



3rd proof w author's corrns • Typesetter: RH • 3 June 2004

Computing hitting time densities for CIR and OU diffusions: applications to mean-reverting models

Vadim Linetsky

Department of Industrial Engineering and Management Sciences, McCormick School of Engineering and Applied Science, Northwestern University, 2145 Sheridan Road, Evanston, IL 60208

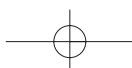
This paper provides explicit analytical characterizations for first hitting time densities for Cox–Ingersoll–Ross (CIR) and Ornstein–Uhlenbeck (OU) diffusions in terms of relevant Sturm–Liouville eigenfunction expansions. The results are applicable to the analysis of mean-reverting CIR and OU models for interest rates, credit spreads, stochastic volatility, commodity convenience yields and other mean-reverting financial variables

1 Introduction

This paper provides explicit analytical characterizations for first hitting time densities for Cox–Ingersoll–Ross (CIR) and Ornstein–Uhlenbeck (OU) diffusions in terms of relevant Sturm–Liouville eigenfunction expansions. Starting with Vasicek (1977) and Cox, Ingersoll and Ross (1985), the Gaussian Ornstein–Uhlenbeck and Feller’s (1951) square-root diffusions are among the most commonly used stochastic processes in finance. Mean-reverting OU and CIR processes are used to model interest rates (eg, Gorovoi and Linetsky, 2004), credit spreads (eg, Duffie and Singleton, 2003), stochastic volatility (eg, Heston, 1993), commodity convenience yields (eg, Schwartz, 1997), market capitalization of growth stocks (Kou and Kou, 2002) and other mean-reverting financial variables. Following the spectral expansion approach to diffusions (McKean, 1956; Itô and McKean, 1976, Section 4.11; Kent, 1980, 1982; Davydov and Linetsky, 2003; Linetsky, 2001, 2004a,b,c), we explicitly compute eigenfunction expansions for hitting time distributions for CIR and OU diffusions in terms of special functions (confluent hypergeometric and Hermite functions, respectively), give large- n asymptotics in terms of elementary functions, and provide several examples of explicit calculations in interest rate, credit risk and stochastic volatility modeling applications implemented in the *Mathematica* package.

For the CIR diffusion, the Laplace transform of the first hitting time is known (Giorno *et al.*, 1986; Göing-Jaesckke and Yor, 2003b; Leblanc and Scaillet, 1998). To recover the density, previous studies have used numerical Laplace transform

This research was supported by the U.S. National Science Foundation under grant DMI-0200429.



2 Vadim Linetsky

inversion algorithms. The eigenfunction expansion approach provides an alternative. When the diffusion has no natural boundaries, the hitting time distribution can be expressed as a Sturm–Liouville eigenfunction expansion series (Kent, 1980, Theorem 6.1). When natural boundaries are present, the situation is more complicated. The spectrum of the relevant Sturm–Liouville problem may contain some non-empty continuous spectrum and, in general, instead of the series expansion one has an integral with respect to the spectral measure (Kent, 1982). From the practical standpoint of applications in finance, the series expansion is generally preferable as no numerical integration is required (see Lewis (1998) and Linetsky (2001, 2004c) for examples of spectral expansions with continuous spectra arising in finance). Linetsky (2004a,b) studies the question of when a diffusion with natural boundaries leads to a purely discrete spectrum and establishes some sufficient conditions to this end. Fortunately, the CIR and OU diffusions both satisfy these conditions, and their hitting time densities can be represented by eigenfunction expansion series of the same form as for diffusions with no natural boundaries studied in Kent (1980). Proposition 2 of the present paper gives explicit analytical forms for these densities in terms of special functions. For the CIR process, the resulting expansions seem to be new. For the Gaussian OU process, similar expansions have previously appeared in Keilson and Ross (1975) and Ricciardi and Sato (1988), but with a different proof relying on inverting the Laplace transform. For both the CIR and OU processes we also provide convenient large- n asymptotics in terms of elementary functions.

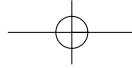
Numerical examples of calculations in *Mathematica* and applications to credit risk, interest rate, and stochastic volatility models are given in Section 3. In Section 3.1 we compute first hitting time densities to reach the long-run mean level for a mean-reverting credit spread following the CIR diffusion and starting from the values above and below its long-run mean. In Section 3.2 we compute first hitting time densities to reach the long-run mean level for a mean-reverting stochastic volatility process following the CIR diffusion. In Section 3.3 we compute the first hitting time density of zero for the Vasicek shadow interest rate started at a negative value currently implied by Black’s (1995) model of interest rates as options for the Japanese economy (see Gorovoi and Linetsky (2004) for details). This interest rate application has been one of our original motivations for the present study.

2 Hitting time densities for CIR and OU diffusions

Consider a non-negative CIR diffusion $\{X_t, t \geq 0\}$ with the infinitesimal generator

$$(\mathcal{G}u)(x) = \frac{1}{2} \sigma^2 x u''(x) + \kappa(\theta - x) u'(x), \quad x > 0 \quad (1)$$

with $\sigma > 0$, $\theta > 0$ and $\kappa > 0$. The boundary at $+\infty$ is non-attracting natural (see, eg, Borodin and Salminen (1996, Chapter II) for boundary classification for one-dimensional diffusions). The boundary at the origin is entrance if the Feller



condition is satisfied:

$$b := \frac{2\kappa\theta}{\sigma^2} \geq 1 \tag{2}$$

In this case the origin is unattainable and is not included in the state space, $I = (0, \infty)$. If $b < 1$, the boundary at the origin is attainable regular, is specified as instantaneously reflecting, and the state space is $I = [0, \infty)$ and $(\mathcal{G}u)(0) := \lim_{x \downarrow 0} (\mathcal{G}u)(x)$. Let $C_b(I)$ be the Banach space of real-valued, bounded continuous functions on I . Then conditional expectation operators $(\mathcal{P}_t u)(x) := \mathbb{E}_x[u(X_t)]$ form a Feller semigroup $\{\mathcal{P}_t, t \geq 0\}$ in $C_b(I)$ with the infinitesimal generator (1). The domain of \mathcal{G} is: $D(\mathcal{G}) := \{u \in C_b(I) : \mathcal{G}u \in C_b(I), \text{ boundary condition at } 0 \text{ for } b < 1\}$. For $b < 1$ the boundary condition at 0 is

$$\lim_{x \downarrow 0} \frac{u'(x)}{\mathfrak{s}(x)} = 0 \tag{3}$$

where $\mathfrak{s}(x)$ is the scale density of the CIR diffusion, $\mathfrak{s}(x) = x^{-b} \exp(2\kappa x / \sigma^2)$. If $b \geq 1$, then 0 is an entrance boundary, and (3) is satisfied *automatically*.

Consider an OU diffusion $\{X_t, t \geq 0\}$ with the infinitesimal generator

$$(\mathcal{G}u)(x) = \frac{1}{2} \sigma^2 u''(x) + \kappa(\theta - x)u'(x), \quad x \in \mathbb{R} \tag{4}$$

with $\sigma > 0, \theta > 0$ and $\kappa > 0$. The state space is $I = \mathbb{R}$ and the boundaries at $-\infty$ and $+\infty$ are non-attracting natural. Let $C_b(\mathbb{R})$ be the Banach space of real-valued, bounded continuous functions on \mathbb{R} . Then conditional expectation operators $(\mathcal{P}_t u)(x) := \mathbb{E}_x[u(X_t)]$ form a Feller semigroup $\{\mathcal{P}_t, t \geq 0\}$ in $C_b(\mathbb{R})$ with the infinitesimal generator (4). The domain of \mathcal{G} is: $D(\mathcal{G}) := \{u \in C_b(\mathbb{R}) : \mathcal{G}u \in C_b(\mathbb{R})\}$ (no boundary conditions are needed).

Fix some $y \in (0, \infty)$ and consider the first hitting time $\mathcal{T}_{x \rightarrow y} := \inf\{t \geq 0 : X_t = y\}$. For $x < y$ (*first hitting time up*) we use the notation $\mathcal{T}_{x \uparrow y}$ and, correspondingly, $\mathcal{T}_{x \downarrow y}$ for $x > y$ (*first hitting time down*). The CIR and OU diffusions possess stationary distributions (with Gamma density in the case of CIR and Gaussian density in the case of OU) and, hence, are positively recurrent:

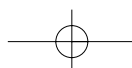
$$\mathbb{P}_x(\mathcal{T}_{x \rightarrow y} < \infty) = 1, \quad \mathbb{E}_x[\mathcal{T}_{x \rightarrow y}] < \infty \tag{5}$$

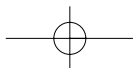
We are interested in the first hitting time density $f_{\mathcal{T}_{x \rightarrow y}}(t)$,

$$\mathbb{P}_x(\mathcal{T}_{x \rightarrow y} \in dt) = f_{\mathcal{T}_{x \rightarrow y}}(t) dt$$

for the CIR and OU diffusions.

We will need the following preliminary results from Linetsky (2004b) on eigenfunction expansions for diffusion hitting times. Consider a one-dimensional, time-homogeneous regular diffusion $\{X_t, t \geq 0\}$ whose state space is some





4 Vadim Linetsky

interval $I \subseteq \mathbb{R}$ with end-points e_1 and e_2 , $-\infty \leq e_1 < e_2 \leq \infty$, and the infinitesimal generator

$$(\mathcal{G}u)(x) = \frac{1}{2} a^2(x) u''(x) + b(x) u'(x), \quad x \in (e_1, e_2)$$

acting on functions on I subject to appropriate regularity and boundary conditions. We assume that the diffusion coefficient $a(x)$ is continuous and strictly positive on the open interval (e_1, e_2) and that drift $b(x)$ is continuous on (e_1, e_2) . The infinitesimal generator can be rewritten in the self-adjoint form

$$(\mathcal{G}u)(x) = \frac{1}{m(x)} \left(\frac{u'(x)}{\tilde{s}(x)} \right)', \quad x \in (e_1, e_2)$$

where $\tilde{s}(x)$ and $m(x)$ are the diffusion scale and speed densities (see, eg, Borodin and Salminen (1996, p. 17)):

$$\tilde{s}(x) := \exp \left(- \int^x \frac{2b(y)}{a^2(y)} dy \right), \quad m(x) := \frac{2}{a^2(x) \tilde{s}(x)} \tag{6}$$

Both $\tilde{s}(x)$ and $m(x)$ are continuous and strictly positive on (e_1, e_2) .

We assume that the endpoints e_i , $i = 1, 2$, are either non-attracting natural, entrance, or regular instantaneously reflecting boundaries for the diffusion X , and that the diffusion is positively recurrent (Equation (5) holds). For regular instantaneously reflecting boundaries, the boundary point e is included in the state space I , $(\mathcal{G}u)(e) := \lim_{x \rightarrow e} (\mathcal{G}u)(x)$, and the boundary condition at e is

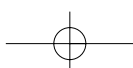
$$\lim_{x \rightarrow e} \frac{u'(x)}{\tilde{s}(x)} = 0 \tag{7}$$

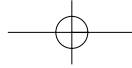
For non-attracting natural and entrance boundaries, the boundary point is not included in I , and the boundary condition (7) is satisfied *automatically*. We also assume that if e_1 and/or e_2 is a natural boundary, it is *non-oscillatory* in the terminology of Linetsky (2004a,b) (both OU and CIR diffusions have non-oscillatory natural boundaries). With these assumptions, we have the following result (it follows from Proposition 2 and Remark 3 in Linetsky (2004b)).

PROPOSITION 1 *Suppose the assumptions above are satisfied. Fix $x, y \in I$ and $t > 0$. Then*

$$\mathbb{P}_x(t < \mathcal{T}_{x \rightarrow y}) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}, \quad t > 0 \tag{8}$$

where $\{\lambda_n\}$, $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and $\{c_n\}$ are explicitly given below.





(i) **Hitting time up** ($\mathcal{T}_{x \uparrow y}$, $x < y$) For $\lambda \in \mathbb{C}$, let $\psi(x, \lambda)$ be the unique (up to a multiple independent of x) non-trivial solution of the Sturm–Liouville (SL) equation (\mathcal{G} is the diffusion infinitesimal generator):

$$-\mathcal{G}u = \lambda u \tag{9}$$

square-integrable with the speed density $m(x)$ (6) near e_1 , satisfying the boundary condition (7) at e_1 with the scale density $s(x)$ (6), and such that $\psi(x, \lambda)$ and $\psi_x(x, \lambda)$ are continuous in x and λ in $(e_1, y] \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each fixed $x < y$. Then the $\{\lambda_n\}_{n=1}^\infty$ in (8) are the eigenvalues of the SL problem for the ODE (9) with the boundary condition (7) at e_1 and the Dirichlet boundary condition at y :

$$u(y) = 0 \tag{10}$$

They can be found as simple positive zeros of the solution $\psi(y, \lambda)$:

$$\psi(y, \lambda_n) = 0 \tag{11}$$

The coefficients $\{c_n\}$ in (8) are given by

$$c_n = \frac{-\psi(x, \lambda_n)}{\lambda_n \psi_\lambda(y, \lambda_n)} \tag{12}$$

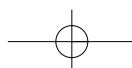
(ii) **Hitting time down** ($\mathcal{T}_{x \downarrow y}$, $x > y$) For $\lambda \in \mathbb{C}$, let $\phi(x, \lambda)$ be the unique (up to a multiple independent of x) non-trivial solution of the SL equation (9) square-integrable with $m(x)$ near e_2 , satisfying the boundary condition (7) at e_2 , and such that $\phi(x, \lambda)$ and $\phi_x(x, \lambda)$ are continuous in x and λ in $[y, e_2) \times \mathbb{C}$ and entire in $\lambda \in \mathbb{C}$ for each fixed $x > y$. Then the $\{\lambda_n\}_{n=1}^\infty$ in (8) are the eigenvalues of the SL problem for the ODE (9) with the boundary condition (7) at e_2 and the Dirichlet boundary condition (10) at y . They can be found as simple positive zeros of the solution $\phi(y, \lambda)$:

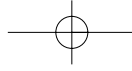
$$\phi(y, \lambda_n) = 0 \tag{13}$$

The coefficients $\{c_n\}$ in (8) are given by

$$c_n = \frac{-\phi(x, \lambda_n)}{\lambda_n \phi_\lambda(y, \lambda_n)} \tag{14}$$

PROOF Specialize Proposition 2 and Remark 3 in Linetsky (2004b) to the assumptions made here. \square





6 Vadim Linetsky

We are now ready to formulate our results for the first hitting time densities $f_{\mathcal{T}_{x \rightarrow y}}(t)$ for the CIR and OU diffusions. First, we need to solve the relevant SL problems with the operators (1) and (4) to find the solutions $\psi(x, \lambda)$ and $\phi(x, \lambda)$ and determine the eigenvalues λ_n and coefficients c_n . Second, to be able to differentiate the series (8) with respect to t to obtain the series representation for the density (15), we need to verify uniform convergence of (15). To this end, we will obtain convenient estimates of λ_n and c_n that will also prove useful in the numerical analysis of the series (15).

PROPOSITION 2 *Fix $x, y \in I$. Then*

$$f_{\mathcal{T}_{x \rightarrow y}}(t) = \sum_{n=1}^{\infty} c_n \lambda_n e^{-\lambda_n t}, \quad t > 0 \quad (15)$$

where $\{\lambda_n\}$, $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$, and $\{c_n\}$ are explicitly given below. For all $t_0 > 0$, the series (15) converges uniformly on $[t_0, \infty)$.

(i) **CIR** \uparrow *Suppose that $0 < x < y < \infty$ (CIR first hitting time up). Introduce the following notation:*

$$b := \frac{2\kappa\theta}{\sigma^2}, \quad \bar{x} := \frac{2\kappa x}{\sigma^2}, \quad \bar{y} := \frac{2\kappa y}{\sigma^2}, \quad a_n := -\frac{\lambda_n}{k}, \quad (16)$$

Then $\{a_n\}$, $0 > a_1 > a_2 > \dots$, $a_n \rightarrow -\infty$ as $n \rightarrow \infty$, are the negative roots of the equation ${}_1F_1(a; b; z)$ is the Kummer confluent hypergeometric function; see the Appendix

$${}_1F_1(a; b; \bar{y}) = 0 \quad (17)$$

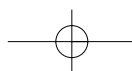
and

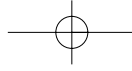
$$c_n = -\frac{{}_1F_1(a_n; b; \bar{x})}{a_n \frac{\partial}{{\partial a}} \left\{ {}_1F_1(a; b; \bar{y}) \right\} \Big|_{a=a_n}} \quad (18)$$

The λ_n and c_n have the following large- n asymptotics:

$$\lambda_n \sim \frac{\kappa\pi^2}{4\bar{y}} \left(n + \frac{b}{2} - \frac{3}{4} \right)^2 - \frac{\kappa b}{2} \quad (19)$$

$$c_n \sim \frac{(-1)^{n+1} 2\pi(n + b/2 - 3/4)}{\pi^2(n + b/2 - 3/4)^2 - 2b\bar{y}} \times$$





$$\times e^{\frac{1}{2}(\bar{x}-\bar{y})} \left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{1}{4}-\frac{b}{2}} \cos\left(\pi\left(n+\frac{b}{2}-\frac{3}{4}\right)\sqrt{\frac{\bar{x}}{\bar{y}}}-\frac{\pi b}{2}+\frac{\pi}{4}\right) \quad (20)$$

(ii) **CIR**↓ Suppose that $0 < y < x < \infty$ (CIR first hitting time down) and the notation (16) is in force. Then $\{a_n\}$ are the negative roots of the equation ($U(a; b; z)$ is the Tricomi confluent hypergeometric function; see the Appendix)

$$U(a; b; \bar{y}) = 0 \quad (21)$$

and

$$c_n = -\frac{U(a_n; b; \bar{x})}{a_n \frac{\partial}{\partial a} \{U(a; b; \bar{y})\} \Big|_{a=a_n}} \quad (22)$$

The λ_n and c_n have the following large- n asymptotics:

$$\lambda_n = \kappa \left(k_n - \frac{b}{2}\right), \quad \text{where } k_n \sim n - \frac{1}{4} + \frac{2\bar{y}}{\pi^2} + \frac{2}{\pi} \sqrt{\left(n - \frac{1}{4}\right)\bar{y} + \frac{\bar{y}^2}{\pi^2}} \quad (23)$$

$$c_n \sim \frac{(-1)^{n+1} \sqrt{k_n}}{(k_n - b/2)(\pi\sqrt{k_n} - \sqrt{\bar{y}})} e^{\frac{1}{2}(\bar{x}-\bar{y})} \left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{1}{4}-\frac{b}{2}} \cos\left(2\sqrt{k_n\bar{x}} - \pi k_n + \frac{\pi}{4}\right) \quad (24)$$

(iii) **OU**↓ Suppose that $-\infty < y < x < \infty$ (OU first hitting time down). Introduce the following notation:

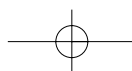
$$\bar{x} := \frac{\sqrt{2\kappa}}{\sigma} (x - \theta), \quad \bar{y} := \frac{\sqrt{2\kappa}}{\sigma} (y - \theta), \quad v_n := \frac{\lambda_n}{\kappa} \quad (25)$$

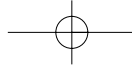
Then $\{v_n\}$, $0 < v_1 < v_2 < \dots$, $v_n \rightarrow \infty$ as $n \rightarrow \infty$, are the positive roots of the equation ($H_v(z)$ is the Hermite function; see the Appendix)

$$H_v(\bar{y}/\sqrt{2}) = 0 \quad (26)$$

and

$$c_n = -\frac{H_{v_n}(\bar{x}/\sqrt{2})}{v_n \frac{\partial}{\partial v} \{H_v(\bar{y}/\sqrt{2})\} \Big|_{v=v_n}} \quad (27)$$





8 Vadim Linetsky

The λ_n and c_n have the following large- n asymptotics:

$$\lambda_n = \kappa v_n$$

$$v_n = 2k_n - \frac{1}{2}, \quad \text{where } k_n \sim n - \frac{1}{4} + \frac{\bar{y}^2}{\pi^2} + \frac{\bar{y}\sqrt{2}}{\pi} \sqrt{n - \frac{1}{4} + \frac{\bar{y}^2}{2\pi^2}} \quad (28)$$

$$c_n \sim \frac{(-1)^{n+1} 2\sqrt{k_n}}{(2k_n - 1/2)(\pi\sqrt{k_n} - 2^{-1/2}\bar{y})} e^{\frac{1}{4}(\bar{x}^2 - \bar{y}^2)} \cos\left(\bar{x}\sqrt{2k_n} - \pi k_n + \frac{\pi}{4}\right) \quad (29)$$

(iv) **OU $\hat{\uparrow}$** Suppose that $-\infty < x < y < \infty$ (OU first hitting time up). Then λ_n and c_n are as in (iii) with the substitution $\bar{x} \rightarrow -\bar{x}$ and $\bar{y} \rightarrow -\bar{y}$.

PROOF (i) We start with the CIR $\hat{\uparrow}$ case. The boundary at 0 is entrance for $b \geq 1$ and regular instantaneously reflecting for $b < 1$. The relevant Sturm–Liouville problem is

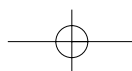
$$\frac{1}{2} \sigma^2 x u'' + \kappa(\theta - x)u' + \lambda u = 0 \quad (30)$$

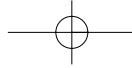
on $(0, y)$ with the boundary conditions (10) at y and (7) at 0. The spectrum $\{\lambda_n\}$ is purely discrete and positive, $0 < \lambda_1 < \lambda_2 < \dots, \lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. The solution $\psi(x, \lambda)$ is: $\psi(x, \lambda) = {}_1F_1(-\lambda/\kappa; b; \bar{x})$. The eigenvalues can be taken in the form $\lambda_n = -\kappa a_n$, where $\{a_n\}$ are the roots of Equation (17) and the coefficients c_n are given by Equation (18). In order to be able to differentiate the series (8) term-by-term to obtain the series (15) for the density, we need to verify that the series (15) converges uniformly on $[t_0, \infty)$ for all $t_0 > 0$. To this end, we note the asymptotics as $a \rightarrow -\infty$, keeping $b > 0$ and $z > 0$ fixed (Slater (1960), Equation (4.4.15), p. 68):

$${}_1F_1(a; b; z) = \pi^{-1/2} \Gamma(b) e^{z/2} (z(b/2 - a))^{1/4 - b/2} \times \cos\left(2\sqrt{z(b/2 - a)} - \pi b/2 + \pi/4\right) \{1 + O(|a|^{-1/2})\} \quad (31)$$

From (31), the eigenvalues λ_n and coefficients c_n have the large- n asymptotics (19) and (20). From these asymptotics, the series (15) is uniformly convergent on $[t_0, \infty)$ for all $t_0 > 0$.

(ii) **CIR \downarrow** By Theorem 3 in Linetsky (2004a), $+\infty$ is a *non-oscillatory* natural boundary for the CIR process (see Section 6.3 in Linetsky (2004a)). The relevant Sturm–Liouville problem is the ODE (30) on (y, ∞) with the boundary condition (10) at y . Since $+\infty$ is a non-oscillatory boundary, the spectrum $\{\lambda_n\}$ is purely discrete (Theorem 2(i) in Linetsky (2004a)). The solution of the ODE (30)





bounded at $+\infty$ is $\phi(x, \lambda) = U(-\lambda/\kappa; b; \bar{x})$. The eigenvalues can be taken in the form $\lambda_n = -\kappa a_n$, where $\{a_n\}$ are the roots of Equation (21), and the coefficients c_n are given by Equation (22). In order to be able to differentiate the series (8) term-by-term to obtain the series (15), we need to verify that the series (15) converges uniformly on $[t_0, \infty)$ for all $t_0 > 0$. To this end, we note the asymptotics as $a \rightarrow -\infty$, keeping $b > 0$ and $z > 0$ fixed (Slater (1960), Equation (4.4.18), p. 69):

$$U(a; b; z) = 2^{1/2} e^{z/2} z^{1/4 - b/2} k^{k-1/4} e^{-k} \cos\left(2\sqrt{kz} - \pi k + \pi/4\right) \{1 + O(k^{-1/2})\} \tag{32}$$

where $k = b/2 - a$. From Equation (32), the eigenvalues λ_n and the coefficients c_n have the large- n asymptotics (23) and (24). From these asymptotics, the series (15) is uniformly convergent on $[t_0, \infty)$ for all $t_0 > 0$.

(iii) **OU**↓ By Theorem 3 in Linetsky (2004a), the natural boundaries $-\infty$ and $+\infty$ are non-oscillatory for the OU process (see Section 6.2 in Linetsky (2004a)). The relevant Sturm–Liouville problem is

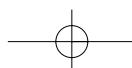
$$\frac{1}{2} \sigma u'' + \kappa(\theta - x)u' + \lambda u = 0 \tag{33}$$

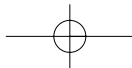
on (y, ∞) with the boundary condition (10) at y . Since $+\infty$ is a non-oscillatory boundary, the spectrum $\{\lambda_n\}$ is purely discrete. The solution of the ODE (33) bounded at $+\infty$ is $\phi(x, \lambda) = H_{\lambda/\kappa}(\bar{x}/\sqrt{2})$. The eigenvalues can be taken in the form $\lambda_n = \kappa v_n$, where $\{v_n\}$ are the roots of Equation (26), and the coefficients c_n are given by Equation (27). In order to be able to differentiate the series (8) term-by-term to obtain the series (15), we need to verify that the series (15) converges uniformly on $[t_0, \infty)$ for all $t_0 > 0$. To this end, we note the asymptotics as $v \rightarrow \infty$, keeping $z \in \mathbb{R}$ fixed (this asymptotics follows from expressing $H_\nu(z) = 2^\nu U(-\nu/2; 1/2; z^2)$ (see Lebedev (1972), p. 285) and Equation (32)):

$$H_\nu(z) = 2^{\nu+1/2} e^{z^2/2} k^{k-1/4} e^{-k} \cos\left(2z\sqrt{k} - \pi k + \pi/4\right) \{1 + O(k^{-1/2})\} \tag{34}$$

where $k = \nu/2 + 1/4$. From (34), the eigenvalues λ_n and the coefficients c_n have the large- n asymptotics (28) and (29). From these asymptotics, the series (15) is uniformly convergent on $[t_0, \infty)$ for all $t_0 > 0$.

(iv) **OU**↑ The relevant Sturm–Liouville problem is (33) on $(-\infty, y)$ with the boundary condition (10) at y . Since $-\infty$ is a non-oscillatory boundary, the spectrum $\{\lambda_n\}$ is purely discrete. The solution of (33) bounded at $-\infty$ is $\psi(x, \lambda) = H_{\lambda/\kappa}(-\bar{x}/\sqrt{2})$. The eigenvalues can be taken in the form $\lambda_n = \kappa v_n$, where $\{v_n\}$ are the roots of Equation (26) with the substitution $\bar{y} \rightarrow -\bar{y}$, and the coefficients c_n are given by Equation (27) with the substitution $\bar{x} \rightarrow -\bar{x}$ and $\bar{y} \rightarrow -\bar{y}$. From the asymptotics (28) and (29) (with the substitution $\bar{y} \rightarrow -\bar{y}$), the series (15) converges uniformly on $[t_0, \infty)$ for all $t_0 > 0$. □





10 Vadim Linetsky

The large- n estimates of the eigenvalues λ_n and coefficients c_n are useful for numerical analysis of the series (15). Fixing $t_0 > 0$, the series (15) is uniformly convergent on $[t_0, \infty)$. The rate of numerical convergence is dictated by the choice of t_0 . Truncating the series after $N - 1$ terms, the absolute value of the first omitted term $|c_N \lambda_N \exp(-\lambda_N t)|$ is maximized at $t = t_0$. For fixed t_0 , from (19)–(20) and (16) for CIR \uparrow we have a large- N estimate

$$|c_N \lambda_N e^{-\lambda_N t_0}| \sim AN e^{-BN^2 t_0} \tag{35}$$

with the constants

$$A = \frac{\sigma^2 \pi}{4y} e^{\frac{1}{2}(\bar{x}-\bar{y})} \left(\frac{x}{y}\right)^{\frac{1}{4}-\frac{b}{2}}, \quad B = \frac{\sigma^2 \pi^2}{8y} \tag{36}$$

This estimate can be used to determine how many terms are needed in the series to achieve some desired error tolerance. For CIR \uparrow , the eigenvalues λ_n grow as n^2 and the series (15) converges fast for t_0 not too small. From (35)–(36), we also observe that the larger the volatility σ and the smaller the level y , the larger the λ_n and the faster the series convergence.

From (23)–(24) and (28)–(29) for CIR \downarrow , OU \downarrow , and OU \uparrow we have a large- N estimate

$$|c_N \lambda_N e^{-\lambda_N t_0}| \sim A e^{-BN t_0} \tag{37}$$

with the constants for CIR \downarrow

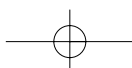
$$A = \frac{\kappa}{\pi} e^{\frac{1}{2}(\bar{x}-\bar{y})} \left(\frac{x}{y}\right)^{\frac{1}{4}-\frac{b}{2}}, \quad B = \kappa \tag{38}$$

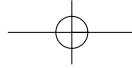
and for OU \downarrow and OU \uparrow

$$A = \frac{2\kappa}{\pi} e^{\frac{1}{4}(\bar{x}^2-\bar{y}^2)}, \quad B = 2\kappa \tag{39}$$

In contrast with the quadratic growth of eigenvalues for CIR \uparrow , in all three cases here the eigenvalues grow linearly with n and the series converges slower. The slower linear rate of growth of the eigenvalues in these cases is due to the fact that the domains of the relevant SL problems are infinite with infinite natural boundaries. The eigenvalues also grow linearly with the rate of mean-reversion, κ ; the faster the mean-reversion, the faster the series convergence. In contrast with CIR \uparrow , volatility σ and level y do not influence the rate of convergence.

REMARK: FIRST HITTING TIME OF THE OU PROCESS TO ITS LONG-RUN LEVEL There is a simple, well-known expression for the first hitting time density of the OU





process to its long-run level (eg, Keilson and Ross (1975), Ricciardi and Sato (1988), Göing-Jaesche and Yor (2003a)):

$$f_{\mathcal{T}_{x \rightarrow \theta}}(t) = \frac{|x - \theta|}{\sigma \sqrt{2\pi}} \left(\frac{\kappa}{\sinh(\kappa t)} \right) \exp \left(\frac{\kappa t}{2} - \frac{\kappa(x - \theta)^2 e^{-\kappa t}}{2\sigma^2 \sinh(\kappa t)} \right) \quad (40)$$

We now verify that Equation (15) collapses to (40) when $y = \theta$. In this case $\bar{y} = 0$ and Equation (26) reduces to

$$H_\nu(0) = \frac{2^\nu \sqrt{\pi}}{\Gamma((1 - \nu)/2)} = 0$$

Since the Gamma function $\Gamma(z)$ has simple poles at $z = -(n - 1)$, $n = 1, 2, \dots$, with the residues $(-1)^{n-1}/(n - 1)!$, the eigenvalues can be determined in closed form:

$$\lambda_n = \kappa(2n - 1), \quad n = 1, 2, \dots$$

and, for $x > \theta$, the coefficients reduce to

$$c_n = \frac{(-1)^{n-1} 2^{-2(n-1)} H_{2n-1}(\bar{x}/\sqrt{2})}{(2n - 1)(n - 1)! \sqrt{\pi}}, \quad n = 1, 2, \dots$$

(where the Hermite functions with integer indexes reduce to Hermite polynomials). Substituting in Equation (15), we have the series

$$f_{\mathcal{T}_{x \rightarrow \theta}}(t) = \frac{\kappa}{\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{-2(n-1)}}{(n - 1)!} H_{2n-1}(\bar{x}/\sqrt{2}) e^{-\kappa(2n-1)t}$$

This series collapses to (40) due to the classical identity for Hermite polynomials:

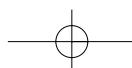
$$\sum_{n=0}^{\infty} \frac{w^n}{n!} H_{2n+1}(z) = \frac{2z}{(1 + 4w)^{3/2}} \exp \left(\frac{4wz^2}{(1 + 4w)} \right)$$

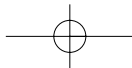
When $y \neq \theta$ ($\bar{y} \neq 0$), there is no simplification and the eigenvalues have to be determined numerically by finding the roots of Equation (26).

3 Applications to mean-reverting models and numerical examples

3.1 How long does it take for a credit spread to revert to its long-run mean?

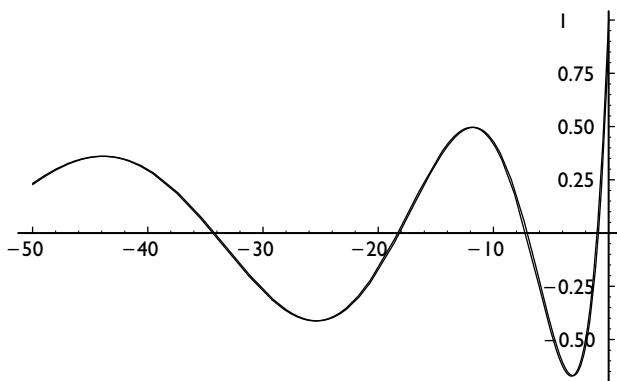
Suppose that an instantaneous credit spread follows a CIR diffusion with the long-run credit spread level of 200 basis points ($\theta = 0.02$), the initial spread level of 100bp ($x = 0.01$), the rate of mean-reversion $\kappa = 0.25$, and the volatility parameter $\sigma/\sqrt{x} = 100\%$, or $\sigma = 0.1$ (see Duffie and Singleton (2003), pp. 66–7). We are interested in the first hitting time density of the long-run level $y = \theta$, starting from $x < y$.





12 Vadim Linetsky

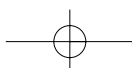
FIGURE 1 Function (17) and its estimate (31) for $b = 1$ and $\bar{y} = 1$.



The normalized parameters are $b = 1$, $\bar{x} = 1/2$, $\bar{y} = 1$. To determine the eigenvalues λ_n we use the root-finding function FindRoot in *Mathematica* (we used *Mathematica* 4.0 running on a Pentium III PC for all calculations in this paper) to determine the roots of Equation (17). For each $n \geq 1$, we use the estimate (19) as a starting point for the FindRoot function. Figure 1 plots the function (17) and its estimate (31). Table 1 gives the exact values of λ_n , $n = 1, 2, \dots, 10$, as well as their estimates (19). *Mathematica* allows one to compute the eigenvalues with any desired accuracy. Table 1 shows that the estimates are quite accurate even for lower eigenvalues. Table 1 also gives the exact values of the expansion coefficients c_n (18), $n = 1, 2, \dots, 10$, as well as their estimates (20).

Fixing $t_0 > 0$, the first hitting time density is given by the series (15) uniformly convergent on $[t_0, \infty)$. The constants in the estimate (35)–(36) are: $A = 0.3637$ and $B = 0.6169$. Fix $t_0 = 0.01$ and error tolerance $\epsilon = 10^{-6}$. From (35) we estimate that we need 52 terms in the series to achieve this accuracy. The maximum absolute value of the first omitted term with $N = 53$ is $|\lambda_{53} c_{53} e^{-\lambda_{53} t_0}| = 6.66 \times 10^{-7}$. Figure 2 plots the first hitting time density on the interval $[0.01, 4]$ with $N = 52$ terms included in the series (15). The mean hitting time is 2.991 years.

Figure 2 also plots the density (15) where we use estimates (19) instead of the exact eigenvalues λ_n computed by numerically solving Equation (17) and estimates (20) instead of exact coefficients (18). We observe that the exact and approximate densities are quite close. From the practical standpoint, if one is interested in roughly estimating the density, one can use estimates (19) and (20) in place of the exact values. The estimates are expressed in terms of elementary functions and do not require any special function calculations. This drastically cuts down on the computation time. Running *Mathematica* 4.0 on a Pentium III PC, it takes several minutes to compute the first 100 exact eigenvalues λ_n and coefficients c_n as *Mathematica* 4.0 invokes its internal routines to compute the confluent hypergeometric functions, while computing the elementary function



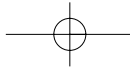
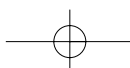
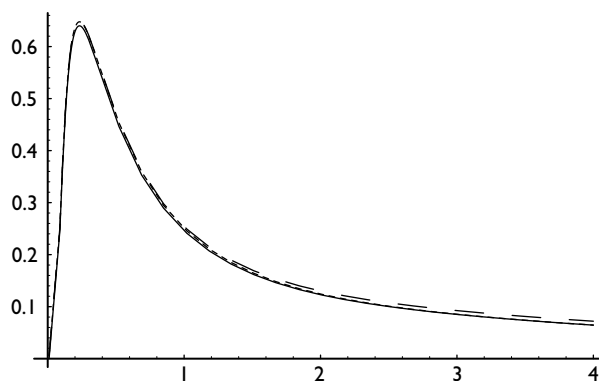


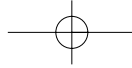
TABLE I The λ_n and c_n for the CIR first hitting time up. CIR parameters: $y = \theta = 0.02, x = 0.01, \kappa = 0.25, \sigma = 0.1$. Normalized parameters: $b = 1, \bar{x} = 1/2, \bar{y} = 1$.

n	Exact λ_n	Estimates (19)	Exact c_n	Estimates (20)
1	0.25000	0.22198	0.69611	0.78231
2	1.79894	1.76410	0.35167	0.36051
3	4.57573	4.53993	0.12457	0.12642
4	8.58556	8.54946	-0.04861	-0.04851
5	13.8289	13.7927	-0.11769	-0.11801
6	20.3059	20.2696	-0.08609	-0.08639
7	28.0166	27.9802	-0.00620	-0.00633
8	36.9609	36.9246	0.05723	0.05725
9	47.1390	47.1026	0.06617	0.06625
10	58.5507	58.5143	0.02647	0.02654

estimates (19) and (20) is essentially instantaneous. Since the accuracy of the estimates increases with n , for a more accurate calculation one can determine the first several exact eigenvalues λ_n and coefficients c_n and use the estimates for the rest. The density with one exact term and the other 51 computed using the estimates is also plotted in Figure 2. It is so close to the exact density that the graphs can hardly be distinguished. The mean hitting time computed using this approximate density is 3.000 years vs. the exact mean of 2.991 years. In our experience,

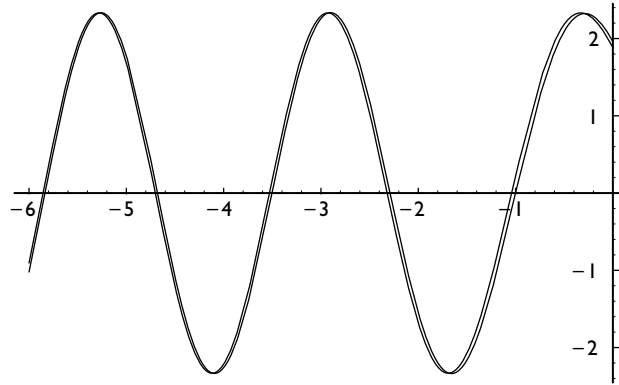
FIGURE 2 CIR first hitting time up density on the interval $[0.01, 4]$. CIR parameters: $y = \theta = 0.02, x = 0.01, \kappa = 0.25, \sigma = 0.1$. Normalized parameters: $b = 1, \bar{x} = 1/2, \bar{y} = 1$. Mean first hitting time: 2.991 years. Solid line: exact density. Dashed line (long dashes): estimated density using estimates (19) and (20) in place of exact λ_n and c_n . Dashed line (short dashes): estimated density using exact λ_1 and c_1 and estimates (19) and (20) in place of exact λ_n and c_n for $n \geq 2$. In all three cases the series (15) is truncated after 52 terms.





14 Vadim Linetsky

FIGURE 3 Function (21) and its estimate (32) for $b = 1$ and $\bar{y} = 1$ (both scaled by the factor $e^k k^{1/4-k}$ for convenience).



the smaller the value of the parameter b , the closer the estimates approximate exact values. For large values of b , the quality of the estimates deteriorates and more exact terms in the series are needed before one switches to the approximate values provided by the estimates.

In our second example, suppose that the initial credit spread level is 400bp ($x = 0.04$) and the other parameters are the same, $y = \theta = 0.02$, $\kappa = 0.25$, $\sigma = 0.1$. We are interested in the first hitting time density of the long-run level $y = \theta$, starting from $x > y$. The normalized parameters are: $b = 1$, $\bar{x} = 2$, $\bar{y} = 1$.

To determine the eigenvalues λ_n we use the root-finding function FindRoot to determine the roots of Equation (21). Figure 3 plots the function (21) and its estimate (32), both scaled by the factor $e^k k^{1/4-k}$ for convenience. The exact function and the estimate are quite close. For each $n \geq 1$, we use the estimate (23) as a starting point for the FindRoot function. Table 2 gives the exact values of λ_n , $n = 1, 2, \dots, 10$, as well as their estimates (23). Table 2 shows that the estimates are quite accurate even for lower eigenvalues. Table 2 also gives the exact values of the expansion coefficients c_n , $n = 1, \dots, 10$, as well as their estimates (24). Fixing $t_0 > 0$, the first hitting time density is given by the series (15) uniformly convergent on $[t_0, \infty)$. From numerical standpoint, the rate of numerical convergence is dictated by the choice of t_0 . The constants in the estimate (37)–(38) are $A = 0.2207$ and $B = 0.25$. Fix $t_0 = 0.07$ and error tolerance $\epsilon = 10^{-5}$. From (37) we estimate that we need 571 terms in the series to achieve this accuracy. The maximum absolute value of the first omitted term with $N = 572$ is $|\lambda_{572} c_{572} e^{-\lambda_{572} t_0}| = 3.6 \times 10^{-6}$. Figure 4 plots the first hitting time density on the interval $[0.07, 5]$ with $N = 571$ terms included in the series (15). The mean hitting time is 2.773 years. Due to the linear growth of CIR \downarrow eigenvalues (23) in contrast with the quadratic growth (20) for CIR \uparrow , the series converges more slowly and more terms are needed.

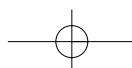
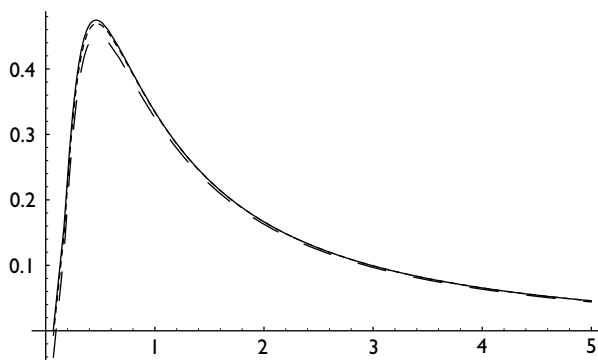


TABLE 2 The λ_n and c_n for the CIR first hitting time down. CIR parameters: $\gamma = \theta = 0.02, x = 0.04, \kappa = 0.25, \sigma = 0.1$. Normalized parameters: $b = 1, \bar{x} = 2, \bar{\gamma} = 1$.

n	Exact λ_n	Estimates (23)	Exact c_n	Estimates (24)
1	0.25000	0.26001	0.50000	0.48869
2	0.57211	0.57971	0.23719	0.23104
3	0.87555	0.88191	0.15306	0.14806
4	1.16993	1.17550	0.10780	0.10359
5	1.45869	1.46371	0.07856	0.07497
6	1.74354	1.74815	0.05786	0.05477
7	2.02547	2.02975	0.04238	0.03972
8	2.30510	2.30912	0.03043	0.02811
9	2.58287	2.58667	0.02097	0.01896
10	2.85909	2.86270	0.01339	0.01164

Figure 4 also plots the density (15) where we use estimates (23) instead of the exact eigenvalues, λ_n , computed by numerically solving Equation (21) and estimates (24) instead of exact coefficients (22). We observe that the exact and approximate densities are quite close. From the practical standpoint, if one is interested in roughly estimating hitting times, one can use estimates (23) and (24) in place of the exact values. The estimates are expressed in terms of elementary functions and do not require any special function calculations. This drastically

FIGURE 4 CIR first hitting time down density on the interval $[0.07, 5]$. CIR parameters: $\gamma = \theta = 0.02, x = 0.04, \kappa = 0.25, \sigma = 0.1$. Normalized parameters: $b = 1, \bar{x} = 2, \bar{\gamma} = 1$. Mean first hitting time: 2.773 years. Solid line: exact density. Dashed line (long dashes): estimated density using the estimates (23) and (24) in place of exact λ_n and c_n . Dashed line (short dashes): estimated density using exact λ_n and c_n for $n = 1, 2, \dots, 10$ and estimates (23) and (24) in place of exact λ_n and c_n for $n \geq 11$. In all three cases the series (15) is truncated after 571 terms.



16 Vadim Linetsky

cuts down on the computation time, as computing the estimates is essentially instantaneous. Since the accuracy of the estimates increases with n , for a more accurate calculation one can determine the first several exact eigenvalues λ_n and coefficients c_n and use the estimates for the rest. The density with 10 exact terms and the remaining 561 terms computed using the estimates is also plotted in Figure 4. It is so close to the exact density that the graphs can hardly be distinguished. The mean hitting time computed using this approximate density is 2.771 years vs. the exact mean of 2.773 years.

3.2 How long does it take for stochastic volatility to revert to its long-run mean?

In our next example we consider a stochastic volatility model. The instantaneous asset price variance is assumed to follow a CIR diffusion. Duffie *et al.* (2000) estimate parameters of the (risk-neutral) CIR process implied in the S&P500 options prices on November 2, 1993: the long-run variance level is $\theta = 0.019$ (corresponding to the long-run volatility level of 13.78%), the initial variance level is $x = 0.01$ (corresponding to the initial volatility level of 10%), the volatility of variance parameter is $\sigma = 0.61$, and the rate of mean-reversion is $\kappa = 6.21$. We are interested in the first hitting time density of the long-run level $y = \theta$, starting from this initial value $x = 0.01 < y = 0.019$. The normalized parameters are: $b = 0.6342$, $\bar{x} = 0.3338$, $\bar{y} = 0.6342$. Table 3 gives the exact values of λ_n and c_n , $n = 1, 2, \dots, 5$, as well as their estimates (19) and (20). In this application the eigenvalues grow faster than in the credit spread application due to the large value of the volatility of variance σ .

Fixing $t_0 > 0$, the first hitting time density is given by the series (15) uniformly convergent on $[t_0, \infty)$. Since in this application the rate of mean-reversion is large, $\kappa = 6.21$, the process rapidly reverts to its mean. The constants in the estimate (35)–(36) are $A = 13.8187$ and $B = 24.1611$. Fix $t_0 = 0.001$ and error tolerance $\epsilon = 10^{-6}$. From (35) we estimate that we need 29 terms in the series to achieve this accuracy. The maximum absolute value of the first omitted term with $N = 30$ is $|\lambda_{30}c_{30}e^{-\lambda_{30}t_0}| = 1.5 \times 10^{-7}$.

TABLE 3 The λ_n and c_n for the CIR first hitting time up. CIR parameters: $y = \theta = 0.019$, $x = 0.01$, $\kappa = 6.21$, $\sigma = 0.61$. Normalized parameters: $b = 0.6342$, $\bar{x} = 0.3338$, $\bar{y} = 0.6342$.

n	Exact λ_n	Estimates (19)	Exact c_n	Estimates (20)
1	6.21000	5.80090	0.60043	0.63468
2	57.9487	57.3650	0.36408	0.36846
3	157.853	157.251	0.17940	0.18051
4	306.067	305.460	0.01028	0.01051
5	502.599	501.990	-0.08990	-0.08995

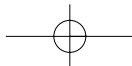


FIGURE 5 CIR first hitting time up density on the interval $[0.001, 0.15]$. CIR parameters: $\gamma = \theta = 0.019$, $x = 0.01$, $\kappa = 6.21$, $\sigma = 0.61$. Normalized parameters: $b = 0.6342$, $\bar{x} = 0.3338$, $\bar{\gamma} = 0.6342$. Mean first hitting time: 0.104 years. Solid line: exact density. Dashed line (short dashes): estimated density using the estimates (19) and (20) in place of exact λ_n and c_n . In both cases the series (15) is truncated after 29 terms.

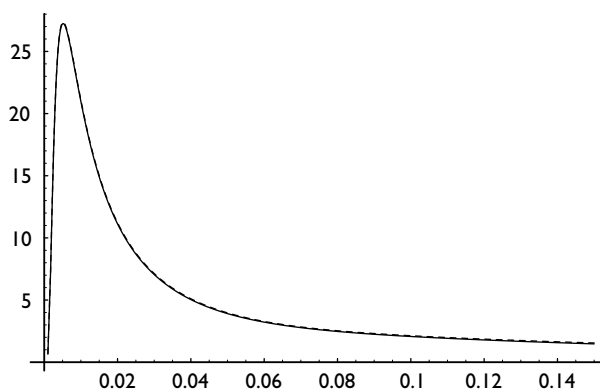
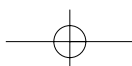


Figure 5 plots the first hitting time density on the interval $[0.001, 0.15]$ with $N = 29$ terms included in the series (15). The mean hitting time is 0.1038 year (the process is fast mean-reverting). Figure 5 also plots the density (15) where we use estimates (19) instead of the exact eigenvalues λ_n computed by numerically solving Equation (17) and estimates (20) instead of exact coefficients (18). We observe that the exact and approximate densities are very close and the graphs can hardly be distinguished. The mean hitting time computed using this approximate density is 0.1167 years. Using the exact λ_1 and c_1 in the first term and the estimates in the remaining 28 terms in the series improves the mean hitting time estimate to 0.1040 years vs. the exact mean of 0.1038 years.

3.3 How long does it take for the Vasicek shadow short rate in Black's model of interest rates as options to hit zero, starting from a negative value?

The current short rate in Japan is zero. Gorovoi and Linetsky (2004) have recently developed a class of models based on Fischer Black's (1995) idea of the non-negative nominal interest rate as an option on the *shadow interest rate* that is allowed to get negative (a similar idea was independently suggested by Rogers (1995)). Assuming that the shadow rate follows a Vasicek process, Gorovoi and Linetsky (2004) estimate that the implied shadow rate in Japan is *negative* at -5.1% ($x = -0.051$) as of February 3, 2002. The rest of the estimated (risk-neutral) Vasicek process parameters are: $\theta = 0.0354$ (3.54%), $\kappa = 0.21$, $\sigma = 0.028$. It is



18 Vadim Linetsky

TABLE 4 The λ_n and c_n for the OU first hitting time up. OU parameters: $\theta = 0.0354$, $\kappa = 0.21$, $\sigma = 0.028$, $x = -0.051$, $y = 0$.

n	Exact λ_n	Estimates	Exact c_n	Estimates
1	0.37563	0.37573	1.09058	0.96507
2	0.86465	0.86548	0.28759	0.22942
3	1.33617	1.33704	0.04336	0.01233
4	1.79907	1.79991	-0.05867	-0.07510
5	2.25660	2.25740	-0.09985	-0.10764
6	2.71038	2.71113	-0.11034	-0.11285
7	3.16134	3.16205	-0.10455	-0.10386
8	3.61009	3.61077	-0.09036	-0.08781
9	4.05704	4.05769	-0.07237	-0.06884
10	4.50250	4.50312	-0.05334	-0.04943

interesting to compute the expected time for the shadow rate to become positive again, starting from the negative value of -5.1% .

To this end, we are interested in the first hitting time density of zero, $y = 0$, starting from $x < y$ (OU \uparrow , case (iv) in Proposition 2). To determine the eigenvalues λ_n we use the root-finding function FindRoot to determine the roots of Equation (26) with $\bar{y} \rightarrow -\bar{y}$. We use the estimates (28) with $\bar{y} \rightarrow -\bar{y}$ as a starting point for the FindRoot function. Table 4 gives the exact values of λ_n , $n = 1, 2, \dots, 10$, as well as their estimates (28). Table 4 shows that the estimates are quite accurate even for lower eigenvalues. Table 4 also gives the exact values of the expansion coefficients c_n , $n = 1, \dots, 10$, as well as their estimates (29). Fixing $t_0 > 0$, the first hitting time density is given by the series (15) uniformly convergent on $[t_0, \infty)$. From numerical standpoint, the rate of numerical convergence is dictated by the choice of t_0 . The constants in the estimate (37) and (39) are $A = 0.3072$ and $B = 0.42$. Fix $t_0 = 0.15$ and error tolerance $\epsilon = 10^{-6}$. From (37) we estimate that we need 200 terms in the series to achieve this accuracy. The maximum absolute value of the first omitted term with $N = 201$ is $|\lambda_{201}c_{201}e^{-\lambda_{201}t_0}| = 6.75 \times 10^{-7}$.

Figure 6 plots the (risk-neutral) density of the first hitting time of zero ($y = 0$), starting from the negative value $x = -0.051$ on the interval $[0.15, 10]$ with $N = 200$ terms included in the series. With these parameters, the (risk-neutral) expected time to hit zero is 3.084 years. Figure 6 also plots the density (15) where we use the estimates instead of the exact values of λ_n and c_n . In this case this approximation for the density is not as good as in the previous examples. However, using the exact values of λ_n and c_n in the first 10 terms in the series and the estimates in the remaining 190 terms with $n = 11, 12, \dots, 200$ produces an approximate density very close to the exact density. The mean hitting time computed using this approximate density is 3.086 years vs. the exact mean of 3.084 years.

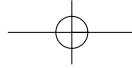
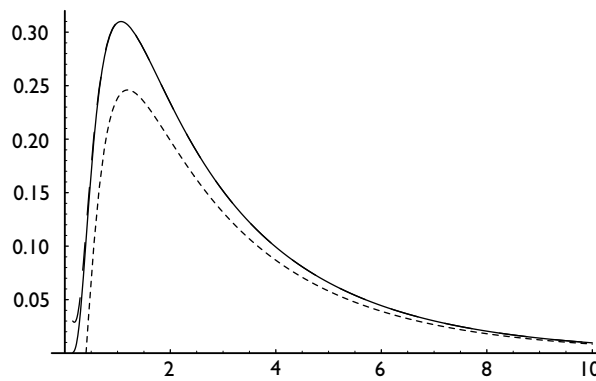


FIGURE 6 OU first hitting time up density on the interval $[0.15, 10]$. OU parameters: $\theta = 0.0354$, $\kappa = 0.21$, $\sigma = 0.028$, $x = -0.051$, $y = 0$. Mean first hitting time: 3.084 years. Solid line: exact density. Dashed line (short dashes): estimated density using the estimates (28) and (29) in place of exact λ_n and c_n . Dashed line (long dashes): estimated density using exact λ_n and c_n for $n = 1, 2, \dots, 10$ and estimates (28) and (29) in place of exact λ_n and c_n for $n \geq 11$. In all three cases the series (15) is truncated after 200 terms.



4 Conclusion

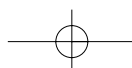
In conclusion, eigenfunction expansions provide easily computed representations for hitting time densities for CIR and OU diffusions by expressing the hitting time density as an infinite series of exponential densities. The results are applicable to the analysis of mean-reverting CIR and OU models for interest rates, credit spreads, stochastic volatility, commodity convenience yields, and other mean-reverting financial variables.

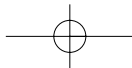
Appendix A: Relevant special functions

The Kummer confluent hypergeometric function ${}_1F_1(a; b; x)$ is defined for all z , $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, by the series $((a)_0 := 1, (a)_n := a(a+1) \dots (a+n-1)$ are the Pochhammer symbols)¹

$${}_1F_1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \tag{41}$$

¹ We remark that in addition to the series representations such as (41), there are a number of other representations for the confluent hypergeometric functions (integral representations, asymptotic expansions, etc.) Depending on the values of a , b and x , some representations may be more computationally efficient than others. Details can be found in *(continued on next page)*





20 Vadim Linetsky

${}_1F_1(a; b; x)$ is available as a built-in function in both *Mathematica* and *Maple* packages (Hypergeometric1F1[a,b,z] in *Mathematica*). To efficiently compute the function, depending on the values of a , b and z , these software packages use several integral representations and asymptotic expansions in addition to the series (see Slater (1960) and Buchholz (1969)). The derivative of the Kummer function with respect to its first index is given by

$$\frac{\partial}{\partial a} \{ {}_1F_1(a; b; z) \} = \sum_{k=0}^{\infty} \frac{(a)_k \Psi(a+k)}{(b)_k k!} z^k - \Psi(a) {}_1F_1(a; b; z) \quad (42)$$

where

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{k=0}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z-1} \right) - \gamma$$

is the digamma function (Abramowitz and Stegun, 1972). This derivative is available in *Mathematica* with the call Hypergeometric1F1^(1,0,0)[a,b,z].

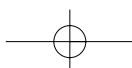
The Tricomi confluent hypergeometric function $U(a; b; z)$ is defined for all a , $z \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, \pm 1, \pm 2, \dots\}$ by (see Slater, 1960, p. 5):

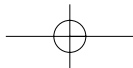
$$U(a; b; x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1(a; b; x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1(1+a-b; 2-b; x) \quad (43)$$

(for $b \in \mathbb{Z}$, $U(a; b; x)$ is defined as a limit, $\lim_{\epsilon \rightarrow 0} U(a; b+\epsilon; x)$, where $U(a; b+\epsilon; x)$ is given by Equation (43); see Slater (1960)). It is available in *Mathematica* with the call HypergeometricU[a,b,z]. The derivative with respect to its first index a is (this derivative is available in *Mathematica* with the call HypergeometricU^(1,0,0)[a,b,z]):

$$\begin{aligned} \frac{\partial}{\partial a} \{ U(a; b; z) \} &= \frac{\Gamma(b-1)z^{1-b}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Psi(a-b+k+1)(a-b+1)_k z^k}{k!(2-b)_k} \\ &+ \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \sum_{k=0}^{\infty} \frac{\Psi(a+k)(a)_k z^k}{k!(b)_k} \\ &- \{ \Psi(a) + \Psi(a-b+1) \} U(a; b; z) \end{aligned}$$

Buchholz (1969) and Slater (1960). In particular, to compute the confluent hypergeometric function for large negative values of a needed for large- n terms in the eigenfunction expansions, one can use the asymptotic expansion in Section 4.4.1, Slater (1960). The estimate (31) is the first term of that asymptotic expansion. *Mathematica* and *Maple* implementations automatically use suitable representations for each parameter set. We used *Mathematica* 4.0 for all calculations in this paper.





The Kummer and Tricomi confluent hypergeometric functions ${}_1F_1(a; b; z)$ and $U(a; b; z)$ are solutions of the confluent hypergeometric equation

$$zu'' + (b - z)u' - au = 0$$

bounded at 0 and $+\infty$, respectively.

The *Hermite function* $H_\nu(z)$ is defined by (see Lebedev, 1972, p. 285):

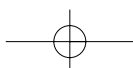
$$H_\nu(z) = 2^\nu \sqrt{\pi} \left\{ \frac{{}_1F_1\left(-\frac{\nu}{2}; \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} - 2z \frac{{}_1F_1\left(\frac{1-\nu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(-\frac{\nu}{2}\right)} \right\} \quad (44)$$

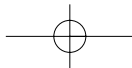
It is available in *Mathematica* with the call `HermiteH[v,z]`. To calculate the derivative of $H_\nu(z)$ with respect to its index, recall Equation (41) (this derivative is available in *Mathematica* with the call `HermiteH(1,0)[v,z]`). The Hermite function $H_\nu(z)$ is a solution of the equation

$$u'' - 2zu' + 2\nu u = 0 \quad (45)$$

REFERENCES

- Abramowitz, M., and Stegun, I. A. (1972). *Handbook of mathematical functions*. Dover, New York.
- Black, F. (1995). Interest rates as options. *Journal of Finance* **50**, 1371–6.
- Borodin, A. N., and Salminen, P. (1996). *Handbook of Brownian motion*. Birkhauser, Boston.
- Buchholz, H. (1969). *The confluent hypergeometric function*. Springer, Berlin.
- Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica* **53**, 385–407.
- Davydov, D., and Linetsky, V. (2003). Pricing options on scalar diffusions: an eigenfunction expansion approach. *Operations Research* **51**, 185–209.
- Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68**, 1343–76.
- Duffie, D., and Singleton, K. (2003). *Credit risk*. Princeton University Press, Princeton, NJ.
- Feller, W., (1951). Two singular diffusion problems. *Annals of Mathematics* **54**, 173–82.
- Giorno, V., Nobile, A. G., Ricciardi, L. M., and Sacerdote, L. (1986). Some remarks on the Rayleigh process. *Journal of Applied Probability* **23**, 398–408.
- Göing-Jaeschke, A., and Yor, M. (2003a). A clarification about hitting time densities for Ornstein–Uhlenbeck processes. Forthcoming in *Finance and Stochastics*.
- Göing-Jaeschke, A., and Yor, M. (2003b). A survey and some generalizations of Bessel processes. *Bernoulli* **9**, 313–49.





22 Vadim Linetsky

- Gorovoi, V, and Linetsky, V. (2004). Black's model of interest rates as options, eigenfunction expansions and Japanese interest rates. *Mathematical Finance* **14**, 49–78.
- Itô, K., and McKean, H. (1974). *Diffusion processes and their sample paths*. Springer, Berlin.
- Keilson, J., and Ross, H. F. (1975). Passage time distributions for Gaussian Markov (Ornstein–Uhlenbeck) statistical processes. In *Selected tables in mathematical statistics, Volume III*, pp. 233–59. American Mathematical Society, Providence, R.I.
- Kent, J. (1980). Eigenvalue expansions for diffusion hitting times. *Zeitschrift Wahrscheinlichkeitstheorie und verwandte Gebiete* **52**, 309–19.
- Kent, J. (1982). The spectral decomposition of a diffusion hitting time. *Annals of Applied Probability* **10**, 207–19.
- Kou, S. C., and Kou, S. G. (2002). A diffusion model for growth stocks. Forthcoming in *Mathematics of Operations Research*.
- Leblanc, B., and Scaillet, O. (1998). Path-dependent options on yield in the affine term structure model. *Finance and Stochastics* **2**, 349–67. (Correction in: Leblanc, B., Renault, O., and Scaillet, O. (2000). A correction note on the first passage time of an Ornstein–Uhlenbeck process to a boundary. *Finance and Stochastics* **4**, 109–11; see also Göing-Jaeschke and Yor (2003a) for a further correction.)
- Lewis, A. (1998). Applications of eigenfunction expansions in continuous-time finance. *Mathematical Finance* **8**, 349–83.
- Linetsky, V. (2001). Spectral expansions for Asian (average price) options. Forthcoming in *Operations Research*.
- Linetsky, V. (2004a). The spectral decomposition of the option value. *International Journal of Theoretical and Applied Finance* **7**(3).
- Linetsky, V. (2004b). Lookback options and diffusion hitting times: a spectral expansion approach. *Finance and Stochastics* **8**(3), 373–98.
- Linetsky, V. (2004c). The spectral representation of Bessel processes with drift: applications in queueing and finance. *Journal of Applied Probability* **41**(2), 327–44.
- McKean, H. (1956). Elementary solutions for certain parabolic partial differential equations. *Transactions of the American Mathematical Society* **82**, 519–48.
- Ricciardi, L. M., and Sato, S. (1988). First-passage-time density and moments of the Ornstein–Uhlenbeck process. *Journal of Applied Probability* **25**, 43–57.
- Rogers, L. C. G. (1995). Which model for term-structure of interest rates should one use? In *Proceedings of IMA Workshop on Mathematical Finance*, IMA Volume 65, pp. 93–116. Springer, New York.
- Schwartz, E. (1997). The stochastic behavior of commodity prices: implications for valuation and hedging. *Journal of Finance* **52**, 923–73.
- Slater, L. J. (1960). *Confluent hypergeometric functions*. Cambridge University Press.
- Vasicek, O. A. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics* **5**, 177–88.

