair of PE parameters necessary to approximate the joint state probabilities in the PMDEs to the first and second conditional partial moments in each space, $E^{(1)}_{v,j}/E^{(0)}_{v,j}$ and $E^{(2)}_{v,j}/E^{(0)}_{v,j}$, for $v = 1, 2$ and $j = 1, 2, \ldots, m_a$. Plugging the current values of the conditional partial moments into (B.2) and (B.3)—and further plugging these parameters into (B.1)—they define the approximation

$$\Pr\{N(t) = s-1,\ J_a(t) = j\} \doteq E^{(0)}_{1,j} \cdot \Pr\{X = s-1\},$$

where X is a PE-distributed random variable with support on $0, 1, \ldots, s-1$ (in this context, "$\doteq$" means "is approximated by"). Similarly,

$$\begin{aligned} \Pr\{N(t) = s,\ J_a(t) = j\} &\doteq E^{(0)}_{2,j} \cdot \Pr\{Y = 0\}, \\ \Pr\{N(t) = c,\ J_a(t) = j\} &\doteq E^{(0)}_{2,j} \cdot \Pr\{Y = c-s\}, \end{aligned}$$

where Y is a shifted PE-distributed random variable with support on $0, 1, \ldots, c-s$. Ong and Taaffe [31] test these approximations versus the exact solutions for the KFEs from the capacitated system and typically find the results satisfactory.

B.2 The MDEs and DDEs for the Finite-Server, Infinite-Buffer Node

In this section we derive the MDEs and DDEs for the finite-server, infinite-buffer node; thus, it is the finite-server equivalent of the model in Appendix A. The equations here are analogous to those in [36] for the $Ph_t/Ph_t/s/c$ node when $s \le c < \infty$; we use the finite-buffer MDEs to validate our equations.

Recall from Section 2.3 that the finite-server MDEs and DDEs are not closed, as they have several terms on their right-hand sides that require approximation. The main consequence is that we require an accurate closure approximation for state probabilities $\Pr\{N(t) = m,\ J_a(t) = j\}$, for $m = s-1,\ s$, and $j = 1, 2, \ldots, m_a$.

B.2.1 PMDEs for the $Ph_t/Ph/s/\infty$ Node

We provide the PMDEs for infinite-buffer models to confirm which probabilities we need to approximate. Notice that we drop the nodal superscript from all partial-node and departure moments for simplicity; the equations derived here can be utilized at any node $n = 1, 2, \ldots, z$.

We introduce the following notation for use in the PMDEs; these are defined for all $j = 1, 2, \ldots, m_a$, $i,\ k = 1, 2, \ldots, m_b$, and $v,\ p = 1, 2$.

- The state of the node at time t is $S(t) \equiv (N_1(t), N_2(t), \ldots, N_{m_b}(t), Q(t), J_a(t))$.
- We split the state space into $2m_a$ partitions, letting $\Omega_1^{(j)}$ denote the partition of all states such that $J_a(t) = j,\ \sum_{i=1}^{m_b} N_i(t) < s$, while $\Omega_2^{(j)}$ the partition of all states such that $J_a(t) = j,\ \sum_{i=1}^{m_b} N_i(t) = s$. Notice that if $S(t) \in \Omega_1^{(j)}$, then $Q(t) = 0$, while if $S(t) \in \Omega_2^{(j)}$, then $N(t) = s + Q(t)$, for $Q(t) \geq 0$. Shorthands for the partial moments in each partition are provided now:

$$\begin{aligned}
P_j^{(v)} &\equiv \Pr\left\{S(t) \in \Omega_v^{(j)}\right\}, \\
L_{i,j}^{(p)} &\equiv \mathbb{E}\left\{(N_i(t))^p,\ S(t) \in \Omega_1^{(j)}\right\}, \\
L_{i,k,j} &\equiv \mathbb{E}\left\{N_i(t)N_k(t),\ S(t) \in \Omega_1^{(j)}\right\}, \\
M_j^{(p)} &\equiv \mathbb{E}\left\{N^p(t),\ S(t) \in \Omega_2^{(j)}\right\}, \\
R_{i,j} &\equiv \mathbb{E}\left\{N_i(t)N(t),\ S(t) \in \Omega_2^{(j)}\right\}.
\end{aligned}$$

Notice we can recapture the notation for the partial-moments specified in (B.4) by defining

$$E_{v,j}^{(0)} \equiv P_j^{(v)}, \tag{B.5}$$

$$E_{1,j}^{(1)} \equiv \sum_{i=1}^{m_b} L_{i,j}^{(1)}, \tag{B.6}$$

$$E_{1,j}^{(2)} \equiv \sum_{i=1}^{m_b} \left[L_{i,j}^{(2)} + \left(\sum_{k=1,k\neq i}^{m_b} L_{i,k,j} \right) \right], \tag{B.7}$$

and

$$E_{2,j}^{(p)} \equiv M_j^{(p)}, \tag{B.8}$$

for $j = 1, 2, \ldots, m_a$, and v, $p = 1, 2$.

Recall that our focus is to identify those terms in the PMDEs that need to be approximated for us to calculate (numerically) the mean and variance of the number of entities at the node, where $\mathbb{E}\{N^p(t)\} = \sum_{j=1}^{m_a} \left(E_{1,j}^{(p)} + E_{2,j}^{(p)} \right)$, for $p = 1, 2$. Further recall that for a nonstationary Ph arrival process, the rows in initial probability matrix $\boldsymbol{\alpha}(t)$ are equal for all $t \geq 0$; without loss of generality we let $\alpha_j(t)$ represent the j^{th} element in the first row of $\boldsymbol{\alpha}(t)$, for $j = 1, 2, \ldots, m_a$.

With all notation defined, we now present the MDEs for the finite-server, infinite-buffer node; we define $F' \equiv d/dt[F]$.

- The phase-partition PMDEs are

$$\begin{aligned} P_j^{(v)\prime} &= -\upsilon_j(t) P_j^{(v)} + \sum_{u=1,u\neq j}^{m_a} \upsilon_u(t) a_{uj}(t) P_u^{(v)} \\ &\quad + \alpha_j(t) \sum_{u=1}^{m_a} \upsilon_u(t) d_u(t) \left(P_u^{(v)} + (-1)^v \Pr\{N(t) = s-1, J_a(t) = u\} \right) \\ &\quad + (-1)^{v+1} \sum_{i=1}^{m_b} \mu_i f_i \mathbb{E}\{N_i(t), N(t) = s, J_a(t) = j\}, \end{aligned} \tag{B.9}$$

  for $j = 1, 2, \ldots, m_a$ and $v = 1, 2$. From these, we can show

$$\begin{aligned} \Pr\{N(t) \leq s-1\}' &= -\sum_{j=1} \upsilon_j(t) d_j(t) \Pr\{N(t) = s-1, J_a(t) = j\} \\ &\quad + \sum_{i=1}^{m_b} \mu_i f_i \mathbb{E}\left\{N_i(t), N(t) = s\right\}. \end{aligned}$$

  Of course, $\Pr\{N(t) \geq s\}' = -\Pr\{N(t) \leq s-1\}'$, for all $t \geq 0$.

- The first PMDEs for the number of entities in the i^{th} phase of service when there are idle servers are

$$
\begin{aligned}
L_{i,j}^{(1)\prime} \quad = \quad & -\Big(\upsilon_j(t)+\mu_i\Big)L_{i,j}^{(1)} + \sum_{u=1,u\neq j}^{m_a} \upsilon_u(t)a_{uj}(t)L_{i,u}^{(1)} \\
& + \alpha_j(t)\sum_{u=1}^{m_a}\upsilon_u(t)d_u(t)\Bigg[\Big(L_{i,u}^{(1)} - \mathbb{E}\left\{N_i(t), N(t)=s-1, J_a(t)=u\right\}\Big) \\
& \quad + \beta_i\left(P_u^{(1)} - \Pr\left\{N(t)=s-1, J_a(t)=u\right\}\right)\Bigg] \\
& + \mu_i f_i\Big(\mathbb{E}\left\{(N_i(t))^2, N(t)=s, J_a(t)=j\right\} - \mathbb{E}\left\{N_i(t), N(t)=s, J_a(t)=j\right\}\Big) \\
& + \sum_{k=1,k\neq i}^{m_b}\mu_k\Big(b_{ki}L_{k,j}^{(1)} + f_k\mathbb{E}\left\{N_i(t)N_k(t), N(t)=s, J_a(t)=j\right\}\Big), \qquad \text{(B.10)}
\end{aligned}
$$

for $j=1,2,\ldots,m_a$ and $i=1,2,\ldots,m_b$.

- The second PMDEs for the number of entities in the i^{th} phase of service when there are idle servers are

$$
\begin{aligned}
L_{i,j}^{(2)\prime} \quad = \quad & -\Big(\upsilon_j(t)+2\mu_i\Big)L_{i,j}^{(2)} + \mu_i L_{i,j}^{(1)} + \sum_{u=1,u\neq j}^{m_a} \upsilon_u(t)a_{uj}(t)L_{i,u}^{(2)} \\
& + \alpha_j(t)\sum_{u=1}^{m_a}\upsilon_u(t)d_u(t)\Bigg[\Big(L_{i,u}^{(2)} - \mathbb{E}\left\{(N_i(t))^2, N(t)=s-1, J_a(t)=u\right\}\Big) \\
& \quad + 2\beta_i\Big(L_{i,u}^{(1)} - \mathbb{E}\left\{N_i(t), N(t)=s-1, J_a(t)=u\right\}\Big) \\
& \quad + \beta_i\left(P_u^{(1)} - \Pr\left\{N(t)=s-1, J_a(t)=u\right\}\right)\Bigg] \\
& + \mu_i f_i\Bigg[\Big(\mathbb{E}\left\{(N_i(t))^3, N(t)=s, J_a(t)=j\right\} - 2\mathbb{E}\left\{(N_i(t))^2, N(t)=s, J_a(t)=j\right\}\Big) \\
& \quad + \mathbb{E}\left\{N_i(t), N(t)=s, J_a(t)=j\right\}\Bigg] \\
& + \sum_{k=1,k\neq i}^{m_b}\mu_k\Bigg[b_{ki}\Big(L_{k,j}^{(1)} + 2L_{i,k,j}\Big) \\
& \quad + f_k\mathbb{E}\left\{(N_i(t))^2N_k(t), N(t)=s, J_a(t)=j\right\}\Bigg],
\end{aligned}
$$

for $j = 1, 2, \ldots, m_a$, $i = 1, 2, \ldots, m_b$.

- The first PMDEs for the number of entities at the node when all servers are busy are

$$\begin{aligned} M_j^{(1)\prime} &= -\upsilon_j(t) M_j^{(1)} + \sum_{u=1, u\neq j}^{m_a} \upsilon_u(t) a_{uj}(t) M_u^{(1)} \\ &\quad + \alpha_j(t) \sum_{u=1}^{m_a} \upsilon_u(t) d_u(t) \left(M_u^{(1)} + P_u^{(2)} + s \Pr\{N(t) = s-1, J_a(t) = u\} \right) \\ &\quad - \sum_{i=1}^{m_b} \mu_i f_i \Bigg(\mathbb{E}\left\{ N_i(t), S(t) \in \Omega_2^{(j)} \right\} \\ &\qquad + (s-1)\mathbb{E}\left\{ N_i(t), N(t) = s, J_a(t) = j \right\} \Bigg), \end{aligned}$$

while the second PMDEs are

$$\begin{aligned} M_j^{(2)\prime} &= -\upsilon_j(t) M_j^{(2)} + \sum_{u=1, u\neq j}^{m_a} \upsilon_u(t) a_{uj}(t) M_u^{(2)} \\ &\quad + \alpha_j(t) \sum_{u=1}^{m_a} \upsilon_u(t) d_u(t) \Bigg(M_u^{(2)} + 2M_u^{(1)} + P_u^{(2)} \\ &\qquad + s^2 \Pr\{N(t) = s-1, J_a(t) = u\} \Bigg) \\ &\quad - \sum_{i=1}^{m_b} \mu_i f_i \Bigg(\mathbb{E}\left\{ N_i(t), S(t) \in \Omega_2^{(j)} \right\} - 2R_{i,j} \\ &\qquad + (s-1)^2 \mathbb{E}\{ N_i(t), N(t) = s, J_a(t) = j \} \Bigg), \end{aligned}$$

for $j = 1, 2, \ldots, m_a$.

- The CMDEs for the product of the number of entities in the i^{th} and k^{th} phases of service (for $i \neq k$), when there are idle servers, are

$$
\begin{aligned}
L'_{i,k,j} \;\; = \;\; & -\Big(\upsilon_j(t) + \mu_i + \mu_k\Big) L_{i,k,j} + \sum_{u=1, u\neq j}^{m_a} \upsilon_u(t) a_{uj}(t) L_{i,k,u} \\
& + \alpha_j(t) \sum_{u=1}^{m_a} \upsilon_u(t) d_u(t) \Bigg[\Big(L_{i,k,u} - \mathbb{E}\left\{N_i(t) N_k(t), N(t) = s-1, J_a(t) = u\right\} \Big) \\
& \quad + \beta_i \left(L^{(1)}_{k,u} - \mathbb{E}\{N_k(t), N(t) = s-1, J_a(t) = u\} \right) \\
& \quad + \beta_k \left(L^{(1)}_{i,u} - \mathbb{E}\{N_i(t), N(t) = s-1, J_a(t) = u\} \right) \Bigg] \\
& + \mu_k b_{ki} \left(L^{(2)}_{k,j} - L^{(1)}_{k,j} \right) + \mu_i b_{ik} \left(L^{(2)}_{i,j} - L^{(1)}_{i,j} \right) \\
& + \sum_{r=1, r\neq i,k}^{m_b} \mu_r \left(b_{rk} L_{r,i,j} + b_{ri} L_{r,k,j} \right) \\
& + \mu_i f_i \mathbb{E}\left\{ (N_i(t))^2 N_k(t), N(t) = s, J_a(t) = j \right\} \\
& + \mu_k f_k \mathbb{E}\left\{ N_i(t) (N_k(t))^2, N(t) = s, J_a(t) = j \right\} \\
& - \left(\mu_i f_i + \mu_k f_k \right) \mathbb{E}\left\{ N_i(t) N_k(t), N(t) = s, J_a(t) = j \right\} \\
& + \sum_{r=1, r\neq i,k}^{m_b} \mu_r \mathbb{E}\left\{ N_i(t) N_k(t) N_r(t), N(t) = s, J_a(t) = j \right\},
\end{aligned}
$$

for $j = 1, 2, \ldots, m_a$ and $i, k = 1, 2, \ldots, m_b$ $(i \neq k)$.

- Finally, the CMDEs for the product of the number of entities in the i^{th} phase of service and the total number of entities at the node when all servers are busy are

$$\begin{aligned}
R'_{i,j} \;=\;& -\Big(\upsilon_j(t)+\mu_i\Big)R_{i,j} + \sum_{u=1,u\neq j}^{m_a} \upsilon_u(t)a_{uj}(t)R_{i,u} + \sum_{k=1,k\neq i}^{m_b} \mu_k b_{ki} R_{k,j} \\
& + \alpha_j(t)\sum_{u=1}^{m_a} \upsilon_u(t)d_u(t)\Bigg[\Big(s\mathbb{E}\left\{N_i(t), N(t)=s-1, J_a(t)=u\right\} + R_{i,u}\Big) \\
& \quad + \Big(s\beta_i \Pr\left\{N(t)=s-1, J_a(t)=u\right\} + \mathbb{E}\Big\{N_i(t), S(t)\in\Omega_2^{(u)}\Big\}\Big)\Bigg] \\
& + \beta_i \sum_{k=1}^{m_b} \mu_k f_k R_{k,j} \\
& - \sum_{k=1,k\neq i}^{m_b} \mu_k f_k \Bigg[\mathbb{E}\Big\{N_i(t)N_k(t), S(t)\in\Omega_2^{(j)}\Big\} \\
& \quad + \beta_i\Big(\mathbb{E}\Big\{N_k(t), S(t)\in\Omega_2^{(j)}\Big\} - (s-1)\mathbb{E}\left\{N_k(t), N(t)=s, J_a(t)=j\right\}\Big) \\
& \quad + (s-1)\mathbb{E}\left\{N_i(t)N_k(t), N(t)=s, J_a(t)=j\right\}\Bigg] \\
& + \mu_i f_i \Bigg[(1-\beta_i)\Big(\mathbb{E}\Big\{N_i(t), S(t)\in\Omega_2^{(j)}\Big\} \\
& \quad - (s-1)\mathbb{E}\left\{N_i(t), N(t)=s, J_a(t)=j\right\}\Big) \\
& \quad - \mathbb{E}\Big\{(N_i(t))^2, S(t)\in\Omega_2^{(j)}\Big\} \\
& \quad - (s-1)\mathbb{E}\left\{(N_i(t))^2, N(t)=s, J_a(t)=j\right\}\Bigg],
\end{aligned}$$

for $j=1,2,\ldots,m_a$ and $i=1,2,\ldots,m_b$.

Notice that the PMDEs we have provided in this section are similar to those presented in Theorems 1–4 of [36]; however, our equations do not include any partial moments or state probabilities with respect to the full state $N(t)=c$, since $c=\infty$ in our model.

B.2.2 Approximating Terms in the PMDEs

Thus, the following terms must be approximated to close the MDEs at the finite-server, infinite-buffer node, for $j=1,2,\ldots,m_a$:

1. $\Pr\{N(t) = m, J_a(t) = j\}$: $m = s-1, s$,

2. $\mathbb{E}\{(N_i(t))^p, N(t) = m, J_a(t) = j\}$: $m = s-1, s$; $i = 1, 2, \ldots, m_b$; $p = 1, 2, 3$,

3. $\mathbb{E}\{(N_i(t))^p N_k(t), N(t) = m, J_a(t) = j\}$: $m = s-1, s$; $i, k = 1, 2, \ldots, m_b$; $p = 1, 2, 3$,

4. $\mathbb{E}\{N_i(t)N_k(t)N_r(t), N(t) = s, J_a(t) = j\}$: $i, k, r = 1, 2, \ldots, m_b$,

5. $\mathbb{E}\left\{(N_i(t))^p, S(t) \in \Omega_2^{(j)}\right\}$: $i = 1, 2, \ldots, m_b$, $p = 1, 2$.

We can simplify our work in approximating the partial-moments in items 2–5 by using a result from Rueda [36], who notes that, for example,

$$\mathbb{E}\{N_i(t), N(t) = s, J_a(t) = j\} = \mathbb{E}\{N_i(t) \,|\, N(t) = s, J_a(t) = j\} \cdot \Pr\{N(t) = s, J_a(t) = j\}, \tag{B.11}$$

for $j = 1, 2, \ldots, m_a$. We ignore the second term on the right-hand side of (B.11) for the moment. The first term is the conditional expectation of the number of entities in service phase i, for $i = 1, 2, \ldots, m_b$, when there are exactly s entities at the node (and thus, s entities in service). The number of these s entities in service that are in fact in a *particular* phase of service can be modeled as a random variable having a multinomial distribution, where s identical balls are placed into any of m_b different boxes. Rueda defines the quantity r_i, such that

$$r_i \equiv \frac{\sum_{j=1}^{m_a} L_{i,j}^{(1)}}{\sum_{j=1}^{m_a} \sum_{k=1}^{m_b} L_{k,j}^{(1)}}.$$

Thus, r_i approximates the probability that a single ball would be placed in the i^{th} box, for $i = 1, 2, \ldots, m_b$. Therefore, we can rewrite the terms in items 2–4 above in their product-form (i.e., a conditional expectation times $\Pr\{N(t) = m, J_a(t) = j\}$, for $m = s-1$, s, $j = 1, 2, \ldots, m_a$), and calculate the conditional expectations using properties of the multinomial distribution and the set of selection probabilities r_i, for $i = 1, 2, \ldots, m_b$.

Thus, we have resolved approximating the terms in the first four items to being able to approximate $\Pr\{N(t) = m, J_a(t) = j\}$, for each arrival phase $j = 1, 2, \ldots, m_a$ and entity quantity $m = s - 1,\ s$. For the terms in item 5, we can write

$$\mathbb{E}\left\{(N_i(t))^p, S(t) \in \Omega_2^{(j)}\right\} = \mathbb{E}\left\{(N_i(t))^p \,|\, S(t) \in \Omega_2^{(j)}\right\} \cdot \Pr\left\{S(t) \in \Omega_2^{(j)}\right\}, \tag{B.12}$$

for $j = 1, 2, \ldots, m_a$. We obtain the first term on the right-hand side of (B.12) by noticing $\{S(t) \in \Omega_2^{(j)}\} \Leftrightarrow \{J_a(t) = j,\ \sum_{i=1}^{m_b} N_i(t) = s\}$. Thus, we use the multinomial distribution again to approximate the conditional term in (B.12). As for the second term in (B.12), we need not approximate this; this is the solution of (B.9) for $v = 2$.

B.2.3 DDEs for the $Ph_t/Ph/s/\infty$ Node

We derive the partial DDEs (PDDEs) for the finite-server, infinite-buffer node with Markovian component processes; as in the infinite-server model, the derivatives in the DDEs are with respect to the interval length τ, and the terms on the right-hand sides are evaluated at time $t + \tau$. We utilize the following notation; these terms are defined for all $j = 1, 2, \ldots, m_a$, $i = 1, 2, \ldots, m_b$, and $p, v = 1, 2$.

- Departure moments

  $$D^{(p)} \equiv \mathbb{E}\left\{(D_t(t+\tau))^p\right\} = \sum_{j=1}^{m_a} D_j^{(p)},$$

  where

  $$D_j^{(p)} \equiv \mathbb{E}\left\{(D_t(t+\tau))^p, J_a(t+\tau) = j\right\} = \sum_{j=1}^{m_a}\sum_{v=1}^{2} D_{j,v}^{(p)},$$

  such that $D_{j,v}^{(p)} \equiv \mathbb{E}\left\{(D_t(t+\tau))^p, S(t+\tau) \in \Omega_v^{(j)}\right\}$.

- $\left[D_j^{(1)} N_i^{(1)}\right]_v \equiv \mathbb{E}\left\{D_t(t+\tau) N_i(t+\tau), S(t+\tau) \in \Omega_v^{(j)}\right\}$.

We present the PDDEs now; we use ψ as shorthand for the interval end-time $t+\tau$. The first PDDEs are

$$\begin{aligned}\frac{d}{d\tau}D_{j,v}^{(1)} &= -\upsilon_j(\psi)D_{j,v}^{(1)} + \sum_{u=1,u\neq j}^{m_a}\upsilon_u(\psi)a_{uj}(\psi)D_{u,v}^{(1)} \\ &\quad + \alpha_j\sum_{u=1}^{m_a}\upsilon_u(\psi)d_u(\psi)\Big(D_{u,v}^{(1)} \\ &\quad + (-1)^v\mathbb{E}\{D_t(\psi), N(\psi)=s-1, J_a(\psi)=u\}\Big) \\ &\quad + \sum_{i=1}^{m_b}\mu_i f_i\Big[\mathbb{E}\{N_i(\psi), S(\psi)\in\Omega_v^{(j)}\} \\ &\quad - (-1)^v\mathbb{E}\{D_t(\psi)N_i(\psi), N(\psi)=s, J_a(\psi)=j\} \\ &\quad - (-1)^v\mathbb{E}\{N_i(\psi), N(\psi)=s, J_a(\psi)=j\}\Big], \end{aligned} \tag{B.13}$$

for $j=1,2,\ldots,m_a$, $v=1,2$. Adding the first PDDEs across partitions, we find

$$\begin{aligned}\frac{d}{d\tau}D_j^{(1)} &= -\upsilon_j(\psi)D_j^{(1)} + \sum_{u=1,u\neq j}^{m_a}\upsilon_u(\psi)\Big(a_{uj}(\psi)+\alpha_j(\psi)d_u(\psi)\Big)D_u^{(1)} \\ &\quad + \sum_{i=1}^{m_b}\mu_i f_i\mathbb{E}\{N_i(\psi), J_a(\psi)=j\}.\end{aligned}$$

Therefore,

$$\frac{d}{d\tau}\mathbb{E}\{D_t(t+\tau)\} = \sum_{i=1}^{m_b}\mu_i f_i\mathbb{E}\{N_i(t+\tau)\},$$

where

$$\mathbb{E}\{N_i(t+\tau)\} = \sum_{j=1}^{m_a}\sum_{v=1}^{2}\mathbb{E}\left\{N_i(t+\tau), S(t+\tau)\in\Omega_v^{(j)}\right\}. \tag{B.14}$$

Notice we can approximate the terms on the right-hand side of (B.14) at time ψ from the discussion in Appendix B.2.2: for $v=1$, these terms are the solutions to (B.10) for each $i=1,2,\ldots,m_b$, while for $v=2$, they are the fifth item in the list of approximated terms (for $p=1$) in Appendix B.2.2.

The second PDDEs are

$$\begin{aligned}\frac{d}{d\tau}D_{j,v}^{(2)} &= -\upsilon_j(\psi)D_{j,v}^{(2)} + \sum_{u=1,u\neq j}^{m_a}\upsilon_u(\psi)a_{uj}(\psi)D_{u,v}^{(2)} \\ &\quad + \alpha_j(\psi)\sum_{u=1}^{m_a}\upsilon_u(\psi)d_u(\psi)\Bigg(D_{u,v}^{(2)} \\ &\qquad + (-1)^v\mathbb{E}\{(D_t(\psi))^2, N(\psi)=s-1, J_a(\psi)=u\}\Bigg) \\ &\quad + \sum_{i=1}^{m_b}\mu_i f_i\Bigg[\Big(\mathbb{E}\{N_i(\psi), S(\psi)\in\Omega_v^{(j)}\} + 2[D_j^{(1)}N_i^{(1)}]_v\Big) \\ &\qquad - (-1)^v\sum_{q=0}^{2}\binom{2}{q}\mathbb{E}\{(D_t(\psi))^q N_i(\psi), N(\psi)=s, J_a(\psi)=j\}\Bigg],\end{aligned}$$

for $j = 1, 2, \ldots, m_a$, $v = 1, 2$. Therefore,

$$\frac{d}{d\tau}\mathbb{E}\{(D_t(t+\tau))^2\} = \sum_{i=1}^{m_b}\mu_i f_i\Big(\mathbb{E}\{N_i(t+\tau)\} + 2\mathbb{E}\{D_t(t+\tau)N_i(t+\tau)\}\Big),$$

where $\mathbb{E}\{N_i(t+\tau)\}$ is calculated in (B.14), while

$$\mathbb{E}\{D_t(t+\tau)N_i(t+\tau)\}\} = \sum_{j=1}^{m_a}\sum_{v=1}^{2}\left[D_j^{(1)}N_i^{(1)}\right]_v.$$

The partial departure-system DEs (PCDDEs) for the product of the number of departures over the interval and the number of entities in the i^{th} phase of service at the end of the interval, when at least one server is idle, are

$$
\begin{aligned}
\frac{d}{d\tau}\left[D_j^{(1)}N_i^{(1)}\right]_1 \;=\;& -\left(\upsilon_j(\psi)+\mu_i\right)\left[D_j^{(1)}N_i^{(1)}\right]_1 \\
&+\sum_{u=1,u\neq j}^{m_a}\upsilon_u(\psi)a_{uj}(\psi)\left[D_u^{(1)}N_i^{(1)}\right]_1+\sum_{k=1,k\neq i}^{m_b}\mu_k b_{ki}\left[D_u^{(1)}N_k^{(1)}\right]_1 \\
&+\alpha_j(\psi)\sum_{u=1}^{m_a}\upsilon_u(\psi)d_u(\psi)\Bigg[\left(\left[D_u^{(1)}N_i^{(1)}\right]_1+\beta_i D_{u,1}^{(1)}\right) \\
&\quad-\mathbb{E}\left\{D_t(\psi)N_i(\psi),N(\psi)=s-1,J_a(\psi)=u\right\} \\
&\quad-\beta_i\mathbb{E}\left\{D_t(\psi),N(\psi)=s-1,J_a(\psi)=u\right\}\Bigg] \\
&+\mu_i f_i\Bigg[\left(\mathbb{E}\left\{(N_i(\psi))^2,S(\psi)\in\Omega_1^{(j)}\right\}-\mathbb{E}\left\{N_i(\psi),S(\psi)\in\Omega_1^{(j)}\right\}\right) \\
&\quad+\mathbb{E}\left\{D_t(\psi)(N_i(\psi))^2,N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad-\mathbb{E}\left\{D_t(\psi)N_i(\psi),N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad+\mathbb{E}\left\{(N_i(\psi))^2,N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad-\mathbb{E}\left\{N_i(\psi),N(\psi)=s,J_a(\psi)=j\right\}\Bigg] \\
&+\sum_{k=1,k\neq i}^{m_b}\mu_k f_k\Bigg[\mathbb{E}\left\{N_i(\psi)N_k(\psi),S(\psi)\in\Omega_1^{(j)}\right\} \\
&\quad+\mathbb{E}\left\{D_t(\psi)N_i(\psi)N_k(\psi),N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad+\mathbb{E}\left\{N_i(\psi)N_k(\psi),N(\psi)=s,J_a(\psi)=j\right\}\Bigg],
\end{aligned}
$$

for $j=1,2,\ldots,m_a$, $i=1,2,\ldots,m_b$.

The PCDDEs for the product of the number of departures over the interval and the number of entities in the i^{th} phase of service at the end of the interval, when all servers are busy, are

$$
\begin{aligned}
\frac{d}{d\tau}\left[D_j^{(1)}N_i^{(1)}\right]_2 &= -\Big(\upsilon_j(\psi)+\mu_i\Big)\left[D_j^{(1)}N_i^{(1)}\right]_2 \\
&+ \sum_{u=1,u\neq j}^{m_a}\upsilon_u(\psi)a_{uj}\left[D_u^{(1)}N_i^{(1)}\right]_2 + \sum_{k=1,k\neq i}^{m_b}\mu_k b_{ki}\left[D_u^{(1)}N_k^{(1)}\right]_2 \\
&+ \alpha_j\sum_{u=1}^{m_a}\upsilon_u(\psi)d_u(\psi)\Bigg[\Big(\left[D_u^{(1)}N_i^{(1)}\right]_2+\beta_i D_{u,2}^{(1)}\Big) \\
&\quad \mathbb{E}\left\{D_t(\psi)N_i(\psi),N(\psi)=s-1,J_a(\psi)=u\right\} \\
&\quad \beta_i\mathbb{E}\left\{D_t(\psi),N(\psi)=s-1,J_a(\psi)=u\right\}\Bigg] \\
&+ \mu_i f_i\Bigg[\mathbb{E}\Big\{(N_i(\psi))^2,S(\psi)\in\Omega_2^{(j)}\Big\} \\
&\quad -(1-\beta_i)\mathbb{E}\Big\{N_i(\psi),S(\psi)\in\Omega_2^{(j)}\Big\} \\
&\quad -\mathbb{E}\Big\{D_t(\psi)(N_i(\psi))^2,N(\psi)=s,J_a(\psi)=j\Big\} \\
&\quad +(1-\beta_i)\mathbb{E}\left\{D_t(\psi)N_i(\psi),N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad -\mathbb{E}\Big\{(N_i(\psi))^2,N(\psi)=s,J_a(\psi)=j\Big\} \\
&\quad +(1-\beta_i)\mathbb{E}\left\{N_i(\psi),N(\psi)=s,J_a(\psi)=j\right\}\Bigg] \\
&+ \sum_{k=1,k\neq i}^{m_b}\mu_k f_k\Bigg[\mathbb{E}\Big\{\left(N_i(\psi)+\beta_i\right)N_k(\psi),S(\psi)\in\Omega_2^{(j)}\Big\} \\
&\quad -\mathbb{E}\left\{\left(D_t(\psi)-1\right)N_i(\psi)N_k(\psi),N(\psi)=s,J_a(\psi)=j\right\} \\
&\quad -\beta_i\mathbb{E}\left\{\left(D_t(\psi)-1\right)N_k(\psi),N(\psi)=s,J_a(\psi)=j\right\}\Bigg],
\end{aligned}
$$

for $j=1,2,\ldots,m_a$, $i=1,2,\ldots,m_b$.

B.2.4 Approximating Terms in the DDEs

As in the PMDEs, we look at the right-hand sides of the PDDEs and the PCDDEs and identify those *new* terms (i.e., terms that involve the number of departures) that we need to approximate to close the system of DDEs at the finite-server node. Using ψ to replace $t+\tau$, we require approximations for

1. $\mathbb{E}\left\{(D_t(\psi))^p, N(\psi)=s-1, J_a(\psi)=j\right\}$: $p=1,2$,
2. $\mathbb{E}\left\{(D_t(\psi))^p N_i(\psi), N(\psi)=m, J_a(\psi)=j\right\}$: $m=s-1,s$; $i,k=1,2,\ldots,m_b$; $p=1,2$,
3. $\mathbb{E}\left\{D_t(\psi)(N_i(\psi))^2, N(\psi)=s, J_a(\psi)=j\right\}$: $i=1,2,\ldots,m_b$,
4. $\mathbb{E}\left\{D_t(\psi)N_i(\psi)N_k(\psi), N(\psi)=s, J_a(\psi)=j\right\}$: $i,k=1,2,\ldots,m_b$,

for $j=1,2,\ldots,m_a$. We utilize the conditional expectation technique discussed in Appendix B.2.2, evaluating the terms at time ψ (we return to the discuss on approximating $\Pr\{N(t)=m, J_a(t)=j\}$, for $m=s-1$, s, $j=1,2,\ldots,m_a$, and all $t\geq 0$, in the next section).

Notice that the first item in the list of approximated terms in this section is different in form from items 2–4; we approach it first. As in [24], we approximate terms of this form by claiming

$$\begin{aligned}\mathbb{E}\left\{(D_t(\psi))^p, N(\psi)=s-1, J_a(\psi)=j\right\} &\approx \mathbb{E}\left\{(D_t(\psi))^p, S(\psi)\in\Omega_1^{(j)}\right\} \\ &\quad\cdot\frac{\Pr\left\{N(\psi)=s-1, J_a(\psi)=j\right\}}{\Pr\left\{S(\psi)\in\Omega_1^{(j)}\right\}} \\ &= D_{j,1}^{(p)}\cdot\frac{\Pr\left\{N(\psi)=s-1, J_a(\psi)=j\right\}}{\Pr\left\{S(\psi)\in\Omega_1^{(j)}\right\}},\end{aligned} \tag{B.15}$$

for $j=1,2,\ldots,m_a$, where '$\approx$' indicates the two terms are approximately equal. The first term on the right-hand side of (B.15) is the solution to (B.13), while the denominator in the second term is the solution to (B.9), both for $v=1$.

The other three items in the list are of the form

$$\mathbb{E}\left\{(D_t(\psi))^p(N_i(\psi))^r(N_k(\psi))^q, N(\psi)=m, J_a(\psi)=j\right\}$$

for $p, r, q = 0, 1, 2$, $i, k = 1, 2, \ldots, m_b$, $j = 1, 2, \ldots, m_a$, and $m = s-1,\ s$ (not all combinations of p, q, and r are utilized). To approximate these terms we assume the departure count on $[t, \psi]$ and the system size at ψ are independent, given $N(\psi)$; this assumption is introduced in [22]. We claim that

$$\begin{aligned}\mathbb{E}\left\{(D_t(\psi))^p(N_i(\psi))^r(N_k(\psi))^q, N(\psi)=m, J_a(\psi)=j\right\} \approx\\ \mathbb{E}\left\{(D_t(\psi))^p, N(\psi)=m, J_a(\psi)=j\right\}\cdot\frac{\mathbb{E}\left\{(N_i(\psi))^r(N_k(\psi))^q, N(\psi)=m, J_a(\psi)=j\right\}}{\Pr\left\{S(\psi)\in\Omega^{(j)}_{m+2-s}\right\}},\end{aligned} \tag{B.16}$$

for $m = s-1,\ s$. Notice that the first term on the right-hand side of (B.16) comes from (B.15), while, for the second term, the numerator is approximated using techniques in Appendix B.2.2 and the denominator is again the solution to (B.9), at time $t+\tau$, for $v = m+2-s$, where $m = s-1,\ s$.

As in the PMDEs, we have described techniques here for approximating those terms required to close the PDDEs *premised on* finding a suitable technique to approximate

$$\Pr\{N(t)=m, J_a(t)=j\}, \tag{B.17}$$

for $m = s-1,\ s$ and $j = 1, 2, \ldots, m_a$. We return now to the question of approximating the $2m_a$ probabilities in (B.17) for a finite-server node with infinite buffer space.

B.3 Using the PE Distribution as a Surrogate in Infinite-Buffer Models

As described in Appendix B.1, employing the PE distribution as a surrogate for the $Ph_t/Ph_t/s/c$ node has proven to be successful for approximating terms necessary for closing the MDEs when $c < \infty$. We require a tool to employ as a surrogate when $c = \infty$. The PE distribution has finite support, and thus applying it directly to the infinite-capacity node is not feasible as we need to approximate terms in partitions whose natural support is infinite. Instead, we consider techniques to truncate the natural support of the state partitions, which would allow us to employ the PE distribution on these truncated spaces.

We define $P_{m,j}(t) \equiv \Pr\{N(t) = m, J_a(t) = j\}$, for $m = s-1,\ s,\ j = 1, 2, \ldots, m_a$. Notice we can approximate $P_{s-1,j}(t)$ for $c = \infty$ as we do for $c < \infty$; that is

$$P_{s-1,j}(t) \doteq E_{1,j}^{(0)} \cdot \Pr\{X = s-1\}, \tag{B.18}$$

for $j = 1, 2, \ldots, m_a$, where X is a PE-distributed r.v. with support $0, 1, \ldots, s-1$ and parameters p and α matched to $E_{1,j}^{(1)}/E_{1,j}^{(0)}$ and $E_{1,j}^{(2)}/E_{1,j}^{(0)}$, according to (B.2) and (B.3); the partial conditional moments $E_{1,j}^{(p)}$ are defined in (B.5) (for $v = 1$), (B.6), and (B.7), respectively (for $p = 0, 1, 2$), for all $j = 1, 2, \ldots, m_a$.

However, we cannot use a similar approximation for $P_{s,j}(t)$, since this would require

$$P_{s,j}(t) \doteq E_{2,j}^{(0)} \cdot \Pr\{Z = 0\}, \tag{B.19}$$

for $j = 1, 2, \ldots, m_a$, where Z is a PE r.v. with support on $0, 1, \ldots, c-s$, having parameters p and α derived analogously. Since $c = \infty$, this yields an infinite support, and the PE distribution is inappropriate as a surrogate since it requires finite support.

We consider instead that the PE distribution may be a valid surrogate for the random variable Z (defined in (B.19)) *if* we can identify a value $R \in \mathbb{Z}^+$ to serve as the upper bound of the support for Z, such that the tail probability (i.e., the probability that the system size is larger than $R + s$) is very small for any time $t < \infty$, provided the system size does not grow without bound.

We might consider setting R arbitrarily large, thus minimizing the tail probability; however, this fails for two reasons. First, setting R too large leads to numerical problems in evaluating the PE probabilities. Second, forcing the technique to match two entity moments with a very large support may compromise the accuracy of the probabilities we obtain from the technique, since the PE distribution is not the actual distribution for the number of entities at the node.

Instead, we develop a technique where R is set to be just large enough so that a small change in R does not cause a large change in the nodal moments. At each time t, we test whether the current truncation level R is sufficient to advance the system (through numerical integration) to time $t+\tau$. To do so, we generate the nodal moments at $t+\tau$ using the current R in our approximation of $P_{s,j}(t)$, for each $j = 1, 2, \ldots, m_a$. We then return to t and repeat the integration, using this time a slightly larger value for the truncation point. If the moments from the larger truncation point are relatively the same as those we calculated using R, we maintain our choice of R and continue to time $t+\tau$. If the relative change is significant in either the mean or variance of the number of entities, we set this slightly larger truncation point as our value for R, and repeat the process again from time t—thus only advancing to $t+\tau$ when we have found a satisfactory level for R.

We enumerate the steps described in the previous paragraph here.

Algorithm B.1. *Dynamically setting upper support limit R*

1. Prior to initiating integration, we set the following parameters:

   - Δ_R: the integer quantity to be added to the standing value of R when necessary (default value: 10).

   - γ_m, γ_v: thresholds to compare relative changes (from adjusting R) in the mean and variance of the number of entities, respectively (default value: 0.5%, for both moments).

2. At time $t = 0$, $R = \max\{20, 2s\}$, where s is the number of servers at the node.

3. Calculate the mean and variance of the number of entities at the node at time $t + \tau$ (given τ from Step 2 of Algorithm 3.1), numerically integrating MDEs ((B.9)–(B.11)) and DDEs ((B.13)–(B.15)).
   - Approximate $P_{s-1,j}(t')$, for $j = 1, 2, \ldots, m_a$, $t' \in [t, t + \tau)$, using (B.18).
   - Approximate $P_{s,j}(t')$, for $j = 1, 2, \ldots, m_a$, $t' \in [t, t + \tau)$, using (B.19), such that the upper support for the PE r.v. Z at time t' is equal to R.
   - Approximate all terms enumerated in Sections B.2.2 and B.2.4 at all $t' \in [t, t+\tau)$.
4. If this is the first time we have run the model on $[t, t + \tau)$, return to Step 3, using $R + \Delta_R$ as the upper support limit. Otherwise, go to Step 5.
5. Compare the nodal mean and variance at $t + \tau$ using upper support $R + \Delta_R$ to the respective moments using R. If the relative change in either moment is larger than its respective threshold (i.e., γ_m or γ_v), set $R = R + \Delta_R$ and return to Step 3. Otherwise, go to Step 6.
6. Set $t = t + \tau$.

We are now able to provide approximations for the $2m_a$ probabilities in (B.17); from these we can derive values for the mean and variance of the number of entities at each node as well as the mean and variance of the number of departures from each upstream node over corresponding intervals.

Before we conclude the discussion of our closure technique, it is worth mentioning that our method is not the only way to model the finite-server, infinite-buffer node with Markovian component processes. For example, an argument may be made that using our dynamic truncation method effectively sets a capacity on the node size, thus creating a model where we treat the node as having finite capacity. However, the MDEs in Appendix B.2.1 only account for this implicitly in the approximations for the full space probability terms. Another method for truncating the infinite support would be to model our $Ph_t/Ph/s/\infty$ as a finite capacity $Ph_t/Ph/s/R+s$ model, using the PMDEs in [36] directly with $c = R+s$ and deriving appropriate finite-buffer DDEs. Again, we would employ our dynamic truncation technique in determining R at each time step; however, this is a different application of our technique, as adjustments to R would alter the nodal capacity in addition to the upper support of the surrogate distribution. At present we have no plan to pursue modeling the infinite-buffer node as a $Ph_t/Ph/s/R+s$ finite-buffer node since the application of our dynamic truncation technique in the infinite-buffer node typically has proven successful.

C Specifying a Ph Distribution to Match m_1 and scv

In Sections 3.2 and 4, we utilize well-known moment–matching techniques in which the parameters of a specific family of Ph renewal processes are specified to match the mean interrenewal time m_1 and the squared coefficient of variation scv of the interrenewal distribution. Many techniques exist that accomplish this; we cite the following:

- If $scv \geq 1$, then we specify an h_2b [41], which implies that X is exponentially distributed with mean λ^{-1} with probability α, or exponentially distributed with mean λ_2^{-1} with probability $1-\alpha$. We say h_2b has "balanced means" if $\alpha/\lambda = (1-\alpha)/\lambda_2$. Thus, h_2b has only two free parameters: α and λ. We back these out of the expressions for the mean m_1 and squared coefficient of variation scv of an h_2b giving
$$\alpha = \frac{1}{2}\left(1+\sqrt{\frac{scv-1}{scv+1}}\right),\ \lambda = \frac{2\alpha}{m_1}.$$
The MAP$_t$ representation $(\mathbf{A}, \boldsymbol{v})$ for the h_2b renewal process is $m_a = 2$,
$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 1-\alpha & 0 & 0 \\ \alpha & 1-\alpha & 0 & 0 \end{pmatrix},\ \boldsymbol{v} = (\lambda,\ \lambda(1-\alpha)/\alpha)^\top.$$

- If $scv < 1$, then we use a MECon distribution [46]. First, we find $k \in \mathbb{Z}^+$ such that $1/k \leq scv < 1/(k-1)$, since scv for an Erlang of order k (denoted by $E_k(\lambda)$) is $1/k$. Then X is $E_{k-1}(\lambda)$ distributed with probability α, or $E_k(\lambda)$ distributed with probability $1-\alpha$. Again, this leaves only two free parameters: the mixing probability α and the common rate λ. We back these out of the expressions for the mean and scv of a MECon giving

$$\alpha = \frac{1}{1+scv}\left(k \cdot scv - \sqrt{k(1+scv) - k^2 scv}\right), \ \lambda = \frac{k-\alpha}{m_1}.$$

The MAP$_t$ representation $(\mathbf{A}, \boldsymbol{\upsilon})$ for the stationary MECon renewal process is $m_a = 2k-1$,

$$a_{jh} = \begin{cases} 1, & \text{if } h = j+1,\ j < k, \\ 1, & \text{if } h = j+1,\ j \geq k+1, \\ 0, & \text{otherwise,} \end{cases} \quad d_j = \begin{cases} 1, & \text{if } j \in \{k, 2k-1\}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{jh} = \begin{cases} 1-\alpha, & \text{if } h = 1, \\ \alpha, & \text{if } h = k+1, \\ 0, & \text{otherwise,} \end{cases}$$

while $\upsilon_j = \lambda$, for $j,\ h = 1, 2, \ldots, 2k-1$.

D Parameters for Sample Networks

We provide the network parameters in Table 3 for seven networks specifically examined in Section 4. Plots for the fitted and true moments in Figures 1–3 correspond to networks 1–3, respectively. Networks 1 and 4–7 are the five sample networks utilized in the analysis of alternative network structures discussed in Section 4.2.

Table 3: Parameter values for seven sample networks utilized in evaluating the matching technique.

Network	b_a	PTM	scv_a	AOL(1)	$scv_s^{(1)}$	AU(1)	AOL(2)	$scv_s^{(2)}$	AU(2)
1	36.9%	88.06	2.037	31.41	1.290	47.6%	27.49	1.442	68.7%
2	54.9%	73.63	2.426	25.38	0.595	42.3%	38.19	0.551	55.3%
3	44.7%	34.83	0.905	20.70	1.524	73.9%	25.68	0.718	40.8%
4	11.6%	52.74	1.821	36.23	1.422	54.9%	20.70	0.723	41.4%
5	19.7%	91.54	0.819	37.29	0.949	39.2%	29.30	1.578	50.5%
6	39.0%	56.72	0.928	36.38	1.373	60.6%	22.96	1.130	60.4%
7	52.1%	26.37	1.191	20.25	0.910	69.8%	30.50	0.638	37.2%