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Steps to the Barrier

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The virtues of barrier options that knock in and knock out gradually.

Knock-out options relate to ordinary options "the way crack relates to cocaine", George Soros said in his 1995 book. He went on to explain why he thought they should be banned: "I would not have said that a few months ago, when I testified before Congress, but we have had a veritable crash in currency markets since then. As I have said before, knock-out options played the same role in the 1995 yen explosion as portfolio insurance did in the stock market crash of 1987, and for the very same reason. Portfolio insurance was subsequently rendered inoperable by the introduction of the so-called circuit breakers. Something similar needs to be done now with knock-out options."

While there is no doubt that knock-out options can have adverse effects on the markets, there is a solution, "step options" - barrier options with a finite knock-out (or knock-in) rate - which we put forward in this article.

Barrier options are attractive to investors because they are cheaper than vanilla contracts. By including a barrier provision in the option contract, an investor can avoid paying for those scenarios he feels are unlikely, and the reduction in premium can be substantial, especially when volatility is high. However, these benefits come at a cost. Knock-out options are extinguished when the price of the underlying asset hits a pre-specified price level (barrier) from above, for down-and-out options, or below, for up-and-out options (Rubinstein and Reiner, 1991, give closed-form pricing formulas for all eight types of barrier options).

The discontinuity at the barrier inherent in knock-out contracts creates risk management problems both for option buyers and sellers. An erroneous price movement through the barrier can extinguish the option, leaving the buyer without his position. Even if the investor is generally right on market direction, an accidental price spike can lead to the loss of his entire investment.

Furthermore, when large positions of options with the same barrier level are accumulated in the market, traders can drive the underlying to the barrier, thus triggering it and creating massive losses. This is vividly illustrated by the events in the foreign exchange market in 1995. According to the *Wall Street Journal* of May 5, 1995: "Knock-out options can roil even the mammoth foreign exchange markets for brief periods. David Hale, chief economist at Kemper Financial in Chicago, notes that in the past year, many Japanese exporters moved to hedge against a falling dollar with currency options. Confident at the time that the dollar would fall no further than 95 yen, the exporters chose options that would knock out at that level. Once the dollar plunged through 95 yen early last month, 'they lost everything', he says. The dollar then tumbled as the Japanese companies, 'which had lost their hedges, scrambled to cover' their large exposures by dumping dollars.

"Making matters more volatile, dealers say that pitched battles often erupt around knock-out barriers, with traders hollering across the trading floor of looming billion-dollar transactions ... In three or four minutes, it is all over. But, in that time, every trade gets sucked into the vortex."

This situation prompted some market participants to appeal for regulation of barrier options. As we have seen, George Soros went as far as to suggest an all-out ban.

The barrier option's delta is discontinuous at the barrier, thus creating hedging problems for options sellers as well. To hedge barrier options, dealers establish positions in a series of standard vanilla options that provide a good hedge for a wide range of underlying prices. However, when the underlying nears the barrier level, these static hedges need to be rebalanced, which results in a flurry of trading activity in vanilla options. This, in turn, results in further trading activity in the underlying asset as dealers who sold vanilla options to hedgers of knock-out options need to hedge their exposure dynamically. This increases market volatility around popular barrier levels and increases the cost of hedging barrier options.

Thus it is desirable to modify the barrier provision to retain as much of the premium savings afforded by the standard barrier provision as possible, but at the same time to achieve continuity of both the option's payout and delta at the barrier. And this is where the "step option" comes in.

Step options

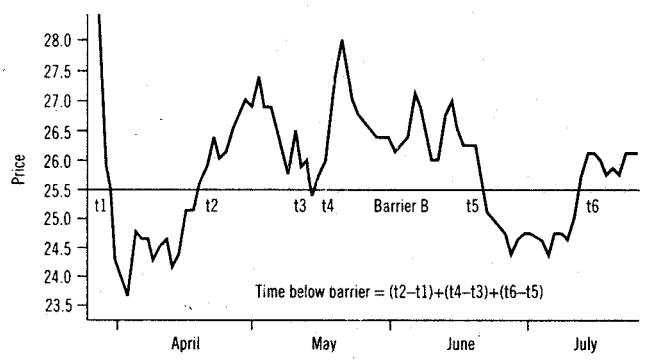
Consider a standard call with strike K and time to expiry $\tau = T - t$, where t and T denote the contract inception and expiry times respectively. A down-and-out provision renders the option worthless as soon as the underlying price hits a pre-specified barrier level B. Accordingly, the payout of a down-and-out call can be written as:[1]

$$1_{\{L_T > B\}} \max(S_T - K, 0) \qquad (1)$$

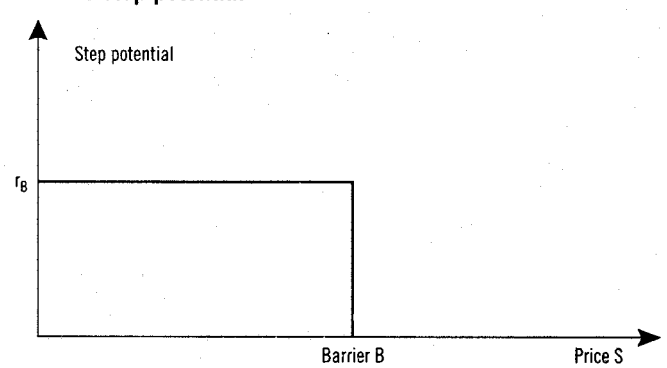
where S_T is the underlying price at expiry, L_T is the lowest price of the underlying between inception t and expiry T, and $1_{\{L_T > B\}}$ is the indicator function equal to one if $L_T > B$ and zero otherwise.

The stochastic model of barrier options is that of Brownian motion instantaneously killed when the barrier is hit. To eliminate the discontinuity, we consider Brownian motion with killing at a finite rate below the barrier (see, for example, Karlin and Taylor, 1981).[2]

1. Calculation of occupation time below a specified barrier level B



2. The step potential



We introduce a finite knock-out rate r_B and define the payout by the formula (Linetsky, 1996):

$$\exp(-r_B \tau_B^-) \max(S_T - K, 0) \qquad (2)$$

where τ_B^- is the total time during the life of the option when the underlying price is lower than the barrier level B (occupation time of the underlying price process). See, for example, Karatzas and Shreve (1992).

Figure 1 illustrates the calculation of τ_B^-. At expiry, the payout of an otherwise identical vanilla option is first determined and then discounted at continuously compounded rate r_B for the time the underlying spent below the barrier during the option's lifetime. We call the discount factor $\exp(-r_B \tau_B^-)$ the knock-out factor. It is defined on price paths and can be represented in the form:

$$\exp(-r_B \tau_B^-) = \exp\left(-\int_t^T r_B H(B - S_u)\,du\right) \qquad (3)$$

where H(x) is the Heaviside step function defined by $H(x) := 1(0)$ for $x \geq 0$ $(x < 0)$. The integrand $r_B H(B-S)$ is called the step potential. It has the shape of a step of height r_B (see Figure 2). Accordingly, we call a one-parameter family of path-dependent options with the payout given by equation (2) *step options*. It is easy to see that the payout (2) coincides with the standard European-style call $\max(S_T - K, 0)$ in the limit of a zero knock-out rate, and tends to the payout of an otherwise identical barrier option (1) in the limit of an infinitely high knock-out rate. Thus we have a sequence of step options approaching the barrier option - hence the headline of this article.[3]

A holder of a down-and-out step option is penalised at rate r_B for the time the underlying price spends below the barrier. For a standard barrier, the knock-out rate is infinitely high and the entire option payout is instantaneously lost by the option holder should the underlying price hit the barrier even momentarily. For any finite knock-out rate, however, it takes some time below the barrier to reduce the option payout to close to zero: the option knocks out gradually.

We define a 90% knock-out time, T_B^-, as time below the barrier needed to reduce the terminal payout of a down-and-out step option by 90%, ie, $\exp(-r_B T_B^-) = 0.1$. That is, the holder of a down-and-out step call receives only 10% of the payout of an otherwise identical vanilla call at expiry if the asset spent time T_B^- below the barrier during the life of the contract. Another useful measure of knock-out speed is a single-day knock-out factor $\beta_B = \exp(-r_B/250)$, where we have assumed 250 trading days a year. This is a factor by which the terminal payout is discounted for every trading day the underlying

spends below the barrier. Obviously, β_B is zero for barrier options and one for vanilla options.

One of the advantages of step options is the ability to structure contracts with any desired knock-out rate. By choosing a finite rate, an option buyer assures himself that the option will never lose its entire value due to a short-term price movement. An investor can customise the option by selecting an knock-out rate appropriate to his risk aversion and the degree of confidence in the barrier not being hit during the option's lifetime. On the other hand, since the step option delta is continuous at the barrier, the advantage for the dealer is the ability to hedge step options by trading the underlying. Thus, step options with finite knock-out rates (gradual knock-outs) have risk management advantages both for buyers and sellers.

Let us also mention another positive effect of regularising barrier options by introducing finite knock-out rates. Since different market participants will select different rates, even though they may all set the barrier at the same obvious support or resistance level, any short-term manipulation by traders will not cause massive simultaneous knock-outs. This would help reduce the volatility around popular barrier levels.

One may also wish to consider other payout structures of the form:

$$f(\tau_B^-)\max(S_T - K,0) \qquad (4)$$

where f is a given function of the occupation time (occupation time-based principal amortising factor). Here, we assume $f(0) = 1$. The *exponential step call* shown in equation (2) is a particularly simple example with $f = \exp(-r_B\tau_B^-)$. It corresponds to continuous amortising at knock-out rate r_B below the barrier. Another choice is a simple linear amortisation:[4]

$$\max(1 - R_B\tau_B^-,0)\max(S_T - K,0) \qquad (5)$$

We call options with the payout (5) linear step options. The optionality in occupation time is needed to limit the option buyer's liability to the premium paid for the option. Thus, linear step options are options on both the terminal asset price and the occupation time. A knock-out time T_B^- for the linear step option is defined as the minimum occupation time below the barrier needed to reduce the option payout to zero, $T_B^- = 1/R_B$. The option buyer just needs to specify the desired knock-out time as a fraction of time to maturity.

Valuation

To value step options we assume the underlying asset price follows a geometric Brownian motion with volatility σ and continuous dividend yield q, and we live in the Black-Scholes world with constant continuously compounded risk-free interest rate r. According to the risk-neutral valuation approach, at inception t the present value of an option with payout (4) is given by the discounted conditional expectation under the risk-neutral measure:

$$e^{-r\tau}E_{t,S}\left[f(\tau_B^-)\max(S_T - K,0)\right] \qquad (6)$$

This expectation can be calculated in closed form for any function f by using the Feynman-Kac approach and solving the corresponding parabolic partial differential equation (Linetsky, 1998). The resulting closed-form pricing formula for the down-and-out step call (Linetsky, 1996) is, in the case $S \geq B$:

$$\mathrm{Step}(S,t) = \mathrm{DOC}(S,t) + \left(\tfrac{B}{S}\right)^{\gamma}\int_0^{\tau}\frac{F(\tau-\tau')e^{-\alpha(\tau-\tau')}}{\sqrt{2\pi}(\tau-\tau')^{3/2}}$$
$$\left(v_2e^{-q\tau'}\left(\tfrac{B^2}{S}\right)N(d_2') - v_1e^{-r\tau'}KN(d_1')\right)d\tau' \qquad (7)$$

where DOC(S,t) is the standard down-and-out call (Rubinstein and Reiner, 1991), N(x) and n(x) are the cumulative standard normal distribution function and its density, respectively, and we have introduced the following notation:

$$\mu = r - q - \tfrac{\sigma^2}{2}, \gamma = \tfrac{2\mu}{\sigma^2},$$
$$\alpha = r + \tfrac{\mu^2}{2\sigma^2}, v_1 = \frac{\mu}{\sigma}, v_2 = v_1 + \sigma,$$
$$d_1' = \tfrac{\ln\left(\frac{B^2}{SK}\right)+\mu\tau'}{\sigma\sqrt{\tau'}}, d_2' = d_1' + \sigma\sqrt{\tau'},$$
$$d_3' = \tfrac{\ln\left(\frac{B}{K}\right)+\mu\tau'}{\sigma\sqrt{\tau'}}, d_4' = d_3' + \sigma\sqrt{\tau'} \qquad (8)$$

In the case $S < B$, we have:

$$\mathrm{Step}(S,t) = \frac{B^{\frac{2}{\gamma}}}{S}\int_0^{\tau}\frac{F(\tau-\tau')e^{-\alpha(\tau-\tau')}}{\sqrt{2\pi}(\tau-\tau')^{3/2}} \times \left\{v_1\rho_1(y)e^{-r\tau'}KN(d_3'\right.$$
$$\left. -v_2\rho_2(y)e^{-q\tau'}BN(d_4') - \sigma ye^{-q\tau'}B(\tau') - \tfrac{1}{\gamma}n(d_4')\right\}$$
$$\exp\left(\frac{y^2}{2(\tau-\tau')}\right)d\tau' \qquad (9)$$

where:

$$y = \frac{1}{\sigma}\ln\left(\frac{S}{B}\right),$$
$$\rho_1(y) = \frac{y^2}{\tau-\tau'} + v_1y - 1, \rho_2(y) = \rho_1(y) + \sigma y \qquad (10)$$

The function $F(\tau - \tau')$ in equations (7) and (9) is obtained from the discount function f in equation (4) by integration:

$$F(\tau - \tau') = \int_0^{\tau-\tau'} f(\tau'')d\tau'' \qquad (11)$$

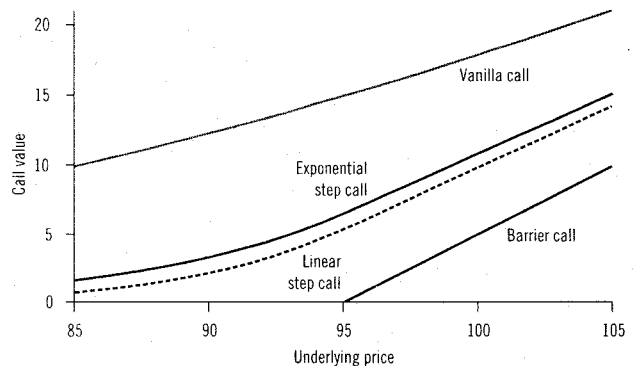
and for the exponential (2) and linear (5) step options it is given by:

$$F_{exp}(\tau - \tau') = \frac{1 - e^{-r_B(\tau-\tau')}}{r_B} \qquad (12)$$

Table 1. Call values and deltas as functions of the underlying price S

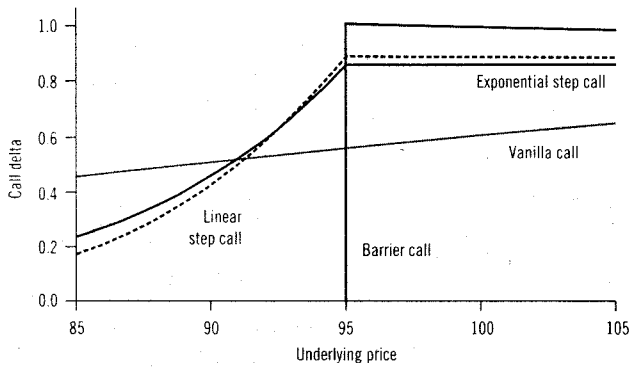
S	C	Δ	EStep	Δ	LStep	Δ	DAO	Δ
85	9.8517	0.4554	1.6062	0.2376	0.7200	0.1730	0	0
90	12.2641	0.5091	3.2951	0.4602	2.1528	0.4291	0	0
95	14.9373	0.5597	6.5008	0.8598	5.3548	0.8908	0	1.0058
100	17.8551	0.6068	10.7942	0.8583	9.7953	0.8862	4.9958	0.9932
105	20.9994	0.6503	15.0904	0.8607	14.2229	0.8855	9.9376	0.9841

3. Vanilla, down-and-out exponential step, linear step and barrier call values*



* As functions of the current asset price S. The option parameters are: K = 100, B = 95, σ = 0.6, r = 0.05, τ = 0.5 (six months). The exponential step call parameters are: β_B = 0.9 (r_B = 26.34, T_B = 21.85 trading days). The linear step call parameters are: ρ_B = 0.1 (R_B = 25, T_B = 10 trading days, β_B = 0.9)

4. Vanilla, down-and-out exponential step, linear step and barrier call deltas*



* As functions of the current asset price S. The option parameters are: K = 100, B = 95, σ = 0.6, r = 0.05, τ = 0.5 (six months). The exponential step call parameters are: β_B = 0.9 (r_B = 26.34, T_B = 21.85 trading days). The linear step call parameters are: ρ_B = 0.1 (R_B = 25, T_B = 10 trading days, β_B = 0.9)

$$F_{lin}(\tau-\tau') = \begin{cases} \frac{1}{2R_B}, & 0 \le \tau' \le \tau - \frac{1}{R_B} \\ (\tau-\tau')\left[1-\frac{R_B}{2}(\tau-\tau')\right], & \tau - \frac{1}{R_B} < \tau' \le \tau \end{cases} \quad (13)$$

We see from equation (7) that the step option price consists of two parts: the standard down-and-out call and a step premium that investors have to pay for the privilege of having the option knock out gradually with the pre-specified finite rate. The higher the knock-out rate, the lower the premium. In the limits $r_B \to \infty$ ($R_B \to \infty$) the options knock out instantly as soon as the barrier is hit, and the step premium is equal to zero.

The lower the knock-out rate, the higher the premium; in the limit of a zero knock-out rate the premium is the highest and such that the step option coincides with an otherwise identical vanilla option. The knock-out rate controls the trade-off between premium savings and knock-out speed. A crucial property of step options is that, unlike standard barrier options, their deltas are continuous at the barrier for any finite knock-out rate r_B (Linetsky, 1996), allowing us to replicate step options dynamically by trading the underlying asset and borrowing.

An example calculation is given in Table 1. The option parameters are: K = 100, B = 95, τ = 0.5 (six months), σ = 0.6, q = 0 and r = 0.05. The continuously compounded knock-out rate r_B for the exponential step call is chosen so that the single-day knock-out factor is β_B = 0.9 (r_B = $-250\ln\beta_B$ = 26.34), ie, 10% of the payout is lost in the first trading day below the barrier. The corresponding 90% knock-out time is about 22 days, ie, 90% of the payout is lost if the underlying spends 22 trading days below the barrier during the contract's life.

For comparison, the simple knock-out rate R_B for the linear step call is also chosen so that 10% of the payout is lost in the first day below the barrier, $R_B = 250(1-\beta_B) = 25$. The knock-out rate per trading day is defined as $\rho_B := R_B/250$, and is equal to 0.1 in our example. The linear step option knocks out faster and is extinguished after 10 days below the barrier. Table 1 and Figures 3 and 4 show vanilla (C), exponential step (EStep), linear step (LStep) and standard barrier (DAO) call values and deltas as functions of the underlying price S. Figure 3 shows that the step option value holds up well when the underlying falls slightly below the barrier, but deteriorates quickly as the underlying continues to fall further, as the probability of getting back up above the barrier decreases and the expected value of the occupation time below barrier increases. Figure 4 illustrates continuity of the step option delta at the barrier.

Related exotic options

Another example of the payout of the form (4) is given by:

$$1_{\{\tau_B^- \le \alpha\tau\}} \max(S_T - K, 0) \quad (14)$$

for a given constant α, $0 \le \alpha \le 1$. This option is a down-and-out call that knocks out when the occu-

pation time below the barrier exceeds the fraction α of the option's lifetime. The closed-form pricing formula is given by the general expressions (7) and (9), where the function F specifies:

$$F_{delay}(\tau-\tau') = \begin{cases} \alpha\tau, & 0 \le \tau' \le (1-\alpha)\tau \\ \tau-\tau', & (1-\alpha)\tau < \tau' \le \tau \end{cases} \quad (15)$$

Delayed barrier options and step options (2) and (5) have quite different properties. Step options knock out gradually, amortising their principal for each unit of time the underlying is below the barrier. In contrast, the holder of a delayed barrier option receives either the full payout from an otherwise identical vanilla option (if the occupation time did not exceed the fraction α of the option's lifetime) or nothing.

Several new classes of exotic options have recently appeared in the literature that either pursue similar goals of reducing the knock-out risk of barrier options or involve occupation times for other purposes. The Parisian barrier options of Chesney et al (1997) are delayed barrier options based on the age of the excursion of price process below (or above) a given barrier. The owner of a down-and-out Parisian option loses his entire position if the underlying falls below the barrier and stays continuously below for longer than a specified delay. It is similar to the occupation time-based delayed barrier (14), but only the current excursion is relevant. Based on Brownian excursions theory, Chesney et al derive an analytical expression for the Laplace transform of a relevant probability density, and Parisian options are priced via numerical inversion of the Laplace transform.

Quantile options, introduced by Miura (1992) and studied by Embrechts, Rogers and Yor (1995), are options on the α-quantile of the price process, which can be interpreted as the lowest barrier level B such that the occupation time τ_B^- is greater than a given fraction α of the option's lifetime. Continuous barrier range (soft barrier) options introduced by Hart and Ross (1994) replace a single discrete barrier level B with a continuous barrier range from B_{min} to B_{max}. The range, switch and corridor options studied by Pechtl (1995) are similar to the range notes popular in the fixed-income market: their payouts are linear in occupation time and can be represented as strips of binary options. A variety of corridor options, as well as related volatility contracts, are studied by Carr (1998).

1 *For brevity in this paper, we discuss down-and-out calls only. Further details are given in Linetsky (1996).*

2 *Another interpretation in physics is that of a quantum particle in an infinitely high potential barrier. Here we consider a finite rather than infinite potential barrier, or a step potential of finite height (see, for example, Messiah, 1961, or Landau and Lifshitz, 1965).*

3 *I am grateful to Jim Bodurtha for suggesting this title*

4 *I am grateful to Vladimir Finkelstein and Eric Reiner for suggesting this payout structure.*

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