# Exotic spectra

Eigenfunction expansions, well known to physicists and engineers, can also be applied to finance. As Vadim Linetsky demonstrates, the method is particularly suited to barrier and Asian options, with convergence properties that compare favourably with Monte Carlo

In this article, we describe a derivatives pricing methodology based on unbundling all European-style derivatives written on a given underlying assumed to follow some diffusion process into portfolios of primitive securities – building blocks called eigensecurities. Eigensecurities are eigenfunctions of the pricing operator.

The eigenfunction expansion method is a powerful tool for generating analytical solutions to partial differential equations (PDEs). It is one of the main tools for solving the Schrödinger equation in quantum mechanics, as well as many other important PDEs of mathematical physics (Morse & Feshback, 1953). In probability theory, applications of the eigenfunction expansion method go back to Karlin & McGregor (1960), Itô & McKean (1974, chapter 4) and Wong (1964), who studied spectral representations for transition probability densities of Markov processes. In finance, the eigenfunction expansion method has an especially appealing intuition. The pricing operator is a fundamental object in finance. Being a linear operator, it is then natural to subject it to spectral analysis and construct a spectral representation of the state-price density. If the spectral representation can be constructed explicitly, it provides analytical solutions to pricing problems.

Here, we demonstrate the power of the method on two specific option pricing applications: barrier options under the constant elasticity of variance (CEV) process with volatility skew and arithmetic Asian options under the lognormal process. The eigenfunction expansion method provides analytical solutions to both problems. For mathematical details, proofs and further results, we refer the reader to Davydov & Linetsky (2001b) and Linetsky (2001).

Among other papers employing the eigenfunction expansion method in finance, Hansen, Scheinkman & Touzi (1998) develop spectral methods to estimate diffusion processes based on discrete observations. Lewis (1998) applies the eigenfunction expansion method to the pricing of options on assets that pay dividends at a constant dollar rate and the pricing of bonds under a short-rate process with non-linear drift. Lewis (2000) applies the eigenfunction expansion method to the analysis of stochastic volatility models. Lipton (2001) develops eigenfunction expansions for foreign exchange applications, including pricing exotic options under stochastic volatility. Beaglehole (1991) and Goldstein & Keirstead (1997) apply the eigenfunction expansion approach to the pricing of bonds when the short-rate process is a scalar diffusion.

To give an idea of the method, consider the pricing of a European-style derivatives security whose payout at expiry $T > 0$ is a function $f$ of a single state variable $X$, which follows a scalar diffusion process under the risk-neutral measure ($B$ is a standard Brownian motion):

$$dX_t = b(X_t)dt + a(X_t)dB_t, \qquad X_0 = x \tag{1}$$

Suppose the derivatives contract has a knock-out provision, ie, if at any time between the contract inception and expiry either a lower barrier $L$ or an upper barrier $U$ is violated, the contract is cancelled. Then the value $V$ of the derivatives security, considered as a function of current time $t$ and state $x$, solves the fundamental pricing PDE:

$$-1/2a^2(x)V_{xx} - b(x)V_x + r(x)V = V_t \tag{2}$$

with the terminal payout condition at expiry $V(T, x) = f(x)$ and two boundary conditions at the barriers $V(t, L) = V(t, U) = 0$ (we assume that the short rate $r$ is also a function only of $x$ and $r(x) \geq 0$ for all $x$). The solution may be written in the form:

$$V(t,x) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n (T-t)} \varphi_n(x) \tag{3}$$

where $\{\lambda_n, n = 1, 2, ...\}$ are the eigenvalues of the Sturm-Liouville boundary value problem:

$$-1/2a^2(x)\varphi_{xx} - b(x)\varphi_x + r(x)\varphi = \lambda\varphi, \qquad \varphi(L) = \varphi(U) = 0 \tag{4}$$

and $\{\varphi_n, n = 1, 2, ...\}$ are the corresponding eigenfunctions. It is classical that the eigenfunctions form a complete and orthogonal basis in the Hilbert space $\mathcal{H} := L^2([L, U], \mathfrak{m})$ of functions on the interval $[L, U]$ square-integrable with the speed density of the underlying diffusion:

$$\mathfrak{m}(x) = 2/a^2(x)\exp\left\{\int^x \left[2b(y)/a^2(y)\right]dy\right\}$$

and with the inner product $\langle f, g\rangle = \int_L^U f(x)g(x)\mathfrak{m}(x)dx$. Any payout $f \in \mathcal{H}$ can be written as a linear combination of these solutions with the coefficients:

$$c_n = \langle f, \varphi_n\rangle / \|\varphi_n\|^2$$

or simply $c_n = \langle f, \varphi_n\rangle$ if the eigenfunctions are normalised so that $\|\varphi_n\|^2 = 1$. We call a security with the payout $\varphi_n$ an eigensecurity. From equation (4), eigensecurities are eigenfunctions of the infinitesimal generator of the underlying diffusion. They are also eigenfunctions of the pricing operator:

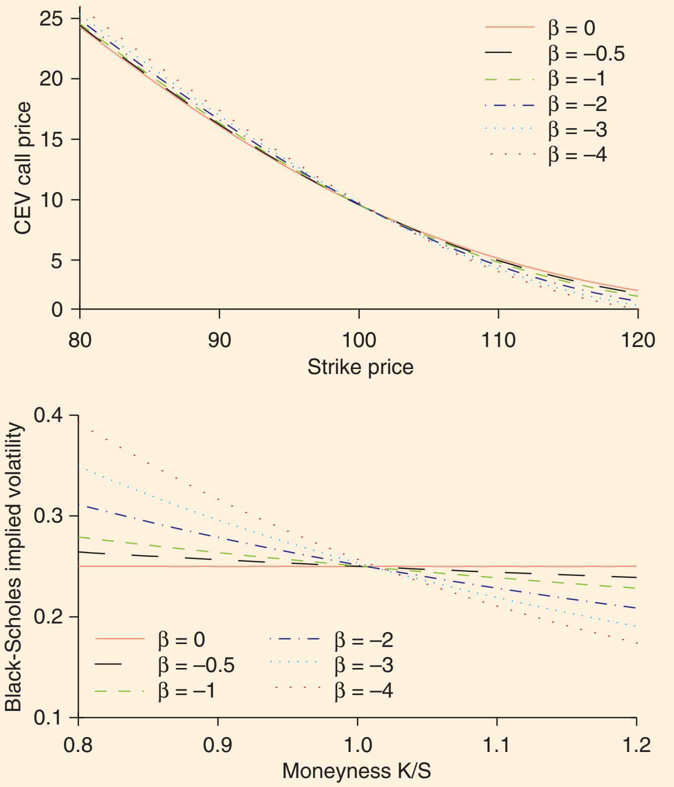
$$E\left[e^{-\int_0^T r(X_u)du}\varphi_n(X_T)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}} \middle| X_0 = x\right] = e^{-\lambda_n T}\varphi_n(x) \tag{5}$$

where $\mathcal{T}_{(L, U)} := \inf\{t \geq 0 : X_t \notin (L, U)\}$ is the first exit time from the interval $(L, U)$. The eigenvalue here is $e^{-\lambda_n T}$, where $\lambda_n$ is the eigenvalue of the infinitesimal generator and $T$ is time to expiry.

In the problem with two absorbing barriers, the eigenvalues $\lambda_n$ grow as $n^2$ as the eigenvalue number increases. An observation of practical interest is that the contributions from higher eigenfunctions in the expansion (3) are suppressed by the factors $e^{-\lambda_n T}$. The longer the time to expiry, the faster the eigenfunction expansion converges. This convergence behaviour contrasts with most other methods, including numerical PDE schemes and Monte Carlo simulation (see Lipton, 2001, chapter 12).

Here we have discussed a regular problem with two knock-out barriers. Both end-points of the interval of interest are regular points for the underlying diffusion process and we have imposed absorbing boundary conditions at the end-points. In cases with only one knock-out barrier or no barriers, the situation is more complicated. Consider a diffusion process (1) on some interval with the left and right end-points $l$ and $r$. Feller's boundary classification classifies end-points into four categories: regular, exit, entrance and natural. Further, natural boundaries can be attracting or unattracting. This refers to the behaviour of the diffusion process near the end-points (Karlin & Taylor, 1981, chapter 15). If there are no natural boundaries, the spectrum is simple and purely discrete. However, if one or both end-points are natural boundaries, the spectrum may be discrete, continuous or mixed (a continuous portion plus some discrete eigenvalues). Furthermore, it may not be

## 1. Vanilla calls with CEV




European-style call price and Black-Scholes implied volatility as functions of strike $K$ under CEV processes with elasticities $\beta = 0, -0.5, -1, -2, -3$ and $-4$. Parameters: $S_0 = 100$, $\sigma_0 = \sigma(100) = \delta(100)^\beta = 0.25$, $r = 0.1$, $q = 0$, $T = 0.5$

simple. There are sufficient conditions available in the singular Sturm-Liouville theory to determine the qualitative nature of the spectrum from the behaviour of the functions $a(x)$, $b(x)$ and $r(x)$ near the end-points.

### Double-barrier options under geometric Brownian motion

To develop our intuition with eigenfunction expansions, here we give a brief treatment of double-barrier options in the standard Black-Scholes framework. Assume that under the risk-neutral measure the underlying asset price follows a geometric Brownian motion process with the initial price $S_0 = x$, constant volatility $\sigma > 0$, risk-free rate $r \geq 0$ and dividend yield $q \geq 0$, and set $\nu := 1/\sigma(r - q - \sigma^2/2)$.

Consider a double-barrier call with strike $K$, expiry date $T$ and two knock-out barriers $L$ and $U$, $0 < L < K < U$. The knock-out provision renders the option worthless as soon as the underlying price leaves the price range ($L$, $U$). There is an extensive literature on double-barrier options under the assumption of geometric Brownian motion (Kunitomo & Ikeda, 1992, Geman & Yor, 1996, Pelsser, 2000, Lipton, 2001, and Davydov & Linetsky, 2002). The double-barrier call payout is $\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}}(S_T - K)^+$.

The Sturm-Liouville problem (4) in this case is:

$$-1/2\sigma^2 x^2 \varphi_{xx} - (r-q)x\varphi_x + r\varphi = \lambda\varphi, \quad \varphi(L) = \varphi(U) = 0 \tag{6}$$

The transformation:

$$y := \sqrt{2}/\sigma \ln(x/L), \quad \varphi(x) = 2^{-\frac{1}{4}}\sqrt{\sigma}\, x^{-\frac{\nu}{\sigma}} \mathrm{v}(y(x)) \tag{7}$$

reduces the Sturm-Liouville problem (6) to the Liouville normal form with the constant potential:

$$-\mathrm{v}_{yy} + Q\mathrm{v} = \lambda \mathrm{v}, \quad \mathrm{v}(0) = 0, \quad \mathrm{v}(B) = 0, \quad B := \sqrt{2}/\sigma \ln(U/L), \quad Q := r + \nu^2/2 \tag{8}$$

It is classical that the functions:

$$\left\{\sqrt{2/B}\sin(n\pi y/B), n = 1,2,\ldots\right\}$$

form a complete set of normalised eigenfunctions for the problem (8) with $Q = 0$ and eigenvalues $n^2\pi^2/B^2$. Then the same functions are eigenfunctions of the problem with constant $Q$, but with the eigenvalues $\lambda_n = Q + n^2\pi^2/B^2$. Inverting the Liouville transformation (7) yields the eigenfunctions of the original problem (6):

$$\varphi_n(x) = \sigma x^{-\frac{\nu}{\sigma}}/\sqrt{\ln(U/L)}\sin\left(\left[\pi n \ln(x/L)\right]/\ln(U/L)\right),$$
$$\lambda_n = r + \nu^2/2 + \left(\sigma^2\pi^2 n^2\right)/\left(2\ln^2(U/L)\right), \quad n = 1,2,\ldots \tag{9}$$

They form a complete orthonormal basis in the Hilbert space $L^2([L, U], \mathfrak{m})$ with the speed density of geometric Brownian motion:

$$\mathfrak{m}(x) = 2/\sigma^2 x^{\frac{2\nu}{\sigma}-1}$$

The call payout $f(x) = (x - K)^+$ on $[L, U]$ can be decomposed in this basis by calculating its inner product with the eigenfunctions:

$$c_n = \langle f, \varphi_n \rangle = L^{\frac{\nu}{\sigma}}/\sqrt{\ln(U/L)}\left[L\psi_n(\nu+\sigma) - K\psi_n(\nu)\right] \tag{10}$$

where:

$$\psi_n(a) := 2/\left(\omega_n^2 + a^2\right)\left[e^{ak}\left(\omega_n\cos(\omega_n k) - a\sin(\omega_n k)\right) - (-1)^n \omega_n e^{au}\right] \tag{11}$$

$$\omega_n := n\pi/u, \quad k := 1/\sigma\ln(K/L), \quad u := 1/\sigma\ln(U/L) \tag{12}$$

Finally, taking the present value of this decomposition of the call payout into the portfolio of eigenpayouts:

$$(S_T - K)^+ \mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} = \sum_{n=1}^{\infty} c_n \varphi_n(S_T)\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}}$$

the double-barrier call price is given by the eigenfunction expansion (3) with the eigenvalues and eigenfunctions (9) and coefficients (10).

### Barrier options under the CEV process with volatility skew

Assume that under the risk-neutral measure the underlying follows Cox's CEV process:

$$dS_t = (r-q)S_t dt + \delta S_t^{\beta+1} dB_t, \quad S_0 = x \tag{13}$$

When $\beta = 0$, the process reduces to the standard geometric Brownian motion. For $\beta < 0$, the local volatility $\sigma(x) = \delta x^\beta$ is a decreasing function of the underlying and the model exhibits volatility skew similar to the skew observed in the equity index options market. Typical values of the CEV elasticity implicit in S&P 500 stock index option prices are around $\beta = -2.5$ to $-4$. For $-1/2 \leq \beta < 0$, zero is an exit boundary. For $\beta < -1/2$, zero is a regular boundary point, and is specified as an absorbing boundary by adjoining an absorbing boundary condition (see Davydov & Linetsky, 2001a and 2001b, for detailed discussions of the CEV process and Albanese *et al*, 2001, for interesting generalisations of the CEV process). Here, we focus on the CEV process with $\beta < 0$ and $r > q$. Figure 1 plots the plain vanilla call price and the Black-Scholes implied volatility as functions of the strike price for different values of the elasticity parameter $\beta$. For each $\beta$, the second CEV parameter $\delta$ is selected so that at-the-money local volatility is 25%, ie, for each $\beta$ we have $\sigma_0 = \sigma(100) = \delta(100)^\beta = 0.25$. The larger the absolute value of $\beta$, the steeper the skew.

The Sturm-Liouville problem now takes the form:

$$-1/2\sigma^2 x^{2\beta+2}\varphi_{xx} - (r-q)x\varphi_x + r\varphi = \lambda\varphi, \quad \varphi(L) = \varphi(U) = 0 \tag{14}$$

The Liouville transformation:

$$y := \sqrt{2}/(\delta|\beta|)x^{-\beta}, \quad \varphi(x) = \sqrt{\delta}\, x^{\frac{1+\beta}{2}}\exp\left(-\mu/\left(2\delta^2|\beta|\right)x^{-2\beta}\right)\mathrm{v}(y(x)) \tag{15}$$

reduces the problem (14) to the Liouville normal form:

$$-\mathrm{v}_{yy} + Q(y)\mathrm{v} = \lambda\mathrm{v}, \quad \mathrm{v}(a) = \mathrm{v}(b) = 0, \quad a = \frac{\sqrt{2}}{\delta|\beta|}L^{-\beta}, \quad b = \frac{\sqrt{2}}{\delta|\beta|}U^{-\beta} \tag{16}$$

with the potential function:

$$Q(y) = b_{-2}/y^2 + b_0 + b_2 y^2, \quad b_{-2} = 1/4\left(1/\beta^2 - 1\right),$$
$$b_0 = r + \mu\left(|\beta| - 1/2\right), \quad b_2 = \mu^2\beta^2/4 \tag{17}$$

The ODE of the form (16–17) is known in mathematical physics as a stationary Schrödinger equation with the radial harmonic oscillator potential. The eigenfunctions can be expressed in terms of the Whittaker functions $M_{k,m}(z)$ and $W_{k,m}(z)$ (Abramowitz & Stegun, 1972, page 505). Introduce the following notation:

$$m := 1/\left(4|\beta|\right), \quad z := \mu/\left(\delta^2|\beta|\right)x^{-2\beta}, \quad l := \mu/\left(\delta^2|\beta|\right)L^{-2\beta},$$
$$u := \mu/\left(\delta^2|\beta|\right)U^{-2\beta}, \quad \kappa := \mu/\left(\delta^2|\beta|\right)K^{-2\beta} \tag{18}$$

and define the function ($\Gamma(z)$ is the standard gamma function):

$$\Delta_{k,m}(x,y) := \frac{\Gamma\left(\frac{1}{2}+m-k\right)}{\Gamma(1+2m)}\left[W_{k,m}(x)M_{k,m}(y) - M_{k,m}(x)W_{k,m}(y)\right] \tag{19}$$

Then the eigenvalues $\{\lambda_n, n = 1, 2, ...\}$ of the spectral problems (14) and (16) are:

$$\lambda_n = r + 2\mu|\beta|\left(k_n - m + 1/2\right) \tag{20}$$

where $\{k_n, n = 1, 2, 3, ...\}$ are the (simple) roots of the equation $\Delta_{k,m}(l, u) = 0$ in the interval $k \in (m - 1/2, \infty)$ (for fixed $l$, $u$ and $m$ defined in (18)). These roots are determined numerically. Let:

$$D_{n,m}(l,u) := \left[\partial\Delta_{k,m}(l,u)/\partial k\right]\Big|_{k=k_n}$$

The corresponding normalised eigenfunctions are:

$$\varphi_n(x) = \mathcal{N}_n x^{\frac{1}{2}+\beta} e^{-\frac{z(x)}{2}}\Delta_{k_n,m}\left(l, z(x)\right), \quad \mathcal{N}_n^2 = \frac{\delta^2|\beta|W_{k_n,m}(u)}{D_{n,m}(l,u)W_{k_n,m}(l)} \tag{21}$$

The eigenvalues (20) and eigenfunctions (21) are CEV counterparts of the eigenvalues and eigenfunctions (9). We now have Whittaker functions instead of trigonometric functions. The eigenfunctions form a complete orthonormal basis in the Hilbert space $L^2([L, U], \mathfrak{m})$ with the speed density of the CEV process:

$$\mathfrak{m}(x) = 2/\delta^2 x^{-2-2\beta}\exp\left(\mu/\left(\delta^2|\beta|\right)x^{-2\beta}\right)$$

The call payout can be decomposed on this basis by calculating its inner product with the eigenfunctions:

$$c_n = \Gamma\left(\tfrac{1}{2}+m-k_n\right)/\Gamma(1+2m)\left[W_{k_n,m}(l)I_n - M_{k_n,m}(l)J_n\right] \tag{22}$$

$$I_n := \frac{1}{\delta\sqrt{\mu|\beta|}}\left[\frac{U^{\frac{1}{2}}}{2m+1}e^{\frac{u}{2}}M_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{2mKU^{-\frac{1}{2}}}{m-k_n-\frac{1}{2}}e^{\frac{u}{2}}M_{k_n+\frac{1}{2},m-\frac{1}{2}}(u)\right.$$
$$\left. - \frac{K^{\frac{1}{2}}}{2m+1}e^{\frac{\kappa}{2}}M_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{2mK^{\frac{1}{2}}}{m-k_n-\frac{1}{2}}e^{\frac{\kappa}{2}}M_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa)\right] \tag{23}$$

$$J_n := \frac{1}{\delta\sqrt{\mu|\beta|}}\left[\frac{U^{\frac{1}{2}}}{k_n+m+\frac{1}{2}}e^{\frac{u}{2}}W_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{KU^{-\frac{1}{2}}}{k_n-m+\frac{1}{2}}e^{\frac{u}{2}}W_{k_n+\frac{1}{2},m-\frac{1}{2}}(u)\right.$$
$$\left. - \frac{K^{\frac{1}{2}}}{k_n+m+\frac{1}{2}}e^{\frac{\kappa}{2}}W_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{K^{\frac{1}{2}}}{k_n-m+\frac{1}{2}}e^{\frac{\kappa}{2}}W_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa)\right] \tag{24}$$

To calculate double-barrier calls, substitute the eigenvalues (20), eigenfunctions (21) and coefficients (22–24) in the expansion (3). The Whittaker functions entering the formulas are related to the confluent hypergeometric functions available in Mathematica and Maple.[1] Note that both the eigenfunctions and the coefficients are given in closed form and we do not need to calculate any integrals numerically or invert Laplace transforms. The only required numerical procedure is the root finding for $k_n$.[2] Table A illustrates convergence of the eigenfunction expansions. The eigenvalues $\lambda_n$ grow as $n^2$ with $n$ and the higher terms are suppressed by $e^{-\lambda_n T}$, so the expansion converges very fast. One term is enough for one year to expiry, and five or six terms are needed for one month. In both cases, computation times are a fraction of a second. The table also gives the results from Davydov & Linetsky (2001a) obtained via numerical Laplace transform inversion. The values for $\beta = 0$ (geometric Brownian motion) are calculated using the analytical formulas in the previous section.

## Arithmetic Asian options

We now turn to arithmetic Asian options. We assume that under the risk-neutral measure the underlying follows a geometric Brownian motion $S_t = S_0 e^{\sigma B_t + (r - q - \sigma^2/2)t}$, $t \geq 0$. Define a continuous arithmetic average process: $\mathcal{A}_0 = S_0$, $\mathcal{A}_t = 1/t \int_0^t S_u\, du$, $t > 0$. An arithmetic Asian call (put) with strike $K$ and expiry $T$ delivers the payout $(\mathcal{A}_T - K)^+$ ($(K - \mathcal{A}_T)^+$). It is sufficient to consider Asian puts. Asian calls can be recovered by applying the well-known put-call parity result for Asian options (Geman & Yor, 1993).

Consider a generalised Asian put payout: $(K - wS_T - (1 - w)\mathcal{A}_T)^+$, where $0 \leq w < 1$ is the weight parameter. For $w = 0$, this payout reduces to the standard Asian put. Following Geman & Yor (1993), standardise the problem as follows:

$$e^{-rT}E\left[\left(K - wS_T - (1-w)\mathcal{A}_T\right)^+\right] = e^{-rT}\left(\frac{4S_0(1-w)}{\sigma^2 T}\right)P^{(\nu)}(x,k,\tau) \tag{25}$$

where the function $P^{(\nu)}(x, k, \tau)$ is defined by:

$$P^{(\nu)}(x,k,\tau) := E\left[\left(k - xe^{2(B_\tau + \nu\tau)} - A_\tau^{(\nu)}\right)^+\right] \tag{26}$$

$A_\tau^{(\nu)}$ is a Brownian exponential functional, $A_\tau^{(\nu)} := \int_0^\tau e^{2(B_u + \nu u)}du$ (Yor, 2001), and the standardised parameters $\tau$ (dimensionless time to expiry), $k$, $x$ and $\nu$ are:

$$\tau := \frac{\sigma^2 T}{4}, \quad k := \frac{\tau K}{S_0(1-w)}, \quad x := \frac{\tau w}{1-w}, \quad \nu := \frac{2(r-q)}{\sigma^2} - 1 \tag{27}$$

This reduction follows from the scaling property of Brownian motion.

Our starting point is the identity in law for any fixed $t \geq 0$ due to Dufresne (1989):

$$xe^{2(B_t + \nu t)} + A_t^{(\nu)} \overset{(law)}{=} X_t \tag{28}$$

where the process $\{X_t, t \geq 0\}$ is defined by:

$$X_t = e^{2(B_t + \nu t)}\left(x + \int_0^t e^{-2(B_u + \nu u)}du\right), \quad t \geq 0 \tag{29}$$

has an Itô differential:

$$dX_t = \left[2(\nu+1)X_t + 1\right]dt + 2X_t dB_t \tag{30}$$

and starts at $x$, $X_0 = x$. Thus, for any fixed $t \geq 0$, the process $xe^{2(B_t + \nu t)} + A_\tau^{(\nu)}$ has the same distribution as a one-dimensional diffusion process on $(0, \infty)$ starting at $x \geq 0$. Hence, the problem of pricing the arithmetic Asian put is reduced to the problem of pricing a vanilla put on the diffusion (30): $P^{(\nu)}(x, k, \tau) = E[(k - X_\tau)^+ | X_0 = x]$. One strategy is to consider an option on this diffusion process stopped at an independent exponential time and paying out at that (random) time. This can be worked out in closed form in terms of Whittaker functions (Donati-Martin, Ghomrasni & Yor, 2001). For an alternative PDE-based derivation, see Lipton (1999). To recover the option expiring at a fixed time $T > 0$ one needs to invert the Laplace transform numerically.

An alternative approach is to develop an eigenfunction expansion (Linetsky, 2001). Diffusion (30) has an entrance boundary at zero and a natural boundary at infinity. For $\nu < 0$, the process has a stationary distribution that is reciprocal gamma (Wong, 1964). The associated Sturm-Liouville problem is:

[1] *We used Mathematica 4.0 for all calculations in this article*

[2] *This can be accomplished with arbitrarily high precision in Mathematica and Maple*

## A. CEV double-barrier calls

| | β = 0 | −0.5 | −1 | −2 | −3 | −4 |
|---|---|---|---|---|---|---|
| N | Double-barrier call T = 1 month | | | | | |
| 1 | 4.8723 | 5.2942 | 5.7393 | 6.7041 | 7.7843 | 9.0208 |
| 2 | 3.0908 | 3.1116 | 3.1043 | 2.9892 | 2.7103 | 2.2176 |
| 3 | 2.9923 | 3.0536 | 3.1039 | 3.1770 | 3.2347 | 3.3242 |
| 4 | 3.0161 | 3.0834 | 3.1402 | 3.2237 | 3.2740 | 3.2999 |
| 5 | 3.0154 | 3.0820 | 3.1376 | 3.2169 | 3.2598 | 3.2764 |
| 6 | 3.0154 | 3.0820 | 3.1376 | 3.2171 | 3.2607 | 3.2794 |
| Laplace | 3.0154 | 3.0820 | 3.1376 | 3.2171 | 3.2606 | 3.2793 |
| N | Double-barrier call T = 12 months | | | | | |
| 1 | 0.1410 | 0.1672 | 0.1994 | 0.2859 | 0.4105 | 0.5827 |
| 2 | 0.1410 | 0.1672 | 0.1994 | 0.2859 | 0.4103 | 0.5822 |
| Laplace | 0.1410 | 0.1673 | 0.1994 | 0.2860 | 0.4104 | 0.5823 |

Convergence of eigenfunction expansions for double-barrier calls under the CEV processes with β = 0, −0.5, −1, −2, −3, −4. For $T = 1$ month, for each option seven values are given: partial sums of the first $N$ terms of the expansion equation (3) ($N = 1, \dots, 6$) and the value obtained by the numerical Laplace inversion. For $T = 12$ months, for each price three values are given: partial sums of the first $N$ terms of the expansion ($N = 1, 2$) and the value obtained by the numerical Laplace inversion. Numerical Laplace inversion values are taken from Davydov & Linetsky (2001a). Parameters: $S_0 = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$. The CEV parameter δ is selected so that at-the-money local volatility is 25% for all β, ie, $\sigma(100) = \delta(100)^{-\beta} = 0.25$ for all β

$$-2x^2\varphi_{xx} - \left[2(\nu+1)x+1\right]\varphi_x = \lambda\varphi, \quad x \in (0,\infty) \tag{31}$$

This problem is singular at both end-points 0 and +∞. Both end-points are of the limit point type in Weyl's classification and no additional boundary conditions are allowed at the end-points. The Liouville transformation:

$$y := 1/\sqrt{2}\ln x, \qquad \varphi(x) = x^{-\frac{\nu}{2}}e^{\frac{1}{4x}}v(y(x))$$

reduces the ODE (31) to the Liouville normal form:

$$-v_{yy} + Q(y)v = \lambda v, \quad Q(y) = \frac{1}{8}e^{-2\sqrt{2}y} + \frac{\nu-1}{2}e^{-\sqrt{2}y} + \frac{\nu^2}{2}, \quad -\infty < y < \infty \tag{32}$$

The ODE (32) has the form of the stationary Schrödinger equation with Morse potential well known in quantum mechanics (Morse, 1929, and Morse & Feshback, 1953, pages 1,671–1,672). For $\nu \geq 0$, this problem has a purely continuous spectrum in $(\nu^2/2, \infty)$. For $\nu < 0$, there are $[|\nu|/2] + 1$ additional discrete eigenvalues in the interval $[0, \nu^2/2)$. The natural boundary at infinity is oscillatory for the ODE (31) for $\lambda > \nu^2/2$ (any solution has an infinite number of zeros increasing towards infinity) and this results in the continuous spectrum in $(\nu^2/2, \infty)$. Therefore, the corresponding eigenfunction expansion contains an integral that has to be calculated numerically.

To avoid numerical integration, fix a large number $b > 0$ and consider an up-and-out put on the diffusion (30) with the knock-out barrier placed at $b$. For sufficiently large $b$ the value of this up-and-out put:

$$P_b^{(\nu)}(x,k,\tau) := E\left[1_{\{\mathcal{T}_b > \tau\}}(k - X_\tau)^+ \middle| X_0 = x\right], \quad \mathcal{T}_b := \inf\{t \geq 0 : X_t = b\}$$

will closely approximate the vanilla put on diffusion (30), $P^{(\nu)}(x, k, \tau) = E[(k - X_\tau)^+ | X_0 = x]$. The problem on the finite interval $(0, b]$ with the absorbing boundary condition at $b$ has a purely discrete spectrum. The eigenfunctions $\varphi_n$ and coefficients $c_n$ can be found in closed form, and we obtain an analytical expression for $P_b^{(\nu)}(x, k, \tau)$. Taking the limit $b \to \infty$, we recover the eigenfunction expansion for the original problem on the infinite interval $(0, \infty)$ with continuous spectrum. As a final step, note that the starting value $x$ of the process (30) depends on $w$ (equation (27)). The standard Asian put payout is obtained in the limit $w \to 0$. This corresponds to starting the process (30) at zero (zero is an entrance boundary and, hence, the process can be started there). Taking the limit $x \to 0$ in the eigenfunction expansions for $P_b^{(\nu)}(x, k, \tau)$ and $P^{(\nu)}(x, k, \tau)$ produces the desired pricing formulas for the standard Asian put upon substituting in equation (26) and setting $w = 0$. We now present the resulting formulas. Details of the proof can be found in Linetsky (2001).

The formulas for the functions $P_b^{(\nu)}(k, \tau) = \lim_{x\to 0} P_b^{(\nu)}(x, k, \tau)$ and $P^{(\nu)}(k, \tau) = \lim_{x\to 0} P^{(\nu)}(x, k, \tau)$ will be expressed in terms of the Whittaker functions, gamma function, the incomplete gamma function $\Gamma(a, z)$ and the generalised Laguerre polynomials $L_n^{(\alpha)}(z)$ (Abramowitz & Stegun, 1972).

Fix a large number $b > k$. Let $\{p_{n,b}, n = 0, 1, 2, \dots\}$ be the zeros on the positive real line $p > 0$ of the Whittaker function $W_{\kappa,\mu}(z)$ with the fixed first index $\kappa = (1 - \nu)/2$, fixed argument $z = 1/(2b)$ and the purely imaginary second index $\mu = ip/2$, $p > 0$. That is, the $p_{n,b}$ are the positive roots of the equation:

$$W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(1/(2b)\right) = 0$$

This equation has an infinite set of simple roots on the positive real axis. If the roots are ordered by $0 < p_{0,b} < p_{1,b} < \dots$, then $p_{n,b} \to \infty$ as $n \to \infty$. Let:

$$\xi_n^{(\nu)}(b) := \left[\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(1/(2b)\right)/\partial p\right]\Big|_{p=p_{n,b}}$$

Furthermore, let $m_\nu(b) \geq 0$ be the total number of roots of the equation:

$$W_{\frac{1-\nu}{2},\frac{q}{2}}\left(1/(2b)\right) = 0$$

in the interval $0 \leq q < |\nu|$ and, for $m_\nu(b) > 0$, let $\{q_{n,b}, n = 0, \dots, m_\nu(b)\}$ be the corresponding roots. Let:

$$\eta_n^{(\nu)}(b) := \left[-\partial W_{\frac{1-\nu}{2},\frac{q}{2}}\left(1/(2b)\right)/\partial q\right]\Big|_{q=q_{n,b}}$$

Then the function $P_b^{(\nu)}(k, \tau)$ is given by the following eigenfunction expansion ($\Sigma_{n=0}^{-1} \equiv 0$ by convention):

$$\begin{aligned} P_b^{(\nu)}(k,\tau) &= \sum_{n=0}^{\infty} e^{-\frac{(\nu^2+p_{n,b}^2)\tau}{2}} \frac{p_{n,b}\Gamma\left(\frac{\nu+ip_{n,b}}{2}\right)}{4\xi_n^{(\nu)}(b)\Gamma(1+ip_{n,b})}(2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2k}\right) M_{\frac{1-\nu}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2b}\right) \\ &+ \sum_{n=0}^{m_\nu(b)-1} e^{-\frac{(\nu^2-q_{n,b}^2)\tau}{2}} \frac{q_{n,b}\Gamma\left(\frac{\nu+q_{n,b}}{2}\right)}{4\eta_n^{(\nu)}(b)\Gamma(1+q_{n,b})}(2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2k}\right) M_{\frac{1-\nu}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2b}\right) \end{aligned} \tag{33}$$

As $b$ increases towards infinity, the eigenvalues are spaced closer and closer together and in the limit merge into the continuous spectrum. In the limit $\lim_{b\to\infty} P_b^{(\nu)}(k, \tau) = P^{(\nu)}(k, \tau)$ the series formula (33) yields the integral formula ($[x]$ denotes the integer part of $x$):

## B. Asian call option prices (q = 0 and K = 2.0)

| Case | r | σ | T | $S_0$ | ν | τ | EE | MC |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.02 | 0.10 | 1 | 2.0 | 3 | 0.0025 | 0.0559860415 (400) | 0.05602 (0.00017) |
| 2 | 0.18 | 0.30 | 1 | 2.0 | 3 | 0.0225 | 0.2183875466 (57) | 0.2185 (0.00059) |
| 3 | 0.0125 | 0.25 | 2 | 2.0 | −0.6 | 0.03125 | 0.1722687410 (41) | 0.1725 (0.00063) |
| 4 | 0.05 | 0.50 | 1 | 1.9 | −0.6 | 0.0625 | 0.1931737903 (24) | 0.1933 (0.00084) |
| 5 | 0.05 | 0.50 | 1 | 2.0 | −0.6 | 0.0625 | 0.2464156905 (23) | 0.2465 (0.00095) |
| 6 | 0.05 | 0.50 | 1 | 2.1 | −0.6 | 0.0625 | 0.3062203648 (23) | 0.3064 (0.00106) |
| 7 | 0.05 | 0.50 | 2 | 2.0 | −0.6 | 0.125 | 0.3500952190 (13) | 0.3503 (0.00146) |

$$\begin{aligned}P^{(\nu)}(k,\tau) &= \frac{1}{8\pi^2}\int_0^\infty e^{-\frac{(\nu^2+p^2)\tau}{2}}(2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{ip}{2}}\left(\frac{1}{2k}\right)\left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p)\,p\,dp \\ &+\mathbf{1}_{\{\nu<0\}}\frac{1}{2\Gamma(|\nu|)}\left\{2k\Gamma(|\nu|,1/(2k))-\Gamma(|\nu|-1,1/(2k))\right\} \\ &+\mathbf{1}_{\{\nu<-2\}}e^{-2(|\nu|-1)\tau}\frac{(|\nu|-2)}{\Gamma(|\nu|)}\Gamma(|\nu|-2,1/(2k)) \\ &+\mathbf{1}_{\{\nu<-4\}}\sum_{n=2}^{[|\nu|/2]} e^{-2n(|\nu|-n)\tau}\frac{(-1)^n(|\nu|-2n)}{2n(n-1)\Gamma(1+|\nu|-n)}(2k)^{\nu+n+1} e^{-\frac{1}{2k}} L_{n-2}^{(|\nu|-2n)}\left(\frac{1}{2k}\right)\end{aligned} \tag{34}$$

For typical parameter values, the series formula approximates the integral formula so well that for $b = 1$ the numerical Asian option prices calculated using the series formula agree with the prices calculated using the integral formula at the precision level of 10 significant digits or better. The eigenfunction expansion formulas (33) and (34) allow exact calculation of the Asian put price (26) (with $w = x = 0$) and provide an alternative to the numerical Laplace transform inversion necessary to calculate the Geman & Yor (1993) formula.

We now use the eigenfunction expansion formulas to calculate Asian option prices. Both Mathematica and Maple have the special functions needed for this calculation. We calculate Asian puts first, and then use the call-put parity for Asian options to calculate calls. We consider seven combinations of parameters, as shown in table B. These combinations were used as test cases for various numerical methods by Eydeland & Geman (1995), Fu, Madan & Wang (1997), Dufresne (2000) and others. Pick $b = 1$ and consider the steps needed to calculate the series formula (33). First, we need to determine the eigenvalues. The eigenvalues need to be determined only once for all options with different strikes, times to expiry and current values of the underlying asset price. Both Mathematica and Maple handle special functions with arbitrary precision arithmetics, so we are able to determine the zeros $p_{n,b}$ and $q_{n,b}$ of the Whittaker function with arbitrarily high precision. After the eigenvalues are determined, the Asian puts are calculated using the series formula.

The integral formula (34) was calculated in Mathematica using the built-in numerical integration routine. The advantage of the series formula is that no numerical integration is required. The results for the Asian calls are reported in the column marked EE (eigenfunction expansion). The numbers in brackets next to the option prices give the number of terms in the series (9) needed to achieve the precision of 10 significant digits. One can see that for case seven, with the largest value of $\tau$, 13 terms are enough to achieve this high level of precision. However, for case one, with the smallest value of $\tau$, 400 terms are needed. Both the series with $b = 1$ and the integral gave identical results in all seven cases at the precision level of 10 significant digits. The integral formula was somewhat slower to compute due to the numerical integration. Table B also gives Asian call prices obtained by simulation in Dufresne (2000) (standard errors are given in brackets).

## Conclusion

The eigenfunction expansion approach is a powerful tool for generating analytical option pricing formulas. While general-purpose numerical methods are required in practice to handle numerous market realities such as time-dependent parameters, discrete dividends, discrete sampling, day count conventions, etc, analytical formulas provide important accuracy benchmarks. Importantly, there is no loss of precision in calculating the Greeks. The analytical eigenfunction expansion approach can also handle long-dated contracts. For longer times to expiry, a few terms are often enough to produce high accuracy. ■

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