Shadow interest

Using a Vasicek process for the shadow rate, Viatcheslav Gorovoi and Vadim Linetsky develop an analytical solution for pricing zero-coupon bonds using eigenfunction expansions, and show how to calibrate their model to the Japanese bond market. This article is not the last word on the subject – in particular, the relationship between shadow interest rates, real rates and inflation should be explored – but we hope it will encourage further research

The current short rate in Japan is zero, while the Federal Funds rate in the US stands at 1% after the June 2003 rate cut. How does one model the term structure of interest rates when the short rate is low or zero? In this article, we report on a class of models recently developed by Gorovoi & Linetsky (2002), based on Black's (1995) idea of the non-negative nominal interest rate as an option on the shadow interest rate that is allowed to get negative.

Gaussian interest rate models, starting with Vasicek (1977), are often used to price interest rate derivatives. It is often said that there is no need to worry about negative rates, as the probability of them occurring is small. While this is true in some cases, Rogers (1996) shows that some derivatives' prices are very sensitive to the possibility of negative rates. For such derivatives, prices obtained with the Gaussian models can be absurd (for example, in the Vasicek model, zero-strike floorlets have substantial values when the short rate is near zero).

An alternative class of models uses diffusion processes for the short rate with the property that zero is an unattainable boundary. The dynamics of the short rate is restricted to the positive half-line and the short rate can never reach zero. The Black & Karasinski (1991) model is an important example. These models have a lognormal instantaneous short-rate volatility and a mean-reverting drift. A common feature of such models is that the volatility declines rapidly as the rate approaches zero, thus switching off the diffusion term and allowing the mean-reverting drift to pull the process away from zero, making zero unattainable. The square-root Cox, Ingersoll & Ross (1985) model is a borderline case. When the mean-reverting drift is large enough relative to the volatility, the rate cannot reach zero. Otherwise, the rate can reach zero, and one must decide on the boundary condition at zero. More precisely, if $2\kappa\theta \geq \sigma^2$, then zero is an unattainable entrance boundary for the process. Otherwise, it is an attainable regular boundary, and a boundary condition must be specified. In both cases, the short-rate volatility declines as the short rate decreases towards zero, because volatility is the result of a square root.

Volatility structures that vanish as the short rate falls to zero contradict empirical evidence. Over the past few years, short-term interest rates in Japan have stayed near zero. However, their volatility remained quite high throughout the period (Goldstein & Keirstead, 1997). Furthermore, after a series of rate cuts, the US federal funds rate stands at 1%. Obviously, both the Gaussian models that allow the short rate to become negative and the lognormal or square-root models with zero an unattainable boundary are inadequate in the current zero short-rate regime in Japan, and are coming into question in the low-rate regimes in the US and Europe.

To model the low-rate regime, one would like a model where the short rate stays non-negative (although it could become zero) and, at the same time, has non-vanishing volatility at low rates. Black (1995) has put forward the following idea to model nominal interest rates as options. He argued that the short rate cannot become negative because currency is an option: when an instrument has a negative rate, we can choose currency instead. Thus, we can treat the short rate itself as an option: we can choose an underlying process that can take negative values and simply replace all the negative values with zeros (take the positive part). We still have a one-factor process: either the short rate (when the underlying process is positive or zero) or what the short rate would be without the currency option (when the underlying process is negative). Black called this process the shadow short rate. In such a model, the shadow short rate can become negative, the nominal short rate is a positive part of the shadow rate, and all term rates are strictly positive. A similar idea was independently discussed by Rogers (1995).

Based on this idea, Gorovoi & Linetsky (2002) have recently developed a class of models where the shadow rate is modelled as a one-dimensional diffusion that is allowed to get negative and the nominal rate is identified with the positive part of the shadow rate. Here, we discuss the model assuming the shadow rate follows the Vasicek process. The general case where the shadow rate is modelled as a one-dimensional diffusion, as well as analytical solutions for affine models, are given in Gorovoi & Linetsky (2002). We refer the reader to that paper for details.

The model

We assume that, under the risk-neutral probability measure, the shadow rate $\{X_t, t \geq 0\}$ follows an Ornstein-Uhlenbeck (OU)/Vasicek process on the real line:

$$dX_t = \kappa(\theta - X_t)dt + \sigma dB_t, \quad X_0 = x \tag{1}$$

where $\theta > 0$ is the long-run level of the shadow rate, $\kappa > 0$ is the rate of mean reversion towards the long-run level and $\sigma > 0$ is the volatility parameter. In Gorovoi & Linetsky (2002), we develop a general framework where the shadow rate is assumed to follow a one-dimensional diffusion process with some infinitesimal drift $\mu(x)$ and variance $\sigma^2(x)$. Here, we restrict ourselves to the Vasicek shadow rate.

Following Black (1995) and Rogers (1995), we assume that the nominal short rate $\{r_t, t \geq 0\}$ is the positive part of the shadow rate ($x^+ \equiv \max\{x, 0\}$):

$$r_t = X_t^+, \quad t \geq 0 \tag{2}$$

A zero-coupon bond pays \$1 at maturity $\tau > 0$. Its price at time zero is given by the risk-neutral expectation:

$$P(x,\tau) = E_x\left[e^{-\int_0^\tau r_u du}\right] = E_x\left[e^{-\int_0^\tau X_u^+ du}\right] \tag{3}$$

where $E_x[\cdot] \equiv \mathrm{E}[\cdot | X_0 = x]$. The corresponding yield-to-maturity $R(x, \tau)$ is defined as usual: $R(x, \tau) := -\ln P(x, \tau)/\tau$. Since the short rate is always non-negative, the zero-coupon bond prices $P(x, \tau)$ given by equation (3) are always strictly less than one for any finite time to maturity $\tau > 0$. Moreover, for each fixed x the zero-coupon bond pricing function $\{P(x, \tau), \tau \geq 0\}$ is strictly decreasing on $[0, \infty)$ with $P(x, 0) = 1$, and all term rates (yields) $R(x, \tau)$ are strictly positive for all $\tau > 0$. This ensures that the model is free from arbitrage opportunities present in models with negative short rates.

We note that the zero-coupon bond price (3) has the form of the Laplace transform (evaluated at the unit value of the transform parameter) of an area functional of the shadow rate diffusion:

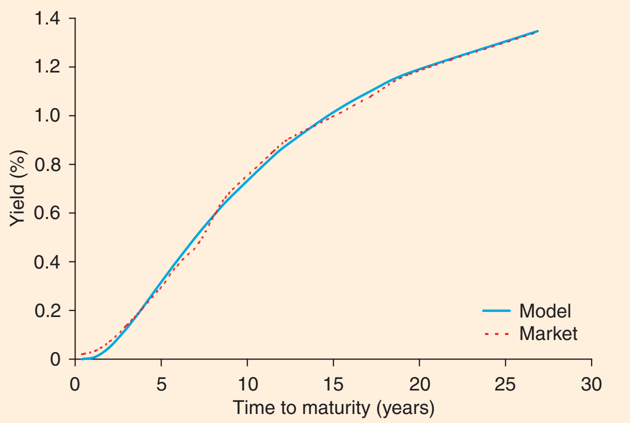
$$A_t := \int_0^t X_u^+ du, \quad t \geq 0 \tag{4}$$

A. Calibration to JGB data

Coupon (%)	Maturity	Price	Bootstrapped yield (%)	Model yield (%)
0	Sep 10, 2003	99.992	0.02	0.00
0.1	Jun 21, 2004	100.080	0.03	0.01
0.1	Apr 20, 2005	100.059	0.07	0.05
0.5	Mar 20, 2006	101.080	0.13	0.12
0.5	Sep 20, 2006	101.140	0.17	0.17
0.5	Mar 20, 2007	101.150	0.21	0.21
1.9	Mar 20, 2008	107.940	0.29	0.31
1.9	Mar 20, 2009	108.970	0.38	0.40
1.9	Jun 21, 2010	110.166	0.48	0.52
1.5	Jun 20, 2011	107.300	0.60	0.60
1.5	Mar 20, 2012	107.290	0.68	0.66
4.5	Sep 22, 2014	140.680	0.85	0.83
3.7	Sep 21, 2015	133.710	0.91	0.89
2.7	Mar 20, 2018	124.460	0.99	1.01
2.2	Jun 22, 2020	118.420	1.08	1.10
1.9	Sep 20, 2022	113.650	1.17	1.18
2.4	Feb 20, 2030	125.850	1.34	1.35

Note: the first three columns give JGB data from Bloomberg, including coupon, maturity and prices on April 9, 2003. The fourth and fifth columns give boot-strapped zero-coupon yields and calibrated model yields. Calibrated Vasicek shadow rate process parameters are: $\theta = 0.008$ (0.8%), $\kappa = 0.18$, $\sigma = 0.026$, $x = -0.056$ (–5.6%)

1. JGB yield curve



Zero-coupon yield curve bootstrapped from JGB data on April 9, 2003 and calibrated Black's model of interest rates as options zero-coupon yield curve. Calibrated Vasicek shadow rate risk-neutral process parameters are: $\theta = 0.008$ (0.8%), $\kappa = 0.18$, $\sigma = 0.026$, $x = -0.056$ (–5.6%). JGB data from Bloomberg

The area functional measures the area below the positive part of a sample path of the process up to time t. Perman & Wellner (1996) study area functionals of standard Brownian motion. For Brownian motion with drift, the area functional has appeared in the finance literature in connection with the valuation of executive stock options in Carr & Linetsky (2000). Here, we are interested in area functionals of the OU process. Namely, we need to calculate the zero-coupon bond price $P(x, \tau) = E_x[e^{-A_\tau}]$.

The eigenfunction expansion solution

To calculate prices of zero-coupon bonds and interest rate derivatives, we use the spectral expansion approach described in detail in Linetsky (2002a) (see also Davydov & Linetsky, 2003; Goldstein & Keirstead, 1997; Gorovoi & Linetsky, 2002; Lewis, 1994 and 1998; Linetsky, 2001, 2002b and 2000c; Lipton, 2001 and 2002; and Lipton & McGhee, 2002, for applications of the eigenfunction expansion method to derivatives pricing). Below, we give the final result for the model with the Vasicek shadow rate. See Gorovoi & Linetsky (2002) for details.

The zero-coupon bond pricing function $P(x, \tau)$ as a function of time to maturity τ and shadow rate x solves the fundamental pricing partial differential equation:

$$\frac{1}{2}\sigma^2 P_{xx} + \kappa(\theta - x)P_x - x^+ P = P_\tau \tag{5}$$

with the initial condition $P(x, 0) = 1$. The solution has the eigenfunction expansion:

$$P(x,\tau) = E_x\left[e^{-A_\tau}\right] = \sum_{n=0}^{\infty} c_n e^{-\lambda_n \tau}\varphi_n(x) \tag{6}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ are the eigenvalues, $0 < \lambda_0 < \lambda_1 < \ldots$, $\lim_{n\uparrow\infty}\lambda_n = \infty$, and $\{\varphi_n\}_{n=0}^{\infty}$ are the corresponding eigenfunctions of the associate Sturm-Liouville (SL) spectral problem:

$$-\frac{1}{2}\sigma^2 u''(x) - \kappa(\theta - x)u'(x) + x^+ u(x) = \lambda u(x), \quad x \in \mathbb{R} \tag{7}$$

The eigenfunctions form a complete, orthonormal basis in the Hilbert space $L^2(\mathbb{R}, \mathfrak{m})$ of real-value functions square-integrable with the speed density $\mathfrak{m}(x)$ of the Vasicek diffusion:

$$\mathfrak{m}(x) = \frac{2}{\sigma^2} e^{-\frac{\kappa(\theta - x)^2}{\sigma^2}} \tag{8}$$

and endowed with the inner product $(f, g) = \int_{-\infty}^{\infty} f(x)g(x)\mathfrak{m}(x)dx$, so that $\|\varphi_n\|^2 \equiv (\varphi_n, \varphi_n) = 1$. Since the payout of the zero-coupon bond is a constant, and constants are in $L^2(\mathbb{R}, \mathfrak{m})$, the expansion coefficients c_n in equation (6) are given by:

$$c_n = (1, \varphi_n) = \int_{-\infty}^{\infty} \varphi_n(x)\mathfrak{m}(x)dx \tag{9}$$

Due to the factors $e^{-\lambda_n \tau}$ in equation (6), the longer the time to maturity τ the faster the eigenfunction expansion converges.

We have expressed the zero-coupon bond pricing function in terms of the eigenvalues λ_n and eigenfunctions φ_n of the SL problem (7). It remains to determine their explicit expressions and plug them into the eigenfunction expansion. The results are expressed in terms of the Weber-Hermite parabolic cylinder function $D_\nu(z)$ (we follow the notation of Erdelyi, 1953, pages 116–130, and Buchholz, 1969, pages 39–49). Introduce the following notation:

$$c := \frac{\sqrt{2\kappa}}{\sigma}, \quad \alpha := \sigma\sqrt{\frac{2}{\kappa^3}}, \quad \beta := c\theta, \quad \gamma := \theta - \frac{\sigma^2}{2\kappa^2} \tag{10}$$

For $\lambda \in \mathbb{C}$, define the function $w(\lambda)$ by:

$$\begin{aligned} w(\lambda) &:= \frac{c\lambda}{\kappa} D_{\frac{\lambda-\gamma}{\kappa}}(\alpha - \beta) D_{\frac{\lambda}{\kappa}-1}(\beta) \\ &+ cD_{\frac{\lambda}{\kappa}}(\beta)\left(\frac{\lambda - \gamma}{\kappa} D_{\frac{\lambda-\gamma}{\kappa}-1}(\alpha - \beta) - \frac{\alpha}{2} D_{\frac{\lambda-\gamma}{\kappa}}(\alpha - \beta)\right) \end{aligned} \tag{11}$$

This function is entire in λ, its zeros are simple and positive, and the eigenvalues λ_n of the SL problem (6) can be identified with its zeros. The corresponding continuous eigenfunctions with continuous first derivatives and normalised so that $\|\varphi_n\|^2 = 1$ are:

$$\varphi_n(x) = \begin{cases} \sqrt{\dfrac{D_{\frac{\lambda_n-\gamma}{\kappa}}(\alpha - \beta)}{w'(\lambda_n) D_{\frac{\lambda_n}{\kappa}}(\beta)}}\, e^{\frac{1}{4}(\beta - cx)^2} D_{\frac{\lambda_n}{\kappa}}(\beta - cx), & x \le 0 \\ \sqrt{\dfrac{D_{\frac{\lambda_n}{\kappa}}(\beta)}{w'(\lambda_n) D_{\frac{\lambda_n-\gamma}{\kappa}}(\alpha - \beta)}}\, e^{\frac{1}{4}(\beta - cx)^2} D_{\frac{\lambda_n-\gamma}{\kappa}}(\alpha - \beta + cx), & x \ge 0 \end{cases} \tag{12}$$

where $w'(\lambda_n) \equiv \frac{dw(\lambda)}{d\lambda}\big|_{\lambda=\lambda_n}$. The Weber-Hermite parabolic cylinder function $D_\nu(z)$ is expressed in terms of the Hermite function $H_\nu(z)$ (Lebedev, 1972, page 284):

$$D_\nu(z) = 2^{-\frac{\nu}{2}} e^{-\frac{z^2}{2}} H_\nu\left(2^{-\frac{1}{2}} z\right) \quad (13)$$

When $\nu = n$ is an integer, the Hermite function reduces to the Hermite polynomial. For any ν, real or complex, the Hermite function can be represented by an (infinite for ν not an integer) series in z and is available as a built-in function in Mathematica (HermiteH[ν, z]). For more details, see the Mathematica documentation available on the Mathematica website (*http://functions.wolfram.com/HypergeometricFunctions/HermiteHGeneral/*). To calculate the eigenfunction expansion coefficients, the single integrals in equation (9) were calculated numerically using the built-in numerical integration routine in Mathematica. For longer times to maturity, several terms in expansion (6) are enough to attain the five-decimal accuracy. As time to maturity decreases, more terms in the eigenfunction expansion have to be added to attain the same accuracy.

From equation (6), we have the following asymptotics for large times to maturity:

$$\lim_{\tau\uparrow\infty} R(x,\tau) = \lim_{\tau\uparrow\infty}\left(-\frac{1}{\tau}\ln P(x,\tau)\right) = \lambda_0 > 0 \quad (14)$$

As time to maturity increases, the yield curve flattens out and approaches the principal eigenvalue $\lambda_0 > 0$ (see Lewis, 1994 and 1998, for general discussions along these lines, as well as some interesting examples). In Black's model of interest rates as options, the principal eigenvalue is guaranteed to be strictly positive. However, in models that allow negative nominal rates, the principal eigenvalue can, in general, be negative. This can lead to absurd economic consequences. As time to maturity increases, the zero-coupon bond price blows up to infinity as the yield curve flattens out and approaches a negative asymptotic yield. In particular, this happens in the Vasicek model with $\sigma^2 > 2\kappa^2\theta$ (in the Vasicek model, $\lambda_0 = \theta - \frac{\sigma^2}{2\kappa^2}$).

Model calibration and analysis

We now calibrate Black's model with Vasicek shadow rate to the Japanese government bond (JGB) data. Table A gives JGB data from Bloomberg, including coupon, maturity and prices on April 9, 2003.[1] The fourth and fifth columns give bootstrapped zero-coupon yields and calibrated model yields. Calibrated Vasicek shadow rate process parameters are: $\theta = 0.008$ (0.8%), $\kappa = 0.18$, $\sigma = 0.026$ and $x = -0.056$ (–5.6%). Figure 1 plots JGB and calibrated model yield curves. The fit of the model to the JGB data is excellent. We have 17 data points (17 bonds), three model parameters (θ, κ and σ) plus the initial shadow rate x. We fit the model by minimising the root mean-squared error between the JGB yield curve and the model yield curve (table A). We use the built-in Mead & Nelder (1965) minimisation algorithm in Mathematica that searches for a global minimum of a function. Typical bid/ask spreads for JGBs reported on Bloomberg are 2 basis points. Our model calibrates to within 2bp for all 17 data points but one.

It is particularly notable that the current implied shadow rate is negative at –5.6%. For comparison, Gorovoi & Linetsky (2002) give calibration results on February 3, 2002: $\theta = 0.0354$, $\kappa = 0.21$, $\sigma = 0.028$ and $x = -0.051$. Comparing this with the April 2003 results presented above, we see that the shadow rate further declined from –5.1% to –5.6%, θ declined from 3.5% to 0.8%, while model parameters κ and σ did not change significantly over the 14-month period (the model seems to be relatively robust with respect to the estimates for parameters κ and σ). The observed decline in the long-run level θ over this 14-month period suggests a further extension of our model to a two-factor model with stochastic θ.

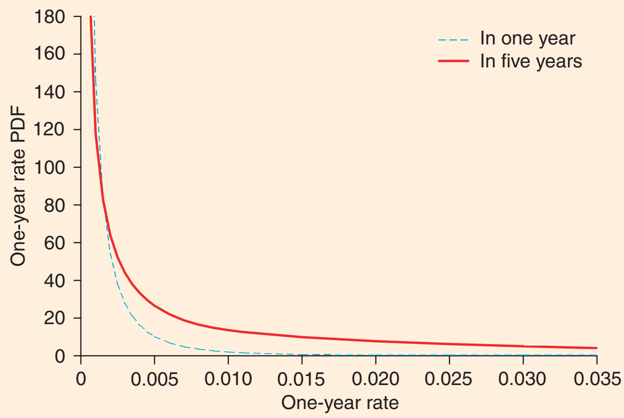
It is interesting to calculate the expected time for the shadow rate to become positive again, starting from the negative value of –5.6%. The distribution of the first hitting time of zero for the Vasicek process can be readily calculated by using the eigenfunction expansion method (see Linetsky, 2003, for details). Figure 2 plots the resulting risk-neutral probability density function of the first hitting time of zero, starting from the negative value $x = -5.6\%$. With these parameters, the risk-neutral expected time to hit zero is calculated as $E_{-0.056}[T_0] = 5.36$ years. The real-world expected time should be shorter, as the risk-neutral parameter $\theta = 0.008$ reflects the risk adjustment $\theta = \theta_{rw} - \sigma\lambda/\kappa$, assuming the constant market price of risk λ. Under the real-world probability measure, $\theta_{rw} = \theta + \sigma\lambda/\kappa > \theta$ and the mean reversion pulls the shadow rate process towards this higher value θ_{rw}.

It is also interesting to analyse distributions of future term rates. Figure 3 plots the probability density functions of the one-year rate (yield on a one-year bond) one year and five years into the future, starting with the initial shadow rate $x = -5.6\%$. While the one-year rate is strictly positive, since the initial shadow rate is negative at $x = -5.6\%$ and the mean hitting time of zero is over five years, there is a significant probability of finding the one-year rate near zero both in one year and five years. For the probability density function one year forward, most probability is concentrated in the interval between zero

2. PDF of first hitting time of zero

Probability density function of the first hitting time of zero, starting from $x = -5.6\%$. Calibrated Vasicek shadow rate process parameters are: $\theta = 0.008$ (0.8%), $\kappa = 0.18$, $\sigma = 0.026$, $x = -0.056$ (–5.6%)

3. PDF of one-year rate in one and five years



PDF of the one-year rate one year and five years in the future, starting from $x = -5.6\%$. Calibrated Vasicek shadow rate process parameters are: $\theta = 0.008$ (0.8%), $\kappa = 0.18$, $\sigma = 0.026$, $x = -0.056$ (–5.6%)

[1] *To avoid the so-called reverse coupon effect, one should try to select bonds with similar coupons. Generally, in the JGB market premium bonds with larger coupons tend to be in greater demand, which can result in some liquidity premium for those bonds*

and 0.5%, while for the probability density function five years forward the right tail is much fatter, meaningfully extending to rates above 3%.

Interest rate derivatives, numerical implementations and multi-factor extensions

While here we only consider the pricing of zero-coupon bonds, the eigenfunction expansion framework provides a methodology for the pricing of general interest rate derivatives in this model. In particular, in Gorovoi & Linetsky (2002) we consider the pricing of bond options. At the same time, the model can also be easily implemented numerically within any standard numerical environment, such as binomial and trinomial trees, numerical partial differential equation schemes, and Monte Carlo simulation. For example, in the binomial or trinomial tree framework, one first needs to implement the underlying Vasicek tree for the shadow rate, and then observe that the short rate is equal to the shadow rate at the nodes of the tree where the shadow rate is non-negative, and is equal to zero at the nodes where the shadow rate is negative. Just as with the pricing of barrier options, to achieve stable numerical convergence, one should make sure that one of the layers of nodes sits exactly at the zero shadow rate boundary $x = 0$. From the practitioner's perspective, such lattice schemes are easy to implement in practice, as they do not require access to programs such as Mathematica to calculate special functions involved in the spectral expansion solution. Furthermore, such numerical implementations readily generalise to time-inhomogeneous or multi-factor versions of our model where no analytical solutions are available. In particular, one can make the long-run level θ and/or volatility σ time-dependent or stochastic, and consider a time-inhomogeneous and/or multi-factor extension of the Gaussian Vasicek model for the shadow rate, then take the positive part to obtain the nominal rate model. Such models would allow one to accurately calibrate to a larger set of instruments, including some benchmark interest rate derivatives in addition to JGBs used here.

Conclusion

The calibrated Black model of interest rates as options, as well as its time-inhomogeneous and multi-factor extensions, can be used for pricing and hedging of interest rate derivatives in low interest rate environments, such as the one currently experienced in Japan. Furthermore, with short-term interest rates near record lows both in the US and Europe, models of this type may be required in these markets as well. ■

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