Proofs for:

The $Ph_t/Ph_t/\infty$ Queueing System:

Part I—The Single Node

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Proofs

Theorem 3 *The $Ph_t/Ph_t/\infty$ arrival-process state-probability differential equations (ADE's) are:*

$$\mathrm{P}(t;\cdot,k)' \;=\; \alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{P}(t;\cdot,\ell) + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t;\cdot,\ell) - \lambda_k(t)\mathrm{P}(t;\cdot,k)$$

for $k = 1, 2, \ldots, m_A$.

Proof: While we can show this result by straightforward and tedious algebraic manipulation of the expression

$$\mathrm{P}(t;\cdot,k)' \equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty}\mathrm{P}(t;n_1,n_2,\ldots,n_{m_B},k)'.$$

We also note that the arrival process is independent of the service process and therefore we can formulate the Kolmogorov forward equations associated with this simple time-dependent Markov arrival process directly. □

Theorem 4 *The $Ph_t/Ph_t/\infty$ $p^{th}-$partial-moment differential equations (p^{th} PMDE's):*

$$\frac{d}{dt}\mathrm{E}^p_{j,k}(t) = \; - \; \lambda_k(\mathrm{t})\;\mathrm{E}^p_{j,k}(t) + [1 - b_{jj}(t)]\;\mu_j(t)\;\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)\;(-1)^{p-q}$$

$$+ \; \alpha_k(t)\;\sum_{\ell=1}^{m_A}\lambda_\ell(t)\;a_{\ell,m_A+1}(t)\;\mathrm{E}^p_{j,\ell}(t)$$

$$+ \quad \alpha_k(t)\ \beta_j(t)\ \sum_{\ell=1}^{m_A} \lambda_\ell(t)\ a_{\ell,m_A+1}(t) \sum_{q=0}^{p-1} \binom{p}{q} \mathrm{E}^q_{j,\ell}(t)$$

$$+ \quad \sum_{\ell=1}^{m_A} \lambda_\ell(t)\ a_{\ell k}(t)\ \mathrm{E}^p_{j,\ell}(t)$$

$$+ \quad \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t)\ b_{\ell j}(t)\ \sum_{q=0}^{p-1} \binom{p}{q} \mathrm{E}^q_{j\ell,k}(t)$$

for $i = 1, 2, \ldots, m_A,\ j = 1, 2, \ldots, m_B,$ *and* $p = 0, 1, 2, \ldots.$

Proof:

$$\frac{d}{dt}\mathrm{E}^p_{j,k}(t) \equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_j^p \mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k)'$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_j^p \times$$

$$\Big\{- \quad [1 - a_{kk}(t)]\lambda_k(t)\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k)$$

$$- \quad \sum_{\ell=1}^{m_B} n_\ell \mu_\ell(t)[1 - b_{\ell\ell}(t)]\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k)$$

$$+ \quad \sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\alpha_k(t)\lambda_\ell(t) \sum_{h=1}^{m_B} \mathrm{I}_{[n_h>0]}\beta_h(t)\mathrm{P}(t; n_1, \ldots, n_h - 1, \ldots, n_{m_B}, \ell)$$

$$+ \quad \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, \ell)$$

$$+ \quad \sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)[n_\ell + 1]\mu_\ell(t)\mathrm{P}(t; n_1, \ldots, n_\ell + 1, \ldots, n_{m_B}, k)$$

$$+ \quad \sum_{\ell=1}^{m_B} \mathrm{I}_{[n_\ell>0]} \sum_{\substack{h=1\\ h\neq\ell}}^{m_B} b_{h\ell}(t)[n_h + 1]\mu_h(t)\mathrm{P}(t; n_1, \ldots, n_\ell - 1, \ldots, n_h + 1, \ldots, n_{m_B}, k) \quad \Big\}$$

$$= \ -[1 - a_{kk}(t)]\lambda_k(t)\mathrm{E}^p_{j,k}(t)$$

$$-\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1 - b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) - \mu_j(t)[1 - b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t)$$

$$+\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t) \left\{ \sum_{\substack{h=1\\ h\neq j}}^{m_B} \beta_h(t)\mathrm{E}^p_{j,\ell}(t) + \beta_j(t) \sum_{n_j=0}^{\infty} [n_j + 1]^p \mathrm{P}(t; n_j, \ell) \right\}$$

$$
\begin{aligned}
& + \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_\ell+1]\mathrm{P}(t;n_{\ell+1},n_j,k) \\
& + b_{j,m_B+1}(t)\mu_j(t)\sum_{n_j=0}^{\infty} n_j^p[n_j+1]\mathrm{P}(t;n_j+1,k) \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\Bigg\{ \sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=1}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_\ell-1,n_j,n_h+1,k) \\
& \qquad + \mu_j(t)b_{j\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_j+1]\mathrm{P}(t;n_\ell-1,n_j+1,k)\Bigg\} \\
& + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_j=1}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_j-1,n_h+1,k) \\
= \; & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t) \\
& + \alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\left\{[1-\beta_j(t)]\mathrm{E}^p_{j,\ell}(t) + \beta_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)\right\} \\
& + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p n_\ell \mathrm{P}(t;n_\ell,n_j,k) \\
& + b_{j,m_B+1}(t)\mu_j(t)\sum_{n_j=1}^{\infty}[n_j-1]^p\; n_j\mathrm{P}(t;n_j,k) \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty} n_j^p[n_h+1]\mathrm{P}(t;n_\ell,n_j,n_h+1,k) \\
& + \mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{n_j=1}^{\infty}[n_j-1]^p n_j\sum_{n_\ell=0}^{\infty}\mathrm{P}(t;n_\ell,n_j,k)
\end{aligned}
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\begin{aligned}
& + \sum_{\substack{h=1\\h\neq j}}^{m_B} b_{hj}(t)\mu_h(t) \sum_{n_h=0}^{\infty}\sum_{n_j=0}^{\infty} [n_j+1]^p [n_h+1] \mathrm{P}(t; n_j, n_h+1, k) \\
= \quad & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& - \sum_{\substack{\ell=1\\\ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t) \\
& + \alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t) \\
& + \alpha_k(t)(1-\beta_j(t))\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\substack{\ell=1\\\ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& + b_{j,m_B+1}(t)\mu_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& + \sum_{\substack{\ell=1\\\ell\neq j}}^{m_B}\sum_{\substack{h=1\\h\neq\ell\\h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=1}^{\infty}\sum_{n_j=0}^{\infty} n_j^p n_h \mathrm{P}(t; n_\ell, n_j, n_h, k) \\
& + \mu_j(t)\sum_{\substack{\ell=1\\\ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{n_j=1}^{\infty}[n_j-1]^p n_j \mathrm{P}(t; n_j, k) \\
& + \sum_{\substack{h=1\\h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_j=0}^{\infty}[n_j+1]^p n_h \mathrm{P}(t; n_j, n_h, k) \\
= \quad & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& - \sum_{\substack{\ell=1\\\ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t) \\
& + \alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)
\end{aligned}
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\begin{aligned}
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& +b_{j,m_B+1}(t)\mu_j(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\ \sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^p_{jh,k}(t) \\
& +\mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& +\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t) \\
= \; & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& -\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^{p+1}_{j,k}(t) \\
& +\alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& +\mu_j(t)[b_{j,m_B+1}(t)+1-b_{jj}(t)-b_{j,m_B+1}(t)]\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q}
\end{aligned}
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\begin{aligned}
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^p_{jh,k}(t) \\
& + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t) \\
= \; & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) \\
& + \alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t) \\
& + \alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& + \mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \sum_{\substack{h=1\\ h\neq\ell\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}^p_{jh,k}(t) \\
& + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t) \\
= \; & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) \\
& + \alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t)
\end{aligned}
$$

$$
\begin{aligned}
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& +\mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& +\sum_{\substack{h=1\\ h\neq j}}^{m_B}[1-b_{hh}(t)-b_{h,m_B+1}(t)-b_{hj}(t)]\mu_h(t)\mathrm{E}^p_{jh,k}(t) \\
& +\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{q=0}^{p}\binom{p}{q}\mathrm{E}^q_{jh,k}(t) \\
= \; & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& -\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t) \\
& +\alpha_k(t)\beta_j(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^q_{j,\ell}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}^p_{j\ell,k}(t) \\
& +\mu_j(t)[1-b_{jj}(t)]\sum_{q=0}^{p-1}\binom{p}{q}\mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& +\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B}\mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}^p_{j\ell,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t) b_{\ell,m_B+1}(t) \mathrm{E}^p_{j\ell,k}(t) \\
& - \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} \mu_\ell(t) b_{\ell j}(t) \mathrm{E}^p_{j\ell,k}(t) \\
& + \sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t) \sum_{q=0}^{p} \binom{p}{q} \mathrm{E}^q_{jh,k}(t) \\
= \quad & -\lambda_k(t)\mathrm{E}^p_{j,k}(t) \\
& +\alpha_k(t)\beta_j(t) \sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t) \sum_{q=0}^{p-1} \binom{p}{q} \mathrm{E}^q_{j,\ell}(t) \\
& +\alpha_k(t) \sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}^p_{j,\ell}(t) \\
& +\mu_j(t)[1-b_{jj}(t)] \sum_{q=0}^{p-1} \binom{p}{q} \mathrm{E}^{q+1}_{j,k}(t)(-1)^{p-q} \\
& + \sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{\ell j}(t)\mu_l(t) \sum_{q=0}^{p-1} \binom{p}{q} \mathrm{E}^q_{jh,k}(t)
\end{aligned}
$$

□

Theorem 5 *The $Ph_t/Ph_t/\infty$ cross-product partial-moment differential equations are*

$$\begin{aligned}
\mathrm{E}_{ij,k}(t)' &= \sum_{\ell=1}^{m_B} b_{\ell i}(t)\mu_\ell \mathrm{E}_{\ell j,k}(t) + \sum_{\ell=1}^{m_B} b_{\ell j}(t)\mu_\ell \mathrm{E}_{\ell i,k}(t) \\
&\quad - b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t) - b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t) \\
&\quad - [\mu_i(t) + \mu_j(t)]\mathrm{E}_{ij,k}(t) - \lambda_k(t)\mathrm{E}_{ij,k}(t) \\
&\quad + \sum_{\ell=1}^{m_A} \lambda_\ell(t) a_{\ell k}(t)\mathrm{E}_{ij,\ell}(t) + \alpha_k(t)\sum_{\ell=1}^{m_A} \lambda_\ell(t) a_{\ell,m_A+1}(t)\mathrm{E}_{ij,\ell}(t) \\
&\quad + \beta_i(t)\alpha_k(t)\sum_{\ell=1}^{m_A} \lambda_\ell(t) a_{\ell,m_A+1}(t)\mathrm{E}_{j,\ell}(t) + \beta_j(t)\alpha_k(t)\sum_{\ell=1}^{m_A} \lambda_\ell(t) a_{\ell,m_A+1}(t)\mathrm{E}_{i,\ell}(t).
\end{aligned}$$

$i = 1, 2, \ldots, m_B$ *and* $j = 1, 2, \ldots, m_B$, $i \neq j$, *and* $k = 1, 2, \ldots, m_A$.

Proof:

$$\begin{aligned}
\mathrm{E}_{ij,k}(t)' &\equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_i n_j \mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k)' \\
&= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\cdots\sum_{n_{m_B}=0}^{\infty} n_i n_j \times \\
&\quad \Bigg\{ - [1 - a_{kk}(t)]\lambda_k(t)\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k) \\
&\quad - \sum_{\ell=1}^{m_B} n_\ell \mu_\ell(t)[1 - b_{\ell\ell}(t)]\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, k) \\
&\quad + \sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\alpha_k(t)\lambda_\ell(t)\sum_{h=1}^{m_B} \mathrm{I}_{[n_h>0]}\beta_h(t)\mathrm{P}(t; n_1, \ldots, n_h - 1, \ldots, n_{m_B}, \ell) \\
&\quad + \sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{P}(t; n_1, n_2, \ldots, n_{m_B}, \ell) \\
&\quad + \sum_{\ell=1}^{m_B} b_{\ell,m_B+1}(t)[n_\ell + 1]\mu_\ell(t)\mathrm{P}(t; n_1, \ldots, n_\ell + 1, \ldots, n_{m_B}, k) \\
&\quad + \sum_{\ell=1}^{m_B} \mathrm{I}_{[n_\ell>0]}\sum_{\substack{h=1\\ h\neq\ell}}^{m_B} b_{h\ell}(t)[n_h + 1]\mu_h(t)\mathrm{P}(t; n_1, \ldots, n_\ell - 1, \ldots, n_h + 1, \ldots, n_{m_B}, k) \Bigg\} \\
&= -[1 - a_{kk}(t)]\lambda_k(t)\mathrm{E}_{ij,k}(t) \\
&\quad - \sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1 - b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t) \\
&\quad - \mu_i(t)[1 - b_{ii}(t)]\mathrm{E}^2_{ij,k}(t) - \mu_j(t)[1 - b_{jj}(t)]\mathrm{E}^2_{ji,k}(t)
\end{aligned}$$

$$+\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\Bigg\{\sum_{\substack{h=1\\h\neq i\\h\neq j}}^{m_B}\beta_h(t)\mathrm{E}_{ij,\ell}(t)$$

$$+\beta_i(t)\sum_{n_i=0}^{\infty}[n_i+1]n_j\mathrm{P}(t;n_i,n_j,\ell)+\beta_j(t)\sum_{n_j=0}^{\infty}n_i[n_j+1]\mathrm{P}(t;n_i,n_j,\ell)\Bigg\}$$

$$+\sum_{\substack{\ell=1\\\ell\neq i\\\ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_\ell+1]\mathrm{P}(t;n_\ell+1,n_i,n_j,k)$$

$$+b_{i,m_B+1}(t)\mu_i(t)\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_i+1]\mathrm{P}(t;n_i+1,n_j,k)$$

$$+b_{j,m_B+1}(t)\mu_j(t)\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_j+1]\mathrm{P}(t;n_i,n_j+1,k)$$

$$+\sum_{\substack{\ell=1\\\ell\neq i\\\ell\neq j}}^{m_B}\Bigg\{\sum_{\substack{h=1\\h\neq\ell\\h\neq i\\h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=1}^{\infty}\sum_{n_h=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j(n_h+1)\mathrm{P}(t;n_\ell-1,n_i,n_j,n_h+1,k)$$

$$+\mu_i(t)b_{i\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_i+1]\mathrm{P}(t;n_\ell-1,n_i+1,n_j,k)$$

$$+\mu_j(t)b_{j\ell}(t)\sum_{n_\ell=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_j+1]\mathrm{P}(t;n_\ell-1,n_i,n_j+1,k)\Bigg\}$$

$$+\sum_{\substack{h=1\\h\neq i\\h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty}n_in_j[n_h+1]\mathrm{P}(t;n_i-1,n_j,n_h+1,k)$$

$$+\sum_{\substack{h=1\\h\neq i\\h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty}n_in_j[n_h+1]\mathrm{P}(t;n_i,n_j-1,n_h+1,k)$$

$$+b_{ji}(t)\mu_j(t)\sum_{n_j=0}^{\infty}\sum_{n_i=1}^{\infty}n_in_j[n_j+1]\mathrm{P}(t;n_i-1,n_j+1,k)$$

$$+b_{ij}(t)\mu_i(t)\sum_{n_j=1}^{\infty}\sum_{n_i=0}^{\infty}n_in_j[n_i+1]\mathrm{P}(t;n_j-1,n_i+1,k)$$

$$+\sum_{\substack{\ell=1\\\ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)$$

$$
\begin{aligned}
= \quad & -[1-a_{kk}(t)]\lambda_k(t)\mathrm{E}_{ij,k}(t) \\
& -\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t) \\
& -\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\Big\{[1-\beta_i(t)-\beta_j(t)]\mathrm{E}_{ij,\ell}(t) \\
& \qquad + \beta_i(t)[\mathrm{E}_{ij,\ell}(t)+\mathrm{E}_{j,\ell}(t)] + \beta_j(t)[\mathrm{E}_{ij,\ell}(t)+\mathrm{E}_{i,\ell}(t)]\Big\} \\
& +\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t) \\
& +b_{i,m_B+1}(t)\mu_i(t)\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty}[n_i-1]n_jn_i\mathrm{P}(t;n_i,n_j,k) \\
& +b_{j,m_B+1}(t)\mu_j(t)\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty}n_i[n_j-1]n_j\mathrm{P}(t;n_i,n_j,k) \\
& +\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\Bigg\{\sum_{\substack{h=1\\ h\neq \ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\sum_{n_\ell=0}^{\infty}\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_in_jn_h\mathrm{P}(t;n_\ell,n_i,n_j,n_h,k) \\
& \qquad\quad + \mu_i(t)b_{i\ell}(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=1}^{\infty}\sum_{n_j=0}^{\infty}[n_i-1]n_jn_i\mathrm{P}(t;n_\ell,n_i,n_j,k) \\
& \qquad\quad + \mu_j(t)b_{j\ell}(t)\sum_{n_\ell=0}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=1}^{\infty}n_i[n_j-1]n_j\mathrm{P}(t;n_\ell,n_i,n_j,k)\Bigg\} \\
& +\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}[n_i+1]n_jn_h\mathrm{P}(t;n_i,n_j,n_h,k) \\
& +\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\sum_{n_h=1}^{\infty}\sum_{n_i=0}^{\infty}\sum_{n_j=0}^{\infty}n_i[n_j+1]n_h\mathrm{P}(t;n_i,n_j,n_h,k) \\
& +b_{ji}(t)\mu_j(t)\sum_{n_j=1}^{\infty}\sum_{n_i=0}^{\infty}[n_i+1][n_j-1]n_j\mathrm{P}(t;n_i,n_j,k)
\end{aligned}
$$

$$
\begin{aligned}
& +b_{ij}(t)\mu_i(t)\sum_{n_j=0}^{\infty}\sum_{n_i=1}^{\infty}[n_i-1][n_j+1]n_i\mathrm{P}(t;n_j,n_i,k) \\
& +\sum_{\substack{\ell=1\\ \ell\neq k}}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) \\
= \quad & -\lambda_k(t)\mathrm{E}_{ij,k}(t) \\
& -\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)[1-b_{\ell\ell}(t)]\mathrm{E}_{ij\ell,k}(t) \\
& -\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t)-\mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A} a_{\ell,m_A+1}(t)\lambda_\ell(t)\Big\{\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\} \\
& +b_{i,m_B+1}(t)\mu_i(t)\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big]+b_{j,m_B+1}(t)\mu_j(t)\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\Big] \\
& +\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{\ell,m_B+1}(t)\mu_\ell(t)\mathrm{E}_{ij\ell,k}(t) \\
& +\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\Bigg\{\sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t) \\
& \qquad +\mu_i(t)b_{i\ell}(t)\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big]+\mu_j(t)b_{j\ell}(t)\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\Big]\Bigg\} \\
& +\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\Big]+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\Big] \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) \\
& +b_{ji}(t)\mu_j(t)\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{j,k}(t)-\mathrm{E}_{j,k}(t)\Big] \\
& +b_{ij}(t)\mu_i(t)\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)+\mathrm{E}^2_{i,k}(t)-\mathrm{E}_{i,k}(t)\Big] \\
= \quad & -\lambda_k(t)\mathrm{E}_{ij,k}(t)
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)\mathrm{E}_{ij\ell,k}(t) + \sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t) b_{\ell\ell}(t)\mathrm{E}_{ij\ell,k}(t) + \sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t) b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t) \\
& -\mu_i(t)[1-b_{ii}(t)]\mathrm{E}^2_{ij,k}(t) - \mu_j(t)[1-b_{jj}(t)]\mathrm{E}^2_{ji,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A}\Big\{a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) + \beta_i(t)\mathrm{E}_{j,\ell}(t) + \beta_j(t)\mathrm{E}_{i,\ell}(t) \\
& +b_{i,m_B+1}(t)\mu_i(t)\Big[\mathrm{E}^2_{ij,k}(t) - \mathrm{E}_{ij,k}(t)\Big] + b_{j,m_B+1}(t)\mu_j(t)\Big[\mathrm{E}^2_{ji,k}(t) - \mathrm{E}_{ij,k}(t)\Big] \\
& +\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\ \sum_{\substack{h=1\\ h\neq \ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t) \\
& +\mu_i(t)\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{i\ell}(t)\Big[\mathrm{E}^2_{ij,k}(t) - \mathrm{E}_{ij,k}(t)\Big] + \mu_j(t)\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\Big[\mathrm{E}^2_{ji,k}(t) - \mathrm{E}_{ij,k}(t)\Big] \\
& +\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t) + \mathrm{E}_{jh,k}(t)\Big] + \sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t) + \mathrm{E}_{ih,k}(t)\Big] \\
& +b_{ji}(t)\mu_j(t)\Big[\mathrm{E}^2_{ji,k}(t) - \mathrm{E}_{ij,k}(t) + \mathrm{E}^2_{j,k}(t) - \mathrm{E}_{j,k}(t)\Big] \\
& +b_{ij}(t)\mu_i(t)\Big[\mathrm{E}^2_{ij,k}(t) - \mathrm{E}_{ij,k}(t) + \mathrm{E}^2_{i,k}(t) - \mathrm{E}_{i,k}(t)\Big] \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) \\
= \; & -\lambda_k(t)\mathrm{E}_{ij,k}(t) \\
& -\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)\mathrm{E}_{ij\ell,k}(t) + \sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t) b_{\ell\ell}(t)\mathrm{E}_{ij\ell,k}(t) + \sum_{\ell=1}^{m_B} \mu_\ell(t) b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t) \\
& -\mu_i(t)\mathrm{E}^2_{ij,k}(t) - \mu_j(t)\mathrm{E}^2_{ji,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A}\Big\{a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) + \beta_i(t)\mathrm{E}_{j,\ell}(t) + \beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\} \\
& -b_{i,m_B+1}(t)\mu_i(t)\mathrm{E}_{ij,k}(t) - b_{j,m_B+1}(t)\mu_j(t)\mathrm{E}_{ij,k}(t)
\end{aligned}
$$

$$+\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B}\ \sum_{\substack{h=1\\ h\neq\ell\\ h\neq i\\ h\neq j}}^{m_B} b_{h\ell}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)$$

$$+\mu_i(t)\Big\{\sum_{\substack{\ell=1\\ \ell\neq i}}^{m_B} b_{i\ell}(t)\Big\}\Big[\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big]$$

$$+\mu_j(t)\Big\{\sum_{\substack{\ell=1\\ \ell\neq j}}^{m_B} b_{j\ell}(t)\Big\}\Big[\mathrm{E}^2_{ji,k}(t)-\mathrm{E}_{ij,k}(t)\Big]$$

$$+\sum_{\substack{h=1\\ h\neq i}}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{jh,k}(t)\Big]-b_{ji}(t)\mu_j(t)\mathrm{E}^2_{ji,k}(t)$$

$$+\sum_{\substack{h=1\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t)+\mathrm{E}_{ih,k}(t)\Big]-b_{ij}(t)\mu_i(t)\mathrm{E}^2_{ij,k}(t)$$

$$-b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t)-b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t)$$

$$+\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)$$

$$=\ -\lambda_k(t)\mathrm{E}_{ij,k}(t)$$

$$-\sum_{\substack{\ell=1\\ \ell\neq i\\ \ell\neq j}}^{m_B} \mu_\ell(t)\mathrm{E}_{ij\ell,k}(t)+\sum_{\ell=1}^{m_B}\mu_\ell(t)b_{\ell m_B+1}(t)\mathrm{E}_{ij\ell,k}(t)$$

$$-\mu_i(t)\mathrm{E}^2_{ij,k}(t)-\mu_j(t)\mathrm{E}^2_{ji,k}(t)$$

$$+\alpha_k(t)\sum_{\ell=1}^{m_A}\Big\{a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)+\beta_i(t)\mathrm{E}_{j,\ell}(t)+\beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\}$$

$$-b_{i,m_B+1}(t)\mu_i(t)\mathrm{E}_{ij,k}(t)-b_{j,m_B+1}(t)\mu_j(t)\mathrm{E}_{ij,k}(t)$$

$$+\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B}\mu_h(t)\mathrm{E}_{ijh,k}(t)-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hi}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{hj}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)$$

$$-\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{h,m_B+1}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t)$$

$$+\mu_i(t)\Big(\mathrm{E}^2_{ij,k}(t)-\mathrm{E}_{ij,k}(t)\Big)\Big(1-b_{i,m_B+1}(t)\Big)$$

$$
\begin{aligned}
& +\mu_j(t)\Big(\mathrm{E}^2_{ji,k}(t) - \mathrm{E}_{ij,k}(t)\Big)\Big(1 - b_{j,m_B+1}(t)\Big) \\
& +\sum_{h=1}^{m_B} b_{hi}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t) + \mathrm{E}_{jh,k}(t)\Big] + \sum_{h=1}^{m_B} b_{hj}(t)\mu_h(t)\Big[\mathrm{E}_{ijh,k}(t) + \mathrm{E}_{ih,k}(t)\Big] \\
& -b_{ji}(t)\mu_j(t)\mathrm{E}^2_{ji,k}(t) - b_{ij}(t)\mu_i(t)\mathrm{E}^2_{ij,k}(t) \\
& -b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t) - b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t) \\
& -\mu_i(t)b_{ii}(t)\mathrm{E}^2_{ij,k}(t) - \mu_j(t)b_{jj}(t)\mathrm{E}^2_{ji,k}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) \\
= \; & -\lambda_k(t)\mathrm{E}_{ij,k}(t) \\
& +\sum_{\ell=1}^{m_B} \mu_\ell(t)b_{\ell,m_B+1}(t)\mathrm{E}_{ij\ell,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A}\Big\{a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) + \beta_i(t)\mathrm{E}_{j,\ell}(t) + \beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\} \\
& -\mu_i(t)\mathrm{E}_{ij,k}(t) - \mu_i(t)b_{i,m_B+1}(t)\mathrm{E}^2_{ij,k}(t) - \mu_j(t)\mathrm{E}_{ij,k}(t) - \mu_j(t)b_{j,m_B+1}(t)\mathrm{E}^2_{ji,k}(t) \\
& -\sum_{\substack{h=1\\ h\neq i\\ h\neq j}}^{m_B} b_{h,m_B+1}(t)\mu_h(t)\mathrm{E}_{ijh,k}(t) \\
& +\sum_{h=1}^{m_B} b_{hi}(t)\mu_h(t)\mathrm{E}_{jh,k}(t) + \sum_{h=1}^{m_B} b_{hj}(t)\mu_h(t)\mathrm{E}_{ih,k}(t) \\
& -b_{ji}(t)\mu_j(t)\mathrm{E}_{ij,k}(t) - b_{ij}(t)\mu_i(t)\mathrm{E}_{ij,k}(t) \\
& +\sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) \\
= \; & \sum_{h=1}^{m_B} b_{hi}(t)\mu_h(t)\mathrm{E}_{jh,k}(t) + \sum_{h=1}^{m_B} b_{hj}(t)\mu_h(t)\mathrm{E}_{ih,k}(t) \\
& -b_{ij}(t)\mu_i(t)\mathrm{E}_{i,k}(t) - b_{ji}(t)\mu_j(t)\mathrm{E}_{j,k}(t) \\
& -\Big[\mu_i(t) + \mu_j(t)\Big]\mathrm{E}_{ij,k}(t) \\
& +\alpha_k(t)\sum_{\ell=1}^{m_A}\Big\{a_{\ell,m_A+1}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t) + \beta_i(t)\mathrm{E}_{j,\ell}(t) + \beta_j(t)\mathrm{E}_{i,\ell}(t)\Big\}
\end{aligned}
$$

$$-\lambda_k(t)\mathrm{E}_{ij,k}(t) + \sum_{\ell=1}^{m_A} a_{\ell k}(t)\lambda_\ell(t)\mathrm{E}_{ij,\ell}(t)$$

□