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PRICING OPTIONS ON SCALAR DIFFUSIONS: AN EIGENFUNCTION EXPANSION APPROACH

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This paper develops an eigenfunction expansion approach to pricing options on scalar diffusion processes. All contingent claims are unbundled into portfolios of primitive securities called *eigensecurities*. Eigensecurities are eigenvectors (eigenfunctions) of the pricing operator (present value operator). All computational work is at the stage of finding eigenvalues and eigenfunctions of the pricing operator. The pricing is then immediate by the linearity of the pricing operator and the eigenvector property of eigensecurities. To illustrate the computational power of the method, we develop two applications: pricing vanilla, single- and double-barrier options under the constant elasticity of variance (CEV) process and interest rate knock-out options in the Cox-Ingersoll-Ross (CIR) term-structure model.

Received November 2000; revision received October 2001; accepted August 2002.

Subject classifications: Finance, asset pricing: option pricing, CEV model, CIR model. Finance, securities: barrier options. Probability, diffusion: spectral theory, barrier crossing, generalized Bessel process.

Area of review: Financial Services.

1. INTRODUCTION

In this paper we develop an option pricing methodology based on unbundling all contingent claims into portfolios of primitive securities called *eigensecurities*. Eigensecurities are eigenvectors of the pricing operator (present value operator).

Arrow-Debreu securities, each paying one dollar in one specific state of nature and nothing in any other state, are the fundamental building blocks in asset pricing theory (see Duffie 2001). In a continuum of states, the prices of Arrow-Debreu securities are defined by the state-price density, which gives for each state x the price of a security paying one dollar if the state falls between x and $x+dx$. If we know the functional form of the state-price density, we can price any European-style contingent claim by integrating the terminal payoff against the state-price density. In the diffusion setting, the state-price density can be found as a fundamental solution of the pricing partial differential equation (PDE) subject to some boundary conditions. Unfortunately, the task of solving the pricing PDE in closed form is often formidable, and no explicit analytical expressions for the state-price density are available in many cases of interest in applications.

Here we develop an alternative valuation methodology. Instead of using Arrow-Debreu securities to span the space of European-style contingent claims written on a scalar diffusion process, we introduce a concept of *eigensecurities*, or eigenvectors of the pricing operator, as fundamental building blocks in our approach.[1] Eigensecurities diagonalize the pricing operator. All other European-style contingent claims with square-integrable payoffs are represented as portfolios of eigensecurities. Furthermore, the connection between eigensecurities and Arrow-Debreu securities can be established as follows. Arrow-Debreu securities themselves can be formally unbundled into portfolios of eigensecurities. This produces an eigenfunction expansion of the state-price density (spectral representation of the state-price density). Depending on the nature of the diffusion process and boundary conditions, the spectrum can be discrete, continuous, or mixed (see McKean 1956; Wong 1964; Ito and McKean 1974, pp. 149–161; Karlin and Taylor 1981, pp. 330–340; and Schoutens 2000 for applications of eigenfunction expansions, called *eigen-differential expansions* by Ito and McKean, to diffusion processes).

The eigenfunction expansion method is a powerful computational tool for derivatives pricing. First, while the state-price density solves the boundary-value problem for the pricing PDE, the eigensecurities are solutions to the *static* pricing equation without the time derivative term. In the scalar diffusion context, this static pricing equation can be interpreted as a second-order *ordinary* differential equation (ODE) of the Sturm-Liouville type (see Dunford and Schwartz 1963, Levitan and Sargsjan 1975, Stakgold 1998, Titchmarsh 1962, and Zwillinger 1998 for the account of the Sturm-Liouville theory). Second, in cases where the state space is a finite interval with two unmixed (e.g., absorbing or reflecting) boundary conditions at the endpoints, the spectrum of the associated Sturm-Liouville problem is guaranteed to be simple, purely discrete, and bounded below. Accordingly, eigenfunction expansions for security prices are infinite series. Moreover, eigenvalues λ_n, $n = 1, 2, \ldots,$ grow as n^2 and eigenfunction expansions converge rapidly, with contributions from the higher eigenfunctions suppressed by the factors $e^{-\lambda_n T}$ (where T is time to maturity). Only a limited number of terms in the expansion are typically needed to achieve high accuracy in applications.

Several applications of the spectral method to problems in financial economics have already been considered in the literature. Hansen et al. (1998) and Florens et al. (1998) develop spectral methods for econometric applications (estimation of scalar diffusions). Beaglehole (1991) and Goldstein and Keirstead (1997) apply the eigenfunction expansion approach to the pricing of bonds when the short-rate process follows a scalar diffusion. In an interesting recent paper, Lewis (1998) applies the eigenfunction expansion approach to solve two problems in continuous-time finance: pricing options on stocks that pay dividends at a constant dollar rate and pricing bonds under a short-rate process with nonlinear drift. Lewis (2000) applies the eigenfunction expansion approach to the analysis of stochastic volatility models.

In this paper, we develop a general eigenfunction expansion method for claims contingent on scalar diffusions and study two specific applications: Pricing vanilla, single-barrier, and double-barrier options under Cox's constant elasticity of variance (CEV) process and interest rate knock-out options in the Cox-Ingersoll-Ross (CIR) term-structure model.

To give a rough idea of the method, consider the pricing of a European-style derivative security whose payoff at expiration time T is a function f of a single state variable X, which follows a scalar diffusion process under the risk-neutral measure:

$$dX_t = b(X_t)\,dt + a(X_t)\,dB_t, \quad X_0 = x.$$

Suppose the derivative contract has a double-barrier provision: If at any time between the contract inception and expiration either a lower barrier L or an upper barrier U is violated, the contract is canceled (*knocked out*). It is well known that the value V of the derivative security, considered as a function of current time t and state x, solves the fundamental pricing PDE

$$\frac{1}{2}a^2(x)\frac{\partial^2 V}{\partial x^2} + b(x)\frac{\partial V}{\partial x} - r(x)V = -\frac{\partial V}{\partial t},$$

with the payoff condition at expiration $V(x, T) = f(x)$ and two boundary conditions at the barriers $V(L, t) = V(U, t) = 0$ (we have assumed that the instantaneous risk-free interest rate r is also a function only of x). We look for solutions in the form

$$V(x, t) = \sum_{n=1}^{\infty} c_n\, e^{-\lambda_n (T-t)} \varphi_n(x).$$

Due to the linear form of the PDE, each of the $\varphi_n(x)$ satisfy the ODE

$$-\frac{1}{2}a^2(x)\frac{d^2\varphi_n}{dx^2} - b(x)\frac{d\varphi_n}{dx} + r(x)\varphi_n = \lambda_n \varphi_n,$$

with the boundary conditions $\varphi_n(L) = \varphi_n(U) = 0$. It is classical that solutions $\{\varphi_n\}_{n=1}^{\infty}$ to this boundary-value ODE problem form a complete and orthogonal basis in the Hilbert space of all square-integrable functions on the interval $[L, U]$ with the weight $\mathfrak{m}(x)$ given by Equation (4), and thus any such function can be written as a linear combination of these solutions (eigenfunctions). In cases with only one barrier or no barriers, the problem may have a continuous spectrum, and there is an integral in place of the sum.

The remainder of this paper is organized as follows. In §2 we formally introduce eigensecurities and develop the general methodology of pricing options on scalar diffusions via eigenfunction expansions. Section 2.1 deals with regular problems. Section 2.2 discusses singular problems. In §3 we apply the method to vanilla, single-barrier, and double-barrier options under the CEV process. Our main result is the analytical inversion of the Laplace transforms in time to expiration for CEV barrier option prices obtained by Davydov and Linetsky (2001). In §4 we apply the method to interest rate knock-out options in the CIR term-structure model. In §5 we give computational results, illustrate convergence of eigenfunction expansions for barrier options, and compare their computational performance with numerical finite-difference PDE schemes. Section 6 concludes the paper. Proofs are collected in the appendix.

2. AN EIGENFUNCTION EXPANSION APPROACH FOR OPTIONS ON SCALAR DIFFUSIONS

2.1. Regular Problems

2.1.1. General Set-Up. In this paper, we take an equivalent martingale measure Q as given and assume that under Q the state variable in our economy follows a one-dimensional, time-homogeneous diffusion process $\{X_t, t \geqslant 0\}$ taking values in some interval $D \subset \mathbb{R}$ with the end-points $\mathfrak{l}$ and $\mathfrak{r}$, $-\infty \leqslant \mathfrak{l} < \mathfrak{r} \leqslant \infty$, and with the infinitesimal generator

$$(\mathcal{G}f)(x) = \frac{1}{2}a^2(x)f''(x) + b(x)f'(x). \tag{1}$$

We assume that diffusion and drift coefficients $a(x)$ and $b(x)$ are continuous and $a(x) > 0$ for all $x \in (\mathfrak{l}, \mathfrak{r})$. The boundary behavior at the end-points $\mathfrak{l}$ and $\mathfrak{r}$ depends on the behavior of functions $a(x)$ and $b(x)$ as $x \to \mathfrak{l}$ and $x \to \mathfrak{r}$.[2] If any of the end-points is a regular boundary, we adjoin a killing boundary condition at that end-point, sending the process to a cemetery state ∂ at the first hitting time of the end-point. We also assume that the instantaneous risk-free interest rate is a function of the state variable, $r_t = r(X_t)$, and $r(x)$ is nonnegative and continuous for all $x \in (\mathfrak{l}, \mathfrak{r})$.

Let $I = (L, U)$ be an interval in the interior of D, $\mathfrak{l} < L < U < \mathfrak{r}$, and assume that the initial state $x \in (L, U)$. Let f be a square-integrable function on I. Consider a double-barrier claim that pays off an amount $f(X_T)$ at expiration $T > 0$ if the process X does not leave the interval (L, U) prior to expiration, and zero otherwise. Then the price of this double-barrier claim at time $t = 0$ is given by the risk-neutral expectation of the discounted payoff

$$V(x,T) = E_x\left[e^{-\int_0^T r(X_t)\,dt} f(X_T)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\right], \tag{2}$$

where the subscript x in E_x signifies that the process X starts at x at $t=0$, $\mathcal{T}_{(L,U)} = \inf\{t \geqslant 0\colon X_t \notin (L,U)\}$ is the first exit time from (L,U), and $\mathbf{1}_{\{A\}}$ is the indicator function of the event A. The following proposition summarizes the eigenfunction expansion method in this setting.

PROPOSITION 1. *Let $\mathfrak{s}$ and $\mathfrak{m}$ be the scale and speed densities*[3] *of the diffusion process* (1):

$$\mathfrak{s}(x) = \exp\left\{-\int^x \frac{2b(y)}{a^2(y)}\,dy\right\}, \tag{3}$$

$$\mathfrak{m}(x) = \frac{2}{a^2(x)\mathfrak{s}(x)}. \tag{4}$$

Let $\mathcal{H} = L^2([L,U],\mathfrak{m})$ be the Hilbert space of functions on (L,U) square-integrable with the speed density $\mathfrak{m}$ and endowed with the inner product

$$\langle f,g\rangle = \int_L^U f(x)g(x)\mathfrak{m}(x)\,dx. \tag{5}$$

(i) *$\mathcal{H}$ admits a complete orthonormal basis $\{\varphi_n(x)\}_{n=1}^\infty$, $\langle\varphi_n,\varphi_m\rangle = 1\,(0)$ if $n=m$ $(n\neq m)$, such that φ_n are eigenvectors* (*eigenfunctions*) *of the pricing operator*

$$E_x\left[e^{-\int_0^T r(X_t)\,dt}\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\varphi_n(X_T)\right] = e^{-\lambda_n T}\varphi_n(x) \tag{6}$$

for some $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$ with $\lambda_n\to\infty$ as $n\to\infty$. Any payoff $f\in\mathcal{H}$ is in the span of eigenpayoffs φ_n:

$$f = \sum_{n=1}^\infty c_n\varphi_n, \tag{7}$$

$$c_n = \langle f,\varphi_n\rangle, \tag{8}$$

and convergence is in the norm of the Hilbert space.

(ii) *Let $\mathcal{A}$ be the second-order differential operator* (*the negative of the infinitesimal generator of the pricing semigroup*)

$$\begin{aligned}(\mathcal{A}f)(x) &:= -\frac{1}{2}a^2(x)f''(x) - b(x)f'(x) + r(x)f(x)\\ &= -\frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{\mathfrak{s}(x)}\right)' + r(x)f(x),\end{aligned} \tag{9}$$

where the second equality in (9) *follows from the definitions of the scale and speed densities* (3) *and* (4). *The eigenvalue–eigenfunction pairs (λ_n,φ_n) solve the second-order ODE with the two Dirichlet boundary conditions* (regular Sturm-Liouville boundary-value problem):[4]

$$(\mathcal{A}u)(x) = \lambda u(x),\quad u(L)=0,\quad u(U)=0. \tag{10}$$

(iii) *The price of the double-barrier claim* (2) *is given by the eigenfunction expansion*

$$V(x,T) = \sum_{n=1}^\infty c_n e^{-\lambda_n T}\varphi_n(x). \tag{11}$$

PROOF. See the appendix.

Proposition 1 unbundles any European-style, double-barrier contingent claim with the payoff in $\mathcal{H}$ into a portfolio of *eigensecurities* with *eigenpayoffs* φ_n. The pricing is then automatic by the linearity of the pricing operator and the eigenvector property of the eigenpayoffs (6). From the practical standpoint, all the work is at the stage of determining the eigenvalues λ_n and the corresponding normalized eigenfunctions φ_n. This is accomplished by solving the regular Sturm-Liouville boundary value problem (9)–(10).

So far we have limited our discussion to payoffs that occur at some prespecified time $T\geqslant 0$. Our results can be straightforwardly extended to continuous dividend streams. Consider a security with dividends paid continuously during $[0, T\wedge\mathcal{T}_{(L,U)}]$. The dividends stop at time T or the first exit time $\mathcal{T}_{(L,U)}$, whichever comes first. Let $f_t = f(X_t)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}}$ be the dividend-rate process, so that the cumulative dividend process of a security is $D_t = \int_0^t f(X_u)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>u\}}du$. Then the risk-neutral pricing formula is (e.g., Duffie 2001, p. 225)

$$V(x,T) = E_x\left[\int_0^T e^{-\int_0^t r(X_u)du} f(X_t)\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}}\,dt\right].$$

Application of Equation (11) and Fubini's theorem yields the result for continuous dividend streams:

$$V(x,T) = \sum_{n=1}^\infty\left(\frac{1-e^{-\lambda_n T}}{\lambda_n}\right)c_n\varphi_n(x),\qquad c_n = \langle f,\varphi_n\rangle.$$

2.1.2. Determining Eigenvalues and Eigenfunctions. From the results in the previous section, the continuous state-price density $p(t;x,y)$ with two killing boundary conditions at L and U has a spectral representation:

$$\begin{aligned}p(t;x,y)\,dy &\equiv E_x\left[e^{-\int_0^t r(X_u)du}\mathbf{1}_{\{\mathcal{T}_{(L,U)}>t\}};\, X_t\in dy\right]\\ &= \sum_{n=1}^\infty e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)\mathfrak{m}(y)\,dy.\end{aligned} \tag{12}$$

Introduce a *resolvent kernel* or *Green's function* as the Laplace transform of the transition probability density $(s>0)$

$$G_s(x,y) := \int_0^\infty e^{-st}p(t;x,y)\,dt. \tag{13}$$

This definition of the Green's function is customary in probability theory (e.g., Borodin and Salminen 1996; note that Borodin and Salminen define the Green's function with respect to the speed measure, while here we define it with respect to the Lebesgue measure). To simplify the subsequent formulae, it is convenient to change the sign of the transform variable $s=-\lambda$ and define the Green's function as follows:

$$g(x,y;\lambda) := G_{-\lambda}(x,y).$$

This definition of the Green's function is customary in the Sturm-Liouville theory (e.g., Stakgold 1998). Note that in

Davydov and Linetsky (2001) we used the notation λ for the transform variable s. Thus, our λ here is the negative of the λ in Davydov and Linetsky (2001).

From Equation (12), the Green's function of the regular Sturm-Liouville problem with two Dirichlet boundary conditions can be represented as

$$g(x,y;\lambda)=\mathfrak{m}(y)\sum_{n=1}^{\infty}\frac{\varphi_n(x)\varphi_n(y)}{\lambda_n-\lambda}. \tag{14}$$

Continuing the right-hand side of Equation (14) to the whole complex λ plane, for each $x, y\in(L,U)$ the Green's function is a meromorphic function of λ with simple poles at $\lambda=\lambda_n$, $n=1,2,\dots$, and with the corresponding residues $-\mathfrak{m}(y)\varphi_n(x)\varphi_n(y)$. In practice, one way to determine the eigenvalues and the corresponding normalized eigenfunctions φ_n of the regular Sturm-Liouville problem is to construct the Green's function in such a way that we can keep track of its dependence on λ, and then find its poles and calculate the residues.

We note that for each $y\in(L,U)$ the Green's function $g(x,y;\lambda)$ is the unique continuous solution of the inhomogeneous ODE with two Dirichlet boundary conditions ($\delta(x)$ is the Dirac delta function):

$$(\mathscr{A}-\lambda)\,g(x,y;\lambda)=\delta(x-y),\quad x\in(L,U),$$
$$g(L,y;\lambda)=g(U,y;\lambda)=0. \tag{15}$$

The solution to this boundary-value problem can be constructed as follows (see Stakgold 1998, p. 441). For each complex λ, let $\xi_\lambda(x)$ and $\eta_\lambda(x)$ be the unique solutions of the homogeneous ODE

$$(\mathscr{A}u)(x)=\lambda u(x),\quad x\in(L,U), \tag{16}$$

with the initial conditions (prime denotes differentiation in x)

$$\xi_\lambda(L)=0,\quad \xi_\lambda'(L)=1 \tag{17}$$

and

$$\eta_\lambda(U)=0,\quad \eta_\lambda'(U)=-1. \tag{18}$$

For each x, the $\xi_\lambda(x)$ and $\eta_\lambda(x)$ are *entire* functions of λ (analytic in the whole λ plane). This follows from the fact that $\xi_\lambda(x)$ and $\eta_\lambda(x)$ satisfy the initial conditions independent of λ and an ODE where λ appears analytically (see Stakgold 1998, p. 441, or Levitan and Sargsjan 1975). By Equations (16) and (9), the Wronskian of the functions $\eta_\lambda(x)$ and $\xi_\lambda(x)$ is of the form

$$W_x(\eta_\lambda,\xi_\lambda)\equiv\eta_\lambda(x)\xi_\lambda'(x)-\xi_\lambda(x)\eta_\lambda'(x)=C(\lambda)\mathfrak{s}(x), \tag{19}$$

where $\mathfrak{s}(x)$ is the scale density (3) and $C(\lambda)$ is independent of x but may depend on λ. Then the Green's function of the regular Sturm-Liouville problem with two Dirichlet boundary conditions can be taken in the form (Stakgold 1998, p. 441) ($x\wedge y:=\min\{x,y\}$, $x\vee y:=\max\{x,y\}$):

$$g(x,y;\lambda)=\mathfrak{m}(y)\frac{\xi_\lambda(x\wedge y)\eta_\lambda(x\vee y)}{C(\lambda)}. \tag{20}$$

Because ξ and η are entire functions of λ, so are ξ', η', W, and C. Let λ be a zero of C, i.e., $C(\lambda)=0$. Then the Wronskian of $\xi_\lambda(x)$ and $\eta_\lambda(x)$ vanishes, and these functions are linearly dependent for this value of λ. In view of their initial values neither function can vanish identically in x. Therefore $\xi_\lambda(x)$ is a nontrivial constant multiple of $\eta_\lambda(x)$, and both functions satisfy the two boundary conditions and the ODE in (10). Thus, λ is an eigenvalue of (10) with eigenfunction (not normalized) $\xi_\lambda(x)$. From (14) it is clear that at an eigenvalue $g(x,y;\lambda)$ has a simple pole, and therefore C must vanish. Thus, we conclude that the (simple) zeros of $C(\lambda)$ are located along the positive real axes and coincide with the eigenvalues of the Sturm-Liouville problem (10). We label the eigenvalues $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$ with $\lambda_n\to\infty$ as $n\to\infty$, and

$$\xi_{\lambda_n}(x)=A_n\eta_{\lambda_n}(x), \tag{21}$$

where A_n is a real nonzero constant. Thus, $\xi_{\lambda_n}(x)$ (or $\eta_{\lambda_n}(x)$) is a real eigenfunction corresponding to the simple positive eigenvalue λ_n. Neither $\xi_{\lambda_n}(x)$ nor $\eta_{\lambda_n}(x)$ is normalized. To find the normalized eigenfunctions, we note that the residue of $g(x,y;\lambda)$ at $\lambda=\lambda_n$ is

$$\mathfrak{m}(y)\frac{\xi_{\lambda_n}(x\wedge y)\eta_{\lambda_n}(x\vee y)}{C'(\lambda_n)}=\mathfrak{m}(y)\frac{\xi_{\lambda_n}(x)\xi_{\lambda_n}(y)}{A_nC'(\lambda_n)}$$
$$=\mathfrak{m}(y)\frac{A_n\eta_{\lambda_n}(x)\eta_{\lambda_n}(y)}{C'(\lambda_n)}, \tag{22}$$

where

$$C'(\lambda_n):=\left.\frac{dC(\lambda)}{d\lambda}\right|_{\lambda=\lambda_n}. \tag{23}$$

On the other hand, from (14) the residue of $g(x,y;\lambda)$ at $\lambda=\lambda_n$ is equal to $-\mathfrak{m}(y)\varphi_n(x)\varphi_n(y)$, and we recognize that the normalized eigenfunction $\varphi_n(x)$ is given by

$$\varphi_n(x)=\pm\frac{\xi_{\lambda_n}(x)}{\sqrt{-A_nC'(\lambda_n)}}=\pm\sqrt{\frac{-A_n}{C'(\lambda_n)}}\,\eta_{\lambda_n}(x). \tag{24}$$

Thus, from the practical standpoint, the problem of finding eigenvalues and eigenfunctions reduces to solving the two initial value problems for the $\xi_\lambda(x)$ and $\eta_\lambda(x)$, calculating their Wronskian, and determining its zeros.

2.1.3. Example: Double-Barrier Options Under the Geometric Brownian Motion Process. Assume that under the risk-neutral measure Q the underlying asset price follows a geometric Brownian motion

$$S_t=Se^{\sigma(B_t+\nu t)},\quad t\geqslant 0, \tag{25}$$

where $\{B_t, t\geqslant 0\}$ is a standard Brownian motion, $\sigma>0$ is the constant volatility, $r\geqslant 0$ is the constant risk-free interest rate, $q\geqslant 0$ is the constant dividend yield, $S>0$ is the initial asset price at $t=0$, and

$$\nu:=\frac{1}{\sigma}\left(r-q-\frac{\sigma^2}{2}\right). \tag{26}$$

The process (25) solves the SDE

$$dS_t = (r-q)S_t\,dt + \sigma S_t\,dB_t, \quad S_0 = S. \tag{27}$$

Consider a double-barrier call option with the strike price K, expiration date T, and two knock-out barriers L and U, $0 < L < K < U$. The knock-out provision renders the option worthless as soon as the underlying price leaves the price range (L, U). Double barrier options under the assumption of geometric Brownian motion have been studied by Kunitomo and Ikeda (1992), Geman and Yor (1996), Zhang (1997), Pelsser (2000), and Schroder (1999). The double-barrier call payoff is

$$\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}(S_T - K)^+, \tag{28}$$

where $\mathcal{T}_{(L,U)} = \inf\{t \geqslant 0 \colon S_t \notin (L,U)\}$, and $x^+ \equiv \max\{x, 0\}$. Then the double-barrier call price at $t = 0$ is given by the risk-neutral expectation of the discounted payoff

$$C(S,T) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}(S_T - K)^+], \tag{29}$$

where the subscript S in E_S signifies that the process is starting at $S_0 = S$ at time $t = 0$.

The scale and speed densities of the geometric Brownian motion (25) have the form

$$\mathfrak{s}(x) = x^{-(2\nu/\sigma)-1}, \quad \mathfrak{m}(x) = \frac{2}{\sigma^2} x^{(2\nu/\sigma)-1}. \tag{30}$$

Proposition 1 specified to this case yields Proposition 2.

PROPOSITION 2. (i) *Functions*

$$\varphi_n(x) = \frac{\sigma}{\sqrt{\ln(U/L)}} x^{-(\nu/\sigma)} \sin\left(\frac{\pi n \ln(x/L)}{\ln(U/L)}\right), \quad n = 1, 2, \ldots, \quad x \in [L, U], \tag{31}$$

form a complete orthonormal basis in $L^2([L,U], \mathfrak{m})$.

(ii) *Functions* φ_n *are eigenfunctions of the pricing operator for the problem with two knock-out barriers*:

$$e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\, \varphi_n(S_T)] = e^{-\lambda_n T} \varphi_n(S), \tag{32}$$

where

$$\lambda_n = r + \frac{\nu^2}{2} + \frac{\sigma^2 \pi^2 n^2}{2\ln^2(U/L)}. \tag{33}$$

(iii) *The call payoff* $f(x) = (x-K)^+$ *on* $[L,U]$ *can be decomposed in this basis according to* (7) *with the coefficients*

$$c_n = \langle f, \varphi_n \rangle = \frac{L^{\nu/\sigma}}{\sqrt{\ln(U/L)}} [L\,\psi_n(\nu+\sigma) - K\,\psi_n(\nu)], \tag{34}$$

where

$$\psi_n(a) := \frac{2}{\omega_n^2 + a^2}\left[e^{ak}(\omega_n \cos(\omega_n k) - a \sin(\omega_n k)) - (-1)^n \omega_n e^{au}\right], \tag{35}$$

$$\omega_n := \frac{n\pi}{u}, \quad k := \frac{1}{\sigma}\ln\left(\frac{K}{L}\right), \quad u := \frac{1}{\sigma}\ln\left(\frac{U}{L}\right). \tag{36}$$

(iv) *The price of the double-barrier call option* (29) *is given by the eigenfunction expansion*

$$C(S,T) = \sum_{n=1}^{\infty} c_n\, e^{-\lambda_n T} \varphi_n(S). \tag{37}$$

PROOF. See the appendix.

Note that the price of the double-barrier option vanishes in the limit $T \to \infty$. Analytically, this follows from the fact that the lowest eigenvalue $\lambda_1 > 0$ is strictly positive even for zero interest rate $r = 0$. Probabilistically, this follows from the fact that the stock price eventually hits one of the barriers with probability one.

2.1.4. Asymptotics of Eigenvalues and Eigenfunctions for Large n. The observation of practical importance is that the eigenvalues λ_n given by Equation (33) grow as n^2 as n increases, and contributions from the higher eigenfunctions are suppressed by the factors $e^{-\lambda_n T}$. As a result, the eigenfunction expansion (37) converges so rapidly that only a few terms are needed to achieve high accuracy in option pricing applications with typical parameter values.

This behavior is characteristic of regular Sturm-Liouville problems. Let us return to the general set-up of Proposition 1. To show the large-n asymptotics of eigenvalues and eigenfunctions in the general case with some functions $a(x)$, $b(x)$, and $r(x)$, we first make the *Liouville transformation* and transform the problem to the *Liouville normal form* (see Fulton and Pruess 1994). Consider the Sturm-Liouville problem (9)–(10). Introduce a new variable

$$y := \sqrt{2}\int_L^x \frac{dz}{a(z)}. \tag{38}$$

We look for solutions in the form

$$u(x) = 2^{-1/4}\sqrt{a(x)\mathfrak{s}(x)}\,v(y(x)) \tag{39}$$

for some function $v = v(y)$. Substituting (39) into the ODE (10), we find that the function v solves the Sturm-Liouville problem in the Liouville normal form (the coefficient in front of the second derivative is equal to (negative) one and the first-derivative term is absent):

$$-v'' + Q(y)v = \lambda v, \quad y \in (0, B), \tag{40}$$

$$B := y(U) = \sqrt{2}\int_L^U \frac{dz}{a(z)}, \quad v(0) = 0, \quad v(B) = 0, \tag{41}$$

where the *potential function* $Q(y)$ is given by (assume that $a''(x)$ and $b'(x)$ exist):

$$Q(y) = \frac{f''(y)}{f(y)} + r(x(y)), \tag{42}$$

where

$$f(y) = \frac{2^{1/4}}{\sqrt{a(x(y))\mathfrak{s}(x(y))}}, \tag{43}$$

and $x = x(y)$ is the inverse of $y = y(x)$ in Equation (38).

Adapting the results in Fulton and Pruess (1994) to our setting, we have the following large-n asymptotics for eigenvalues and normalized eigenfunctions:

$$\lambda_n = \frac{n^2\pi^2}{B^2} + a_0 + \frac{a_2}{n^2\pi^2} + \frac{a_4}{n^4\pi^4} + O\left(\frac{1}{n^6}\right), \tag{44}$$

$$a_0 = \frac{1}{B}\int_0^B Q(y)\,dy, \tag{45}$$

$$a_2 = -\frac{B^2}{4}a_0^2 + \frac{B}{4}\left\{\int_0^B Q^2(y)\,dy - Q'(B) + Q'(0)\right\}, \tag{46}$$

$$a_4 = -\frac{B^4}{8}a_0^3 - \frac{3}{2}B^2 a_0 a_2 + \frac{B^3}{16} \times\left\{\int_0^B \left(2Q^3(y) + (Q'(y))^2\right) dy - 6\left[Q(B)Q'(B) - Q(0)Q'(0)\right] + Q'''(B) - Q'''(0)\right\}, \tag{47}$$

$$\varphi_n(x) = \pm\frac{2^{\frac{1}{4}}}{\sqrt{B}}\sqrt{a(x)\mathfrak{s}(x)}\left\{\sin\left(\frac{n\pi y(x)}{B}\right) + \frac{B}{n\pi}\left(\frac{a_0}{2}y(x) - \frac{1}{\sqrt{2}}\int_L^x \frac{Q(y(z))}{a(z)}\,dz\right) \times\cos\left(\frac{n\pi y(x)}{B}\right)\right\} + O\left(\frac{1}{n^2}\right). \tag{48}$$

The result (44) is very useful. It closely approximates eigenvalues of the regular Sturm-Liouville problem with two Dirichlet boundary conditions even for moderate values of n. This greatly facilitates numerical work of finding accurate eigenvalues as zeros of the Wronskian $C(\lambda)$. We can use the estimate (44) as a starting point of some numerical search procedure to find the accurate value of λ_n. The result (48) gives an estimate of the corresponding normalized eigenfunction.

2.2. Singular Sturm-Liouville Problems

In the preceding discussion we limited ourselves to double-barrier options that knock out as soon as the underlying state variable exits some pre-specified finite interval in the interior of the state space D. In this case the pricing problem reduces to the regular Sturm-Liouville problem with two Dirichlet boundary conditions at the end-points of the interval. Consider now a contingent claim without knock-out barriers and a terminal payoff $f \in L^2(D, \mathfrak{m})$. The pricing problem reduces to the Sturm-Liouville problem on the entire state space D with the end-points $\mathfrak{l}$ and $\mathfrak{r}$ (finite or infinite). If the interval D is finite and $\mathfrak{s}(x)$, $\mathfrak{m}(x)$, and $r(x)$ are absolutely integrable near both end-points $\mathfrak{l}$ and $\mathfrak{r}$, then the Sturm-Liouville problem is said to be *regular*. Otherwise, the problem is *singular*. For a regular problem with two Dirichlet boundary conditions the spectrum is simple, purely discrete, and strictly positive for $r(x) \geqslant 0$, and Proposition 1 holds true. In contrast, the spectrum of a singular problem can be discrete, continuous, or mixed, and further analysis is needed to determine the nature of the spectrum in each case. For problems with a single upper (lower) knock-out barrier, the domain of the problem has the end-points $\mathfrak{l}$ and U with the Dirichlet boundary condition at U (L and $\mathfrak{r}$ with the Dirichlet boundary condition at L) and the nature of the spectrum will depend on the behavior of the functions $\mathfrak{s}(x)$, $\mathfrak{m}(x)$, and $r(x)$ at the left end-point $\mathfrak{l}$ (right end-point $\mathfrak{r}$), respectively. For problems without barriers, the nature of the spectrum will depend on the behavior near both end-points $\mathfrak{l}$ and $\mathfrak{r}$.

A classification scheme for singular Sturm-Liouville problems based on the celebrated *limit-point/limit-circle alternative* due to Weyl (1910) and *oscillatory/nonoscillatory classification* can be found in Fulton et al. (1996) and Zwillinger (1998, pp. 97–98). The analysis proceeds by first transforming the singular problem to the Liouville normal form as we have done for the regular problem.[5] Then one investigates the behavior of the potential function $Q(y)$ near the singular end-points and applies the well-known criteria to determine the character of each singular end-point according to the limit-point/limit-circle and oscillatory/nonoscillatory classifications (details can be found in Dunford and Schwartz 1963; Levitan and Sargsjan 1975; Pryce 1993; Fulton et al. 1996; Stakgold 1998; and Zwillinger 1998). When the character of each singular end-point is determined, one can apply the spectrum determination criteria (see Fulton et al. 1996 and Zwillinger 1998, pp. 97–98). Finally, when the nature of the spectrum is determined, one can proceed to find the corresponding eigenfunctions. As in the regular case, one way to proceed is to construct the Green's function $g(x, y; \lambda)$ for the singular problem, and then analyze it as a function of the complex variable λ. The procedures to construct the Green's function in singular cases are outlined in Titchmarsh (1962), Levitan and Sargsjan (1975), and Stakgold (1998). In §3.3, 3.4, and 3.6 we apply this analysis to up-and-out, down-and-out, and vanilla options under the CEV diffusion. In all three cases singular end-points are of the nonoscillatory limit-point type. Here we show that Proposition 1 directly generalizes to the cases with singular end-points of the nonoscillatory limit-point type.

First, we need some facts from the singular Sturm-Liouville theory. We follow the exposition of Fulton et al. (1996). Consider the Sturm-Liouville ODE (16) with $x \in (\mathfrak{l}, \mathfrak{r})$ and λ—an arbitrary complex number, $\lambda \in \mathbb{C}$. There are two fundamental disjoint types into which the Sturm-Liouville equation is classified at each end-point: (1) limit-point or limit-circle which is independent of $\lambda \in \mathbb{C}$, and (2) nonoscillatory or oscillatory for real value of λ, which can vary with λ. For simplicity we give the definitions at $\mathfrak{l}$ only, as the definitions for $\mathfrak{r}$ are entirely similar. The Sturm-Liouville equation is said to be *limit-circle* at $\mathfrak{l}$ if and only if every solution $u(x)$ is square-integrable with the weight $\mathfrak{m}(x)$ near the end-point $\mathfrak{l}$. Otherwise the equation is called *limit-point* at $\mathfrak{l}$. This classification due to

Weyl is mutually exclusive and independent of λ. That is, if the ODE (16) has two linearly independent solutions which are square-integrable for one value of λ, then it will have two linearly independent solutions for all $\lambda \in \mathbb{C}$. When the limit-point case occurs, for $\mathrm{Im}(\lambda) \neq 0$ there exists only one solution that is square-integrable, while for real values of λ there may be one or no solution that is square integrable. To generate self-adjoint operators in the Hilbert space $L^2((\mathfrak{l}, \mathfrak{r}), \mathfrak{m})$, whenever the limit-circle occurs, a boundary condition must be imposed at that end-point, while no additional boundary condition is required at a limit-point end-point.

The oscillatory/nonoscillatory classification is of fundamental importance in determining the qualitative nature of the spectrum. For a given real λ, the Sturm-Liouville ODE is *oscillatory* at $\mathfrak{l}$ if and only if every solution has infinitely many zeros clustering at $\mathfrak{l}$. Otherwise it is called *nonoscillatory* at $\mathfrak{l}$. This classification is mutually exclusive for a fixed λ, but can vary with λ. All regular end-points are limit-circle and nonoscillatory.

In this paper all singular end-points will be of the nonoscillatory limit-point type. If both end-points are nonoscillatory, the spectrum is simple, purely discrete, and bounded below just as in the regular case. Therefore, when we replace the interval $(L, U) \in D$ with the entire state space D (we assume that singular end-points are nonoscillatory and limit-point), Proposition 1 does hold with one modification. At a limit-point end-point, we do not need to impose a boundary condition as in Equation (10). The solution is automatically singled out by the square integrability requirement alone. At a regular end-point, we impose the Dirichlet boundary condition as before.

Next, we need to modify our Green's function based procedure to determine eigenvalues and normalized eigenfunctions to cover the case of singular end-points of the non-oscillatory limit-point type. When both end-points are regular, we have constructed the Green's function (20) on the interval $[L, U]$ using the two solutions $\xi_\lambda(x)$ and $\eta_\lambda(x)$ to the initial value problems (16), (17) and (16), (18), respectively, that are guaranteed to be entire functions of λ for fixed x. Now consider the down-and-out problem on $[L, \mathfrak{r})$, where $\mathfrak{r}$ is a singular nonoscillatory limit-point end-point. The solution $\xi_\lambda(x)$ is still constructed by solving the initial value problem (16), (17). Furthermore, by Theorem 18 in Fulton et al. (1996), if $\mathfrak{r}$ is limit-point nonoscillatory, then there exists a solution of the Sturm-Liouville ODE (16), which is square-integrable with $\mathfrak{m}(x)$ in a neighborhood of $\mathfrak{r}$ for all complex λ and is entire in λ for fixed x. We select this solution to be our $\eta_\lambda(x)$ and use the two solutions $\xi_\lambda(x)$ and $\eta_\lambda(x)$ to construct the Green's function (20) in the down-and-out case. The rest of the discussion in §2.1.2 goes through without modification, as in the regular case. The case of the up-and-out problem with the domain $(\mathfrak{l}, U]$, where $\mathfrak{l}$ is limit-point nonoscillatory, is similar. The $\eta_\lambda(x)$ solves the initial value problem (16), (18), while $\xi_\lambda(x)$ is the solution entire in λ for fixed x and square-integrable with $\mathfrak{m}$ near $\mathfrak{l}$. Finally, consider the problem on D, where both end-points $\mathfrak{l}$ and $\mathfrak{r}$ are limit-point nonoscillatory. The $\xi_\lambda(x)$ and $\eta_\lambda(x)$ are selected as solutions entire in λ and square-integrable with $\mathfrak{m}$ near $\mathfrak{l}$ and $\mathfrak{r}$, respectively.

To conclude this section, we mention that for singular problems with continuous spectra eigenfunction expansions for option prices contain an integral in place of the sum. In particular, Linetsky (2001, 2002a) shows that the pricing problem for arithmetic Asian options in the Black-Scholes framework with $r - q < \sigma^2/2$ has a mixed spectrum with a finite number of discrete eigenvalues in the interval $[0, c)$ plus a continuous spectrum in $[c, \infty)$ with $c = 2(((r-q)/\sigma^2) - \frac{1}{2})^2$. A further discussion of diffusion problems with continuous spectrum can be found in Linetsky (2002b).

3. BARRIER OPTIONS UNDER THE CEV PROCESS

3.1. The CEV Process

In this section we specialize our discussion to the *constant elasticity of variance* (CEV) process of Cox (1975). We assume that under the risk-neutral measure Q the asset price follows the CEV process

$$dS_t = \mu S_t\, dt + \delta S_t^{\beta+1}\, dB_t, \quad 0 < t < \mathcal{T}_0,$$
$$S_0 = S > 0, \tag{49}$$

where the risk-neutral drift rate is $\mu = r - q$ ($r \geqslant 0$ is the constant risk-free rate and $q \geqslant 0$ is the dividend yield). Note that our notation is slightly different from Cox (1975). Our parameter β is defined as the elasticity of the local volatility function. Cox's parameter θ in $dS_t = \mu S_t\, dt + \delta S_t^{\theta/2}\, dB_t$ is defined as the elasticity of the instantaneous variance of the asset price. The two parameters are related by $\beta + 1 = \theta/2$. The CEV specification (49) nests the lognormal model of Black and Scholes (1973) and Merton (1973) ($\beta = 0$) and the absolute diffusion ($\beta = -1$) and square-root ($\beta = -\frac{1}{2}$) models of Cox and Ross (1976) as particular cases. For $\beta < 0$ ($\beta > 0$), the local volatility $\sigma(S) = \delta S^\beta$ is a decreasing (increasing) function of the asset price. The two model parameters β and δ can be interpreted as the *elasticity of the local volatility function*, $\beta = \sigma' S/\sigma$, and the *scale parameter* fixing the initial instantaneous volatility at time $t = 0$, $\sigma_0 = \sigma(S_0) = \delta S_0^\beta$ (it is assumed that $\delta > 0$). Cox (1975) originally studied the case $\beta < 0$. Emanuel and MacBeth (1982) extended his analysis to the case $\beta > 0$. Cox originally restricted the elasticity parameter to the range $-1 \leqslant \beta \leqslant 0$. However, Reiner (1994) and Jackwerth and Rubinstein (1998) find that typical values of the CEV elasticity implicit in the post-crash S&P 500 stock index option prices are as low as $\beta = -3$ or -4. They call the model with $\beta < -1$ *unrestricted CEV*.

According to Feller's classification of boundaries for diffusions, for $\beta < 0$ infinity is a natural boundary for the CEV diffusion. For $-\frac{1}{2} \leqslant \beta < 0$, the origin is an exit boundary. For $\beta < -\frac{1}{2}$, the origin is a regular boundary point, and is specified as a killing boundary by adjoining a killing boundary condition (the process is sent to the cemetery or, in financial terms, "bankruptcy" state ∂ at the first hitting time of zero, $\mathcal{T}_0 = \inf\{t \geqslant 0: S_t = 0\}$). For $\beta > 0$, the origin is a natural boundary and infinity is an entrance boundary (see Davydov and Linetsky 2001, Appendix B, for the treatment of the $\beta > 0$ case). In this paper we will focus on the CEV process with $\beta < 0$ and $\mu > 0$ ($r > q$). This process is used to model the so-called *volatility (half)smile* or *skew* effect in the equity index options market. *From now on we always assume* $\beta < 0$ *and* $\mu > 0$.

The closed-form pricing formulas for vanilla calls and puts under the CEV process are derived by Cox (1975) (see also Schroder 1989 and Davydov and Linetsky 2001 and references therein). The problem of pricing single- and double-barrier options under the CEV process is examined by Boyle and Tian (1999) in the numerical trinomial lattice framework and by Davydov and Linetsky (2001) in the analytical framework. Davydov and Linetsky (2001) derive closed-form expressions for Laplace transforms of single- and double-barrier option prices in time to maturity. The Laplace transforms are then inverted numerically using the Euler numerical inversion algorithm of Abate and Whitt (1995) (see Fu et al. 1997 and Davydov and Linetsky (2001/2002) for applications of the Euler inversion algorithm to option pricing problems). In this paper we develop eigenfunction expansions for single and double barrier option prices under the CEV process. These eigenfunction expansions invert the Laplace transforms of Davydov and Linetsky (2001) in *closed form*.

The CEV process is related to several classical diffusions. Let $\{S_t, t \geqslant 0\}$ be the CEV process. Define a new process $\{Y_t, t \geqslant 0\}$ by: $Y_t = (1/(\delta^2\beta^2))S_t^{-2\beta}$ for $t < \mathcal{T}_0$ and $Y_t = \partial$ for $t \geqslant \mathcal{T}_0$, $\mathcal{T}_0 = \inf\{t \geqslant 0: S_t = 0\}$. The process Y is a square-root diffusion (Feller 1951)

$$dY_t = (aY_t + b)\,dt + 2\sqrt{Y_t}\,dB_t, \quad 0 < t < \mathcal{T}_0,$$
$$a = 2\mu|\beta|, \quad b = 2 + \frac{1}{\beta} \tag{50}$$

with the killing boundary at zero. Further, take the square root of the process Y: $Z_t = \sqrt{Y_t} = (1/(\delta|\beta|))S_t^{-\beta}$ for $t < \mathcal{T}_0$ and $Z_t = \partial$ for $t \geqslant \mathcal{T}_0$. The process $\{Z_t, t \geqslant 0\}$ is a *generalized Bessel diffusion*:

$$dZ_t = \left(\frac{1+\beta}{2\beta}\frac{1}{Z_t} - \mu\beta Z_t\right)dt + dB_t, \quad 0 < t < \mathcal{T}_0, \tag{51}$$

with the killing boundary at zero (see Shiga and Watanabe 1973, Eie 1983, Going-Jaeschke and Yor 1999, and Giorno et al. 1986 for related diffusion processes). For further discussion of the generalized Bessel process see Linetsky (2002b).

3.2. Double-Barrier Options

Consider a double-barrier call with two knock-out barriers L and U. To price this option, we need to compute the discounted risk-neutral expectation (29) with the underlying process (49). We will proceed according to the recipe of §2. The scale and speed densities of the CEV process are

$$\mathfrak{s}(S) = \exp\left(-\frac{\mu}{\delta^2|\beta|}S^{-2\beta}\right),$$
$$\mathfrak{m}(S) = \frac{2}{\delta^2 S^{2+2\beta}}\exp\left(\frac{\mu}{\delta^2|\beta|}S^{-2\beta}\right). \tag{52}$$

To find explicit expressions for the eigenfunctions, we need to find $\xi_\lambda(S)$ and $\eta_\lambda(S)$ solving the initial value problems (16)–(18) with the negative of the infinitesimal generator of the CEV diffusion

$$\mathcal{A} = -\frac{1}{2}\delta^2 S^{2+2\beta}\frac{d^2}{dS^2} - \mu S\frac{d}{dS}. \tag{53}$$

Introduce a new variable

$$x := \frac{\mu}{\delta^2|\beta|}S^{-2\beta}. \tag{54}$$

We look for solutions to the ODE (16) with the CEV operator (53) in the form

$$u(S) = S^{\frac{1}{2}+\beta}e^{-x(S)/2}w(x(S)) \tag{55}$$

for some unknown function w. Substituting this functional form into Equation (16), we arrive at the ODE for w:

$$\frac{d^2w}{dx^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4}-m^2}{x^2}\right)w = 0, \quad x \in (l, u), \tag{56}$$

where the parameters k and m and the end-points of the interval l and u corresponding to the barriers L and U are

$$m := \frac{1}{4|\beta|}, \quad k := m - \frac{1}{2} + \frac{\lambda}{2\mu|\beta|},$$
$$l := \frac{\mu}{\delta^2|\beta|}L^{-2\beta}, \quad u := \frac{\mu}{\delta^2|\beta|}U^{-2\beta}. \tag{57}$$

This is the Whittaker's form of the confluent hypergeometric equation (see Abramowitz and Stegun 1972, p. 505; Slater 1960, p. 9; and Buchholz 1969, p. 11). Then the functions $\xi_\lambda(S)$ and $\eta_\lambda(S)$ can be written in the form

$$\xi_\lambda(S) = \frac{\delta^2}{2\mu}(SL)^{\frac{1}{2}+\beta}e^{(l-x(S))/2}g_{k(\lambda)}(x(S)),$$
$$\eta_\lambda(S) = \frac{\delta^2}{2\mu}(SU)^{\frac{1}{2}+\beta}e^{(u-x(S))/2}h_{k(\lambda)}(x(S)), \tag{58}$$

where $g_k(x)$ and $h_k(x)$ are unique solutions of the Whittaker equation (56) with the initial conditions

$$g_k(l) = 0, \quad g_k'(l) = 1, \quad h_k(u) = 0, \quad h_k'(u) = -1. \tag{59}$$

For any complex k and real $m > 0$, $a > 0$, $b > 0$, introduce the following notation:

$$\Delta_{k,m}(a,b) := W_{k,m}(a)W_{-k,m}(e^{i\pi}b)e^{ik\pi} - W_{k,m}(b)W_{-k,m}(e^{i\pi}a)e^{ik\pi}, \quad (60)$$

where $W_{k,m}(x)$ is the Whittaker function (Abramowitz and Stegun 1972, p. 505; Slater 1960, p. 10; and Buchholz 1969, p. 19). Then for any complex k and real $x > 0$, the functions $g_k(x)$ and $h_k(x)$ can be taken in the form

$$g_k(x) = \Delta_{k,m}(l,x), \quad h_k(x) = \Delta_{k,m}(x,u), \quad x \in [l,u]. \quad (61)$$

Functions $W_{k,m}(x)$ and $W_{-k,m}(e^{i\pi}x)$ provide two linearly independent solutions of the Whittaker equation (56) for any values of k and m (real or complex) with the Wronskian $e^{-ik\pi}$ (Buchholz 1969, p. 25). The Wronskian of $\eta_\lambda(S)$ and $\xi_\lambda(S)$ is given by Equation (19) with the CEV scale density (52) and

$$C(\lambda) = \frac{\delta^2}{2\mu}(LU)^{\frac{1}{2}+\beta}e^{(l+u)/2}\Delta_{k(\lambda),m}(l,u). \quad (62)$$

The eigenvalues λ_n are found numerically as zeros of $C(\lambda)$. Specifically, we need to find the roots $\{k_n\}_{n=1}^{\infty}$ of the equation

$$\Delta_{k,m}(l,u) = 0. \quad (63)$$

Then, from the second definition in Equation (57), the eigenvalues λ_n are

$$\lambda_n = 2\mu|\beta|\left(k_n - m + \frac{1}{2}\right). \quad (64)$$

The roots of Equation (63) are found numerically. Because by Proposition 1 all λ_n are positive, $k_n > m - \frac{1}{2}$ for all n. For each k_n, $g_{k_n}(x)$ and $h_{k_n}(x)$ are linearly dependent and

$$\xi_{\lambda_n}(S) = A_n \eta_{\lambda_n}(S),$$
$$A_n = -e^{(l-u)/2}\left(\frac{L}{U}\right)^{\frac{1}{2}+\beta}\frac{W_{k_n,m}(l)}{W_{k_n,m}(u)}. \quad (65)$$

Then, by Equation (24), the normalized eigenfunctions $\varphi_n(S)$ can be taken in the form

$$\varphi_n(S) = N_n S^{\frac{1}{2}+\beta}e^{-x(S)/2}\Delta_{k_n,m}(l,x(S)), \quad (66)$$

where the normalization factors are given by

$$N_n = \sqrt{\frac{\delta^2|\beta|W_{k_n,m}(u)}{D_{n,m}(l,u)W_{k_n,m}(l)}},$$
$$D_{n,m}(l,u) = \left[\frac{\partial\Delta_{k,m}(l,u)}{\partial k}\right]\bigg|_{k=k_n}. \quad (67)$$

To differentiate $\Delta_{k,m}(a,b)$ we need to compute derivatives of the Whittaker functions $M_{k,m}(z)$ and $W_{k,m}(z)$ with respect to the first index k. This can be done numerically because there are no simple analytical formulas for these derivatives.

For numerical calculations, two alternative representations of the function $\Delta_{k,m}(a,b)$ are useful (Buchholz 1969, p. 20):

$$\Delta_{k,m}(a,b) = \frac{\pi}{\sin(2m\pi)}\left[\frac{M_{k,-m}(a)}{\Gamma(1-2m)}\frac{M_{k,m}(b)}{\Gamma(1+2m)} - \frac{M_{k,-m}(b)}{\Gamma(1-2m)}\frac{M_{k,m}(a)}{\Gamma(1+2m)}\right] \quad (68)$$

$$= \frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\left[W_{k,m}(a)M_{k,m}(b) - M_{k,m}(a)W_{k,m}(b)\right], \quad (69)$$

where $\Gamma(x)$ is the Gamma function (Abramowitz and Stegun 1972, p. 255).

PROPOSITION 3. *The double-barrier call price under the CEV process is given by the eigenfunction expansion* ($0 < L \leqslant S \leqslant U < \infty$)

$$C_{DB}(S,T,K,L,U) = e^{-rT}E_S[\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}(S_T-K)^+] = \sum_{n=1}^{\infty} c_n e^{-(r+\lambda_n)T}\varphi_n(S), \quad (70)$$

with the eigenvalues (64), *normalized eigenfunctions* (66)–(67), *and coefficients*

$$c_n = N_n\frac{\Gamma(\frac{1}{2}+m-k_n)}{\Gamma(1+2m)}\left[W_{k_n,m}(l)I_n - M_{k_n,m}(l)J_n\right], \quad n = 1,2,\ldots, \quad (71)$$

where

$$I_n := \frac{1}{\delta\sqrt{\mu|\beta|}}\left[\frac{U^{\frac{1}{2}}}{2m+1}e^{u/2}M_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{2mKU^{-\frac{1}{2}}}{m-k_n-\frac{1}{2}}e^{u/2}M_{k_n+\frac{1}{2},m-\frac{1}{2}}(u) - \frac{K^{\frac{1}{2}}}{2m+1}e^{\kappa/2}M_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{2mK^{\frac{1}{2}}}{m-k_n-\frac{1}{2}}e^{\kappa/2}M_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa)\right], \quad (72)$$

$$J_n := \frac{1}{\delta\sqrt{\mu|\beta|}}\left[\frac{U^{\frac{1}{2}}}{k_n+m+\frac{1}{2}}e^{u/2}W_{k_n+\frac{1}{2},m+\frac{1}{2}}(u) - \frac{KU^{-\frac{1}{2}}}{k_n-m+\frac{1}{2}}e^{u/2}W_{k_n+\frac{1}{2},m-\frac{1}{2}}(u) - \frac{K^{\frac{1}{2}}}{k_n+m+\frac{1}{2}}e^{\kappa/2}W_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{K^{\frac{1}{2}}}{k_n-m+\frac{1}{2}}e^{\kappa/2}W_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa)\right], \quad (73)$$

$$\kappa := \frac{\mu}{\delta^2|\beta|}K^{-2\beta}. \quad (74)$$

Proof. See the appendix.

The eigenvalues (64) are determined by numerically finding the roots of Equation (63). To facilitate numerical work, the result (44) provides accurate estimates of the eigenvalues. The Liouville transformation

$$y = \frac{\sqrt{2}}{\delta|\beta|}(S^{|\beta|} - L^{|\beta|}), \tag{75}$$

$$u(S) = \sqrt{\delta} S^{(1+\beta)/2} \exp\left(-\frac{\mu}{2\delta^2|\beta|} S^{-2\beta}\right) v(y(S)) \tag{76}$$

reduces the Sturm-Liouville problem (10) with the CEV operator (53) to the Liouville normal form (40) with the right end-point

$$B = \frac{\sqrt{2}}{\delta|\beta|}(U^{|\beta|} - L^{|\beta|}) \tag{77}$$

and the potential function

$$Q(y) = \frac{b_{-2}}{(y+c)^2} + b_0 + b_2(y+c)^2, \tag{78}$$

$$b_{-2} = \frac{1}{4}\left(\frac{1}{\beta^2} - 1\right), \quad b_0 = \mu\left(|\beta| - \frac{1}{2}\right),$$

$$b_2 = \frac{\mu^2\beta^2}{4}, \quad c = \frac{\sqrt{2}}{\delta|\beta|} L^{|\beta|}. \tag{79}$$

Now the eigenvalue estimates (44) can be readily computed and used as starting points for the accurate numerical search procedure to find the eigenvalues λ_n. For large n, the leading terms in the estimate (44) are

$$\lambda_n = \frac{\delta^2\beta^2\pi^2 n^2}{2(U^{|\beta|} - L^{|\beta|})^2} + a_0 + O\left(\frac{1}{n^2}\right), \tag{80}$$

$$a_0 = \mu\left(|\beta| - \frac{1}{2}\right) + \frac{\delta^2}{8}(1-\beta^2)(LU)^{-|\beta|}$$

$$+ \frac{\mu^2}{6\delta^2}(L^{2|\beta|} + U^{2|\beta|} + (LU)^{|\beta|}),$$

and large-n terms in the expansion (70) are suppressed by the factors $e^{-\lambda_n T}$. The result (48) gives the estimate of φ_n for large n.

3.3. Up-and-Out Options

Consider an up-and-out call with some upper knock-out barrier U. The payoff is $\mathbf{1}_{\{\mathcal{T}_U > T\}}(S_T - K)^+$, where $\mathcal{T}_U$ is the first hitting time of the upper barrier, $\mathcal{T}_U = \inf\{t \geqslant 0: S_t = U\}$. For $\beta < -\frac{1}{2}$, zero is a regular boundary point for the CEV diffusion, and we impose a killing boundary condition. The corresponding Sturm-Liouville problem with two Dirichlet boundary conditions at 0 and U is regular, and Proposition 1 holds in the limit $L = 0$ when $\beta < -\frac{1}{2}$ (note that both the CEV scale and speed densities (52) are absolutely integrable near zero in this case).

For $-\frac{1}{2} \leqslant \beta < 0$, zero is an exit boundary for the CEV diffusion. The corresponding Sturm-Liouville problem is singular at zero (the CEV speed density (52) is not integrable near zero). First, we note that the Liouville transformation (75) with $L = 0$ transforms the problem to the Liouville normal form with the potential (78)–(79) where $c = 0$. Examining the behavior of the potential $Q(y)$ near zero, we conclude that the problem with $-\frac{1}{2} \leqslant \beta < 0$ is nonoscillatory and limit-point at zero. Thus, as we have discussed in §2.2, the spectrum of the up-and-out problem with $-\frac{1}{2} \leqslant \beta < 0$ is simple, purely discrete, and positive, just as in the case of the regular up-and-out problem with $\beta < -\frac{1}{2}$.

Proposition 4. *The up-and-out call price is given by the eigenfunction expansion* $(0 < S \leqslant U < \infty)$

$$C_{UO}(S,T,K,U) = e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_U > T\}}(S_T - K)^+]$$

$$= \sum_{n=1}^{\infty} e^{-(r+\lambda_n)T} c_n \varphi_n(S), \tag{81}$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lambda_n \to \infty$ *as* $n \to \infty$, *related by Equation* (64) *to the roots* $\{k_n\}_{n=1}^{\infty}$ *of the equation* (*to be solved numerically*)

$$M_{k,m}(u) = 0, \tag{82}$$

the corresponding normalized eigenfunctions

$$\varphi_n(S) = N_n S^{\frac{1}{2}+\beta} e^{-x(S)/2} M_{k_n,m}(x(S)), \quad S \in (0, U], \tag{83}$$

$$N_n = \sqrt{\frac{\delta^2|\beta|\Gamma(\frac{1}{2}+m-k_n) W_{k_n,m}(u)}{\Gamma(1+2m)\left[\partial M_{k,m}(u)/\partial k\right]\big|_{k=k_n}}}, \tag{84}$$

and the coefficients

$$c_n = N_n I_n, \tag{85}$$

where I_n *are given by Equation* (72).

Proof. See the appendix.

Equation (82) has to be solved numerically. Buchholz (1969, pp. 185–186) shows that the Whittaker function $M_{k,m}(x)$ considered as a function of the complex variable k, keeping $m > -\frac{1}{2}$ and $x > 0$ fixed (in our case $m > 0$ and $x > 0$), has all its zeros concentrated along the positive real line. Moreover, all zeros are simple, occur in an infinite set $0 < k_1 < k_2 < \cdots$, and are decreasing as the value of x increases. Furthermore, Slater (1960, p. 70) gives the following asymptotics as $k \to \infty$ (the asymptotics is valid for complex k and x such that $\arg(kx) < 2\pi$):

$$M_{k,m}(x) = \Gamma(1+2m) x^{\frac{1}{4}} \pi^{-\frac{1}{2}} k^{-m-\frac{1}{4}} \cos\left(2\sqrt{kx} - \pi m - \frac{\pi}{4}\right)$$

$$\times \left\{1 + O\left(\frac{1}{\sqrt{|k|}}\right)\right\}. \tag{86}$$

Thus, asymptotically for large n the zeros k_n of the Whittaker function $M_{k,m}(u)$ are given by

$$k_n \sim \frac{(n+m-\frac{1}{4})^2\pi^2}{4u}. \tag{87}$$

Then, using Equations (57) and (64), we obtain the large-n asymptotics for the eigenvalues

$$\lambda_n \sim \frac{1}{2}\delta^2\beta^2\pi^2 U^{2\beta}\left(n+m-\frac{1}{4}\right)^2 + 2\mu|\beta|\left(\frac{1}{2}-m\right). \quad (88)$$

The eigenvalues for the up-and-out problem grow as n^2 for large n.

3.4. Down-and-Out Options

Down-and-Out Put. Consider now a down-and-out call with some lower knock-out barrier L. The payoff is $\mathbf{1}_{\{\mathcal{T}_L>T\}}(S_T-K)^+$, $\mathcal{T}_L=\inf\{t\geqslant 0\colon S_t=L\}$. The domain of the problem is now unbounded, $[L,\infty)$. The associated Sturm-Liouville problem is singular. Transforming to the Liouville normal form (Equations (75)–(76)) and examining the potential (78) as $y\to\infty$, we conclude that the problem is nonoscillatory and limit-point at infinity. Thus, again, the spectrum is simple, purely discrete, and bounded below.

The complication here is that the call payoff is *not* in $L^2([L,\infty),\mathfrak{m})$, and thus the down-and-out call is *not* in the span of the L^2-eigensecurities. However, the down-and-out put payoff is in $L^2([L,\infty),\mathfrak{m})$. We will price the down-and-out put first, and then find the price of the down-and-out call by appealing to a put-call parity result for down-and-out options.

PROPOSITION 5. *The down-and-out put price is given by the eigenfunction expansion* $(0<L\leqslant S<\infty)$

$$P_{DO}(S,T,K,L) = e^{-rT}E_S[\mathbf{1}_{\{\mathcal{T}_L>T\}}(K-S_T)^+] = \sum_{n=1}^{\infty} c_n\, e^{-(r+\lambda_n)T}\varphi_n(S), \quad (89)$$

with the eigenvalues $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, $\lambda_n\to\infty$ *as* $n\to\infty$, *related by Equation* (64) *to the roots* $\{k_n\}_{n=1}^{\infty}$ *of the equation* (*to be solved numerically*)

$$W_{k,m}(l)=0, \quad (90)$$

the corresponding normalized eigenfunctions

$$\varphi_n(S) = N_n S^{\frac{1}{2}+\beta} e^{-x(S)/2} W_{k_n,m}(x(S)), \quad S\in[L,\infty), \quad (91)$$

$$N_n = \sqrt{\frac{\delta^2|\beta|\Gamma(\frac{1}{2}+m-k_n)M_{k_n,m}(l)}{\Gamma(1+2m)\big[\partial W_{k,m}(l)/\partial k\big]\big|_{k=k_n}}}, \quad (92)$$

and the coefficients

$$c_n = \frac{N_n}{\delta\sqrt{\mu|\beta|}}\left[\frac{L^{\frac{1}{2}}}{k_n+m+\frac{1}{2}} e^{\frac{l}{2}} W_{k_n+\frac{1}{2},m+\frac{1}{2}}(l) - \frac{KL^{-\frac{1}{2}}}{k_n-m+\frac{1}{2}} e^{\frac{l}{2}} W_{k_n+\frac{1}{2},m-\frac{1}{2}}(l) - \frac{K^{\frac{1}{2}}}{k_n+m+\frac{1}{2}} e^{\kappa/2} W_{k_n+\frac{1}{2},m+\frac{1}{2}}(\kappa) + \frac{K^{\frac{1}{2}}}{k_n-m+\frac{1}{2}} e^{\kappa/2} W_{k_n+\frac{1}{2},m-\frac{1}{2}}(\kappa)\right], \quad (93)$$

where κ *is defined in Equation* (74).

PROOF. See the appendix.

We now need to estimate the eigenvalues. Slater (1960, p. 70) gives the following asymptotics for the Whittaker function $W_{k,m}(x)$ as $k\to\infty$ (the asymptotics is valid for complex k and x such that $|\arg(k)|<\pi$ and $|\arg(kx)|<2\pi$):

$$W_{k,m}(x) = \sqrt{2}x^{\frac{1}{4}}k^{-\frac{1}{4}}k^k e^{-k}\cos\left(2\sqrt{kx}-\pi k+\frac{\pi}{4}\right) \times\left\{1+O\left(\frac{1}{\sqrt{|k|}}\right)\right\}. \quad (94)$$

Thus, asymptotically for large n the zeros k_n of the Whittaker function $W_{k,m}(l)$ are given by

$$k_n = n-\frac{1}{4}+\frac{2l}{\pi^2}+\frac{2}{\pi}\sqrt{\left(n-\frac{1}{4}\right)l+\frac{l^2}{\pi^2}}. \quad (95)$$

Then, using Equation (64) we obtain the large-n asymptotics for the eigenvalues

$$\lambda_n \sim 2\mu|\beta|\left[n+\frac{2l}{\pi^2}+\frac{1}{4}-m+\frac{2}{\pi}\sqrt{\left(n-\frac{1}{4}\right)l+\frac{l^2}{\pi^2}}\right]. \quad (96)$$

The major difference with the up-and-out and double-barrier cases is that the eigenvalues grow *linearly* with n, in contrast with the *quadratic* growth for the former cases. Thus the convergence of eigenfunction expansions for down-and-out option prices is slower.

Down-and-Out Call. Because the down-and-out call payoff is not in $L^2([L,\infty),\mathfrak{m})$, it is not in the span of the eigenpayoffs (91). To price the down-and-out call, we first decompose its payoff as follows:

$$\mathbf{1}_{\{\mathcal{T}_L>T\}}(S_T-K)^+ = \mathbf{1}_{\{\mathcal{T}_L>T\}}(K-S_T)^+ + (S_T-K) - \mathbf{1}_{\{\mathcal{T}_L\leqslant T\}}(S_T-K). \quad (97)$$

The first term on the right-hand side is the payoff of a down-and-out put, the second term is the payoff from a forward contract with the delivery price K, and the last term $\mathbf{1}_{\{\mathcal{T}_L\leqslant T\}}(S_T-K)$ can be interpreted as a *down-and-in forward contract* that is activated if and only if the underlying asset price hits the lower barrier L prior to and including maturity T and pays the amount equal to (S_T-K) at T if activated. Taking the present values of both sides of the equality (97), we have for the prices at $t=0$:

$$C_{DO} = P_{DO} + (e^{-qT}S - e^{-rT}K) - f_{DI}. \quad (98)$$

We have already priced the down-and-out put P_{DO}. To price the down-and-out call C_{DO}, we need to price the down-and-in forward contract f_{DI}.

PROPOSITION 6. *The price of the down-and-in forward contract is given by* $(0 < L \leqslant S)$

$$\begin{aligned} f_{DI} &= e^{-rT} E_S[\mathbf{1}_{\{\mathcal{T}_L \leqslant T\}}(S_T - K)] \\ &= e^{-qT} S\left(\frac{G(-2m, x)}{G(-2m, l)}\right) - e^{-rT} K\left(\frac{G(2m, x)}{G(2m, l)}\right) \\ &\quad + \sum_{n=1}^{\infty} e^{-(r+\lambda_n)T}\left[\frac{L}{\mu+\lambda_n} - \frac{K}{\lambda_n}\right] \\ &\quad \times \frac{2\mu|\beta| S^{\frac{1}{2}+\beta} e^{-x/2} W_{k_n,m}(x)}{L^{\frac{1}{2}+\beta} e^{-\frac{l}{2}}\left[\partial W_{k,m}(l)/\partial k\right]\big|_{k=k_n}}, \end{aligned} \tag{99}$$

where λ_n *are the eigenvalues of the Sturm-Liouville problem on* $[L, \infty)$ *determined in Proposition* 5, k_n *are the roots of Equation* (90), x *is defined in Equation* (54), l *is defined in Equation* (57), *and* $G(\nu, a)$ *is the complementary Gamma distribution function*

$$G(\nu, a) = \frac{1}{\Gamma(\nu)} \int_a^{\infty} e^{-t}\, t^{\nu-1}\, dt. \tag{100}$$

PROOF. See the appendix.

Now we can compute the down-and-out call price using the put-call parity relationship (98).

3.5. Capped Options

In addition to their popularity over-the-counter, several types of barrier options are traded on securities exchanges. Capped call (and put) options on the S&P 100 and S&P 500 indices were introduced by the Chicago Board of Options Exchange (CBOE) in November, 1991. A *capped call* is an up-and-out call with a cash rebate equal to the difference between the upper barrier (*cap*) and the strike price (see Broadie and Detemple 1995). It combines a European exercise feature and an automatic exercise feature. The automatic exercise is triggered when the index value first exceeds the cap. The cash amount (rebate) equal to the intrinsic value of the call is paid at the time the index first exceeds the cap. The price of the capped call can be represented as a sum of the up-and-out call price and the price of the rebate (U is the cap price):

$$\begin{aligned} &\mathit{CappedCall}(S, T, K, U) \\ &\quad = C_{UO}(S, T, K, U) + (U - K) E_S[e^{-r\mathcal{T}_U} \mathbf{1}_{\{\mathcal{T}_U \leqslant T\}}]. \end{aligned} \tag{101}$$

We have already priced the up-and-out call in §3.3. To price capped calls, we need to evaluate the price of rebate. In Davydov and Linetsky (2001) the price of rebate was expressed as the inverse Laplace transform of a known function (Equations (4), (16), and (37)). Here we invert the Laplace transform by applying the method developed in this paper.

PROPOSITION 7. *The price of the rebate is* $(0 < S \leqslant U < \infty)$

$$\begin{aligned} E_S[e^{-r\mathcal{T}_U} \mathbf{1}_{\{\mathcal{T}_U \leqslant T\}}] &= \frac{S^{\frac{1}{2}+\beta} e^{-x/2}}{U^{\frac{1}{2}+\beta} e^{-u/2}} \\ &\quad \times \left\{ \frac{M_{\varkappa,m}(x)}{M_{\varkappa,m}(u)} + \sum_{n=1}^{\infty} \frac{2\mu|\beta| e^{-(r+\lambda_n)T} M_{k_n,m}(x)}{(r+\lambda_n)\left[\partial M_{k,m}(u)/\partial k\right]\big|_{k=k_n}} \right\}, \end{aligned} \tag{102}$$

where k_n *are the roots of Equation* (82), λ_n *are the eigenvalues of the Sturm-Liouville problem on* $[0, U]$ *related to* k_n *by Equation* (64), x *is defined in Equation* (54), u *is defined in Equation* (57), *and*

$$\varkappa := m - \frac{1}{2} - \frac{r}{2\mu|\beta|}. \tag{103}$$

PROOF. See the appendix.

3.6. Vanilla Options

Now consider the problem of pricing vanilla options (without barriers). The domain of the problem is $(0, \infty)$. The associated Sturm-Liouville problem is singular. It is nonoscillatory at both end-points 0 and ∞, and the spectrum is again simple, purely discrete, and bounded below. Similar to the down-and-out call, the vanilla call payoff is not in $L^2((0, \infty), \mathfrak{m})$, and thus the vanilla call is not in the span of the eigensecurities. However, the put payoff is in $L^2((0, \infty), \mathfrak{m})$. We will price the put first, and then find the price of the call by appealing to the put-call parity.

PROPOSITION 8. (i) *The spectral representation of the continuous transition probability density for the CEV process on* $(0, \infty)$ *with* $\beta < 0$ *and* $\mu > 0$ *is*

$$p(T; S, S_T) = \mathfrak{m}(S_T) \sum_{n=1}^{\infty} e^{-\lambda_n T} \varphi_n(S) \varphi_n(S_T), \tag{104}$$

where the eigenvalues and the corresponding normalized eigenfunctions are $(n = 1, 2, \dots)$

$$\begin{aligned} &\lambda_n = 2\mu|\beta| n, \quad \varphi_n(S) = N_n S e^{-x(S)} L_{n-1}^{(2m)}(x(S)), \\ &N_n = \sqrt{\frac{(n-1)!\mu}{\Gamma(2m+n)}} \left(\frac{\mu}{\delta^2|\beta|}\right)^m, \end{aligned} \tag{105}$$

where $L_n^{(\nu)}(x)$ *are the generalized Laguerre polynomials* (*Abramowitz and Stegun* 1972).

(ii) *The probability of hitting zero prior to time* T *is given by*

$$\Pr(\mathcal{T}_0 \leqslant T | S_0 = S) = G\left(2m, \frac{x(S)}{1 - e^{2\mu\beta T}}\right), \tag{106}$$

where $G(\nu, a)$ *is the complementary Gamma distribution function* (100).

(iii) *The price of the plain vanilla put is given by the eigenfunction expansion*

$$P(S,T,K) = e^{-rT} K G\left(2m, \frac{x(S)}{1-e^{2\mu\beta T}}\right) + \sum_{n=1}^{\infty} c_n e^{-(r+\lambda_n)T} \varphi_n(S), \tag{107}$$

with the coefficients

$$c_n = \frac{N_n K}{\mu}\left[\frac{\Gamma(2m+n)}{\Gamma(2m)n!} - L_n^{2m-1}(\kappa) - \frac{\kappa}{2m+n} L_{n-1}^{2m+1}(\kappa)\right], \tag{108}$$

where κ *is defined in Equation* (74).

(iv) *The price of the plain vanilla call is found from the put-call parity relationship*

$$C(S,T,K) = P(S,T,K) + e^{-qT} S - e^{-rT} K. \tag{109}$$

PROOF. See the appendix.

What is the relationship of our result with the classic CEV option pricing formula of Cox (1975)? Cox's formula expresses the CEV option prices in terms of the complementary noncentral chi-square distribution function, while our formula expresses CEV option prices as series of Laguerre polynomials. The equivalence is established by appealing to the *Hille-Hardy formula* (Erdelyi 1953, p. 189) (for all $|t| < 1$, $\nu > -1$, $a, b > 0$)

$$(ab)^{\nu/2} \sum_{n=0}^{\infty} \frac{t^{n+(\nu/2)} n!}{\Gamma(n+\nu+1)} L_n^{(\nu)}(a) L_n^{(\nu)}(b) = \frac{1}{1-t} \exp\left\{-\frac{(a+b)t}{1-t}\right\} I_\nu\left(\frac{2\sqrt{tab}}{1-t}\right), \tag{110}$$

where $I_\nu(a)$ is the modified Bessel function of the first kind. Applying this summation formula to the spectral representation (104) and identifying $t = e^{2\mu\beta T}$ yields the standard form of the continuous CEV density used by Cox (1975) ($x = (\mu/(\delta^2|\beta|))S^{-2\beta}$, $x_T = (\mu/(\delta^2|\beta|))S_T^{-2\beta}$):

$$p(T;S,S_T) = \frac{2\mu S_T^{-2\beta-\frac{3}{2}} S^{\frac{1}{2}}}{\delta^2(e^{-2\mu\beta T}-1)} \exp\left(\frac{x_T + x e^{-2\mu\beta T}}{1-e^{-2\mu\beta T}} + \frac{\mu T}{2}\right) \times I_{1/(2|\beta|)}\left(\frac{\sqrt{x x_T}}{\sinh(\mu|\beta|T)}\right). \tag{111}$$

Integrating this density against the option payoff leads to Cox's formula expressed in terms of the complementary chi-square distribution function (see Schroder 1989 and Davydov and Linetsky 2001):

$$C(S,T,K) = e^{-qT} S Q(y_0; d, \zeta) - e^{-rT} K(1 - Q(\zeta; d-2, y_0)), \tag{112}$$

where

$$d := 2 + \frac{1}{|\beta|}, \quad \zeta := \frac{2\mu S^{-2\beta}}{\delta^2\beta(e^{2\mu\beta T}-1)}, \quad y_0 := \frac{2\mu K^{-2\beta}}{\delta^2\beta(1-e^{-2\mu\beta T})}, \tag{113}$$

K is the strike price of the call, and $Q(x; u, v)$ is the complementary noncentral chi-square distribution function with u degrees of freedom and the noncentrality parameter v.

4. INTEREST RATE KNOCK-OUT OPTIONS IN THE CIR TERM STRUCTURE MODEL

4.1. The CIR Process

In this section we consider interest rate options with barriers. A zero-coupon *knock-out bond* pays one dollar at maturity $T > 0$ if some reference interest rate (e.g., three-month LIBOR) does not leave a prespecified range (corridor) prior to maturity, and zero otherwise.

Suppose that under the risk-neutral measure Q the instantaneous risk-free interest rate follows the Cox-Ingersoll-Ross (CIR) diffusion process on $(0, \infty)$

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dB_t, \quad r_0 = r > 0, \tag{114}$$

where $\{B_t, t \geqslant 0\}$ is a standard Brownian motion, $\theta > 0$ is the long-run level, $\kappa > 0$ is the rate of mean reversion to the long-run level, $\sigma > 0$ is the volatility parameter, and the initial interest rate is $r > 0$. To ensure that the origin is inaccessible (the short rate stays strictly positive), the parameters are assumed to satisfy Feller's condition $2\kappa\theta \geqslant \sigma^2$. For this choice of parameters, the origin is an entrance boundary and infinity is a natural boundary. CIR (1985) derive a closed-form expression for the time $t = 0$ price of a zero-coupon bond that pays one dollar at maturity $T > 0$:

$$P(r,T) = E_r\left[e^{-\int_0^T r(t)\,dt}\right] = A(T) e^{-B(T)r}, \tag{115}$$

where

$$A(T) := \left(\frac{2\gamma e^{(\kappa+\gamma)T/2}}{(\gamma+\kappa)(e^{\gamma T}-1)+2\gamma}\right)^b, \quad B(T) := \frac{2(e^{\gamma T}-1)}{(\gamma+\kappa)(e^{\gamma T}-1)+2\gamma}, \tag{116}$$

$$\gamma := \sqrt{\kappa^2 + 2\sigma^2}, \quad b := \frac{2\kappa\theta}{\sigma^2}. \tag{117}$$

CIR also derive closed-form expressions for European call and put options on zero-coupon bonds.

4.2. Eigenfunction Expansions for Knock-Out Bonds

In this section we focus on pricing knock-out contracts where the reference interest rate is the LIBOR rate $L(t, t+\delta)$ (e.g., three-month LIBOR). These contracts knock out when the LIBOR leaves some prespecified corridor $(\underline{L}, \overline{L})$. In the CIR model there is an analytical one-to-one relationship between the LIBOR rate[6] and the short rate

$$L(t, t+\delta) = \frac{1}{\delta}[A^{-1}(\delta)e^{B(\delta)r_t} - 1]. \tag{118}$$

Then the event {*LIBOR leaves the corridor* $(\underline{L}, \overline{L})$} is equivalent to the event {*short rate leaves the corridor* (L, U)}, where

$$L = B^{-1}(\delta)\ln(A(\delta)(1+\delta\underline{L})),$$
$$U = B^{-1}(\delta)\ln(A(\delta)(1+\delta\overline{L})). \tag{119}$$

To price a zero-coupon knock-out bond that is knocked out when the LIBOR rate exits the corridor $(\underline{L}, \overline{L})$ (short rate exits the corridor (L, U)), we need to evaluate the expectation

$$P(r, T, L, U) = E_r\left[e^{-\int_0^T r(t)\,dt}\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}}\right], \tag{120}$$

where $\mathcal{T}_{(L,U)} = \inf\{t \geqslant 0\colon r_t \notin (L, U)\}$.

The speed density of the CIR diffusion is (b is defined in Equation (117))

$$\mathfrak{m}(r) = \frac{2}{\sigma^2}r^{b-1}\exp\left(-\frac{2\kappa r}{\sigma^2}\right) \tag{121}$$

and is used to define the inner product in the space of all square-integrable functions on $[L, U]$. To find explicit expressions for the eigenfunctions, we need to find the functions $\xi_\lambda(r)$ and $\eta_\lambda(r)$ solving the initial value problems (16)–(18) with the CIR operator

$$\mathcal{A} = -\frac{1}{2}\sigma^2 r\frac{d^2}{dr^2} - \kappa(\theta - r)\frac{d}{dr} + r. \tag{122}$$

Introduce a new variable,

$$x := \frac{2\gamma r}{\sigma^2}. \tag{123}$$

We look for solutions to the ODE (16) with the CIR operator (122) in the form

$$u(r) = r^{-b/2}\exp\left(\frac{\kappa r}{\sigma^2}\right)w(x(r)), \tag{124}$$

for some unknown function $w(x)$. Substituting this functional form into the ODE, we arrive at the Whittaker Equation (56) with the parameters k and m and end-points l and u of the interval corresponding to short-rate barriers L and U given by

$$m := \frac{b-1}{2}, \quad k := \frac{1}{\gamma}\left(\frac{\kappa b}{2} + \lambda\right),$$
$$l := \frac{2\gamma L}{\sigma^2}, \quad u := \frac{2\gamma U}{\sigma^2}. \tag{125}$$

Then, from the second definition in Equation (125), the eigenvalues $\{\lambda_n, n = 1, 2, \ldots\}$ for this problem are

$$\lambda_n = \gamma k_n - \frac{\kappa b}{2}, \tag{126}$$

where k_n are the roots of Equation (63). The corresponding normalized eigenfunctions are

$$\varphi_n(r) = N_n r^{-b/2}\exp\left(\frac{\kappa r}{\sigma^2}\right)\Delta_{k_n,m}(l, x(r)), \tag{127}$$

$$N_n = \sqrt{\frac{\sigma^2 W_{k_n,m}(u)}{2D_{n,m}(l,u)W_{k_n,m}(l)}}, \tag{128}$$

where $D_{n,m}(l, u)$ is defined in Equation (67).

Finally, the knock-out bond price is given by the eigenfunction expansion

$$P(r, T, L, U) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n T}\varphi_n(r), \tag{129}$$

with the coefficients (in the case of knock-out bonds the payoff function is $f(r_T) = 1$)

$$c_n = \langle 1, \varphi_n\rangle = \int_L^U \varphi_n(y)\mathfrak{m}(y)\,dy, \quad n = 1, 2, \ldots \tag{130}$$

In contrast with the case of double-barrier options under the CEV process, the integrals in Equation (130) cannot be calculated analytically and must be computed numerically.

More generally, any interest rate derivative with some interest rate dependent payoff at maturity and knock-out barriers can be priced by the eigenfunction expansion method. For example, a *knock-out cap* is a cap that knocks out at the first time the LIBOR leaves the corridor $(\underline{L}, \overline{L})$ (all remaining caplets are extinguished as soon as either $\underline{L}$ or $\overline{L}$ is hit). An individual caplet pays an amount $\delta(L(T, T+\delta) - K)^+$ at time $T + \delta$, where $L(T, T+\delta)$ is the LIBOR for the period $[T, T+\delta]$ observed at time T (see Hull 2000). A *knock-out caplet* payoff is $\delta(L(T, T+\delta) - K)^+\mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})}>T\}}$, $\mathcal{T}_{(\underline{L},\overline{L})} = \inf\{t\colon L(t, t+\delta) \notin (\underline{L}, \overline{L})\}$. This time-$(T+\delta)$ cash flow is equivalent to a time-T cash flow:

$$\begin{aligned}&\frac{\delta(L(T, T+\delta) - K)^+}{1 + \delta L(T, T+\delta)}\mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})}>T\}}\\ &\quad = (1 - (1+\delta K)P(T, T+\delta))^+\mathbf{1}_{\{\mathcal{T}_{(\underline{L},\overline{L})}>T\}}\\ &\quad = (1 - (1+\delta K)A(\delta)e^{-B(\delta)r_T})^+\mathbf{1}_{\{\mathcal{T}_{(L,U)}>T\}},\end{aligned}$$

$\mathcal{T}_{(L,U)} = \inf\{t \geqslant 0\colon r_T \notin (L, U)\}$. The present value of this cash flow at $t = 0$ is given by the eigenfunction expansion of the form Equation (129) with the coefficients

$$c_n = \int_{r^*}^U [1 - (1+\delta K)A(\delta)e^{-B(\delta)y}]\varphi_n(y)\mathfrak{m}(y)\,dy, \quad n = 1, 2, \ldots, \tag{131}$$

$$r^* = B^{-1}(\delta)\ln(A(\delta)(1+\delta K)). \tag{132}$$

Single-barrier up-and-out and down-and-out interest rate options can be priced similarly to the single-barrier CEV options of §§3.3 and 3.4 by solving singular Sturm-Liouville problems for the Whittaker equation on the intervals $(0, u]$ and $[l, \infty)$.

4.3. Eigenfunction Expansions for Vanilla Zero-Coupon Bonds

Consider again the problem of pricing vanilla zero-coupon bonds (no barriers). The solution is given by the CIR formula (115)–(117). How does the CIR formula emerge in the eigenfunction expansion framework?

Proposition 9. (i) *The spectral representation of the state-price density in the CIR model is* (*in contrast to the rest of the paper, here we label the eigenvalues and eigenfunctions starting at zero,* $\{\lambda_n\}_{n=0}^{\infty}$; *this simplifies the subsequent formulas*):

$$p(T; r, r_T) = \mathfrak{m}(r_T) \sum_{n=0}^{\infty} e^{-\lambda_n T} \varphi_n(r)\varphi_n(r_T), \tag{133}$$

where the eigenvalues and the corresponding normalized eigenfunctions are $(n = 0, 1, \ldots)$

$$\lambda_n = \gamma n + \frac{b}{2}(\gamma - \kappa), \tag{134}$$

$$\varphi_n(r) = N_n e^{((\kappa-\gamma)r)/\sigma^2} L_n^{(b-1)}(x(r)),$$

$$N_n = \sqrt{\frac{\sigma^2\, n!}{2\Gamma(b+n)}} \left(\frac{2\gamma}{\sigma^2}\right)^{b/2}, \tag{135}$$

where $L_n^{(\nu)}(x)$ *are the generalized Laguerre polynomials and* $x = 2\gamma r/\sigma^2$.

(ii) *The zero-coupon bond price is given by the eigenfunction expansion*

$$P(r, T) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n T} \varphi_n(r), \tag{136}$$

$$c_n = \frac{2N_n \Gamma(b+n)}{\sigma^2\, n!} \left(\frac{\sigma^2}{\gamma+\kappa}\right)^b \left(\frac{\kappa-\gamma}{\kappa+\gamma}\right)^n. \tag{137}$$

Proof. See the appendix.

The claims with payoffs $\varphi_n(r_T)$ form a complete set of eigensecurities in the space of all T-maturity $L^2((0, \infty); \mathfrak{m})$-claims in the CIR economy. The expression (136) unbundles the zero-coupon bond into the portfolio of eigensecurities. Finally, the CIR bond pricing formula (115)–(117) is recovered by performing the summation in Equation (136) using the classical identity (Gradshteyn and Ryzhik 1994, p. 1063) (for all $|z| < 1$, $\nu > -1$)

$$\sum_{n=0}^{\infty} z^n L_n^{(\nu)}(x) = (1-z)^{-\nu-1} \exp\left(\frac{xz}{z-1}\right), \tag{138}$$

and identifying $z = e^{-\gamma T}((\kappa-\gamma)/(\kappa+\gamma))$ in Equation (138).

5. COMPUTATIONAL RESULTS

5.1. Double-Barrier Options

Table 1 gives the first six eigenvalues for the double-barrier problem with the barriers placed at $L = 90$ and $U = 120$.

Table 1. Eigenvalues.

n	$\beta = -0.5$	-1	-3
	Double-Barrier Eigenvalues		
1	3.66908 (3.66919)	3.56537 (3.56575)	3.11009 (3.10916)
2	14.4087 (14.4087)	13.8465 (13.8466)	11.4541 (11.4533)
3	32.3079 (32.3079)	30.9813 (30.9813)	25.3606 (25.3603)
4	57.3668 (57.3668)	54.9699 (54.9699)	44.8302 (44.8300)
5	89.5854 (89.5854)	85.8124 (85.8125)	69.8626 (69.8624)
6	128.964 (128.964)	123.509 (123.509)	100.458 (100.458)
n	Up-and-Out Eigenvalues		
1	0.12625 (0.10040)	0.29608 (0.26418)	0.97218 (0.89557)
10	6.77796 (6.75082)	21.5068 (21.4684)	90.2394 (90.1390)
20	26.3758 (26.3487)	85.7620 (85.7236)	366.026 (365.926)
50	162.276 (162.248)	535.549 (535.510)	2308.93 (2308.83)
n	Down-and-Out Eigenvalues		
1	0.23393 (0.24363)	0.38170 (0.38860)	0.89662 (0.91391)
10	1.37271 (1.37572)	2.50572 (2.50816)	6.76357 (6.77038)
50	5.79628 (5.79762)	11.0751 (11.0762)	31.5661 (31.5692)
150	16.3560 (16.3567)	31.8264 (31.8271)	92.6214 (92.6232)
250	26.7411 (26.7417)	52.3433 (52.3438)	153.347 (153.348)

Notes. For the double-barrier problem under the CEV processes with $\beta = -0.5, -1, -3$, the first six eigenvalues are given. Next to each "exact" eigenvalue determined by numerically finding the roots of Equation (63), an estimate (80) is given in parentheses. For the up-and-out problem, the eigenvalues λ_n, $n = 1$, 10, 20, 50 are given. Next to each "exact" eigenvalue determined by numerically finding the roots of Equation (82), an estimate (88) is given in parentheses. For the down-and-out problem, the eigenvalues λ_n, $n = 1$, 10, 50, 150, 250 are given. Next to each "exact" eigenvalue determined by numerically finding the roots of Equation (90), an estimate (96) is given in parentheses. Parameters: $L = 90$, $U = 120$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

Table 2. Convergence of eigenfunction expansions for double-barrier call prices under the CEV processes with $\beta = 0, -0.5, -1, -2, -3, -4$ ($\beta = 0$ corresponds to geometric Brownian motion) and $T = 1, 3,$ and 12 months.

	$\beta = 0$	-0.5	-1	-2	-3	-4
N	Double-Barrier Call $T = 1$ Month					
1	4.8723	5.2942	5.7393	6.7041	7.7843	9.0208
2	3.0908	3.1116	3.1043	2.9892	2.7103	2.2176
3	2.9923	3.0536	3.1039	3.1770	3.2347	3.3242
4	3.0161	3.0834	3.1402	3.2237	3.2740	3.2999
5	3.0154	3.0820	3.1376	3.2169	3.2598	3.2764
6	3.0154	3.0820	3.1376	3.2171	3.2607	3.2794
Laplace	3.0154	3.0820	3.1376	3.2171	3.2606	3.2793
N	Double-Barrier Call $T = 3$ Months					
1	2.5586	2.8248	3.1157	3.7778	4.5590	5.4818
2	2.4135	2.6303	2.8579	3.3354	3.8193	4.2705
3	2.4131	2.6301	2.8579	3.3370	3.8268	4.2959
Laplace	2.4131	2.6300	2.8578	3.3370	3.8268	4.2959
N	Double-Barrier Call $T = 12$ Months					
1	0.1410	0.1672	0.1994	0.2859	0.4105	0.5827
2	0.1410	0.1672	0.1994	0.2859	0.4103	0.5822
Laplace	0.1410	0.1673	0.1994	0.2860	0.4104	0.5823

Notes. For $T = 1$ month, for each price seven values are given: Partial sums of the first N terms of the expansion Equation (70) ($N = 1, \ldots, 6$) and the value obtained by the numerical Laplace inversion. For $T = 3$ months, for each price four values are given: Partial sums of the first N terms of the expansion ($N = 1, 2, 3$) and the value obtained by the numerical Laplace inversion. For $T = 12$ months, for each price three values are given: Partial sums of the first N terms of the expansion ($N = 1, 2$) and the value obtained by the numerical Laplace inversion. All numerical Laplace inversion values are taken from Davydov and Linetsky (2001). Parameters: $S = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

The instantaneous risk-free interest rate is 10% ($r = 0.1$) and the underlying asset does not pay dividends ($q = 0$). The CEV process parameters are selected in the following way. For each elasticity β ($\beta = -0.5, -1, -3$), the scale parameter δ is selected so that the instantaneous volatility $\sigma(S) = \delta S^{\beta}$ is equal to 0.25 (volatility of 25%) when $S = 100$ (see Boyle and Tian 1999 and Davydov and Linetsky 2001). The approximate eigenvalues estimated using Equation (80) are given in parentheses next to each "exact" eigenvalue. The "exact" eigenvalues are determined using a numerical root finding procedure for Equation (63). One can see that the estimates (80) are quite accurate even for small n.

Table 2 illustrates convergence of eigenfunction expansions for prices of double-barrier calls with 1, 3, and 12 months to expiration and $S = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$. For each elasticity β ($\beta = 0, -0.5, -1, -2, -3, -4$), the scale parameter δ is selected so that the instantaneous volatility $\sigma(S) = \delta S^{\beta}$ is equal to 0.25 when $S = 100$. The case $\beta = 0$ corresponds to the geometric Brownian motion process of §2. The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2001) are provided for comparison. The agreement between the eigenfunction expansion and the numerical Laplace inversion is remarkable. For 12 months to expiration, only the first two or three terms in the eigenfunction expansion are required to achieve the accuracy of five significant digits for double-barrier call prices. For three-month options, three or four terms are required. For one-month options, five or six terms are required.

5.2. Up-and-Out and Capped Options

Table 1 gives the eigenvalues λ_n, $n = 1, 10, 20, 50$, for the up-and-out problem with $U = 120$. The approximate eigenvalues estimated using Equation (88) are given in parenthesis next to each "exact" eigenvalue. The "exact" eigenvalues are determined using a numerical root finding procedure for Equation (82). The estimates (88) are quite accurate.

Table 3 illustrates convergence of the eigenfunction expansion for up-and-out and capped call prices. The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2001) are provided for comparison.

5.3. Down-and-Out Options

Table 1 gives the eigenvalues λ_n, $n = 1, 10, 50, 150, 250$ for the down-and-out problem with $L = 90$. The approximate eigenvalues estimated using Equation (96) are given in parentheses next to each "exact" eigenvalue. The "exact" eigenvalues are determined using a numerical root finding procedure for Equation (90). The estimates (96) are quite accurate. The convergence for down-and-out options is slower than for double-barrier and up-and-out options

Table 3. Convergence of eigenfunction expansions for up-and-out and capped call prices under the CEV processes with $\beta = -0.5, -1, -2, -3, -4$ and $T = 1$ and 12 months.

	$\beta = -0.5$	-1	-2	-3	-4
N	Up-and-Out Call $T = 1$ Month				
1	0.0499	0.2883	1.4853	3.5512	6.2527
10	4.7555	3.1529	3.2334	3.2754	3.3011
20	3.0674	3.1445	3.2271	3.2757	3.3011
50	3.0870	3.1440	3.2271	3.2757	3.3011
Laplace	3.0870	3.1440	3.2271	3.2756	3.3010
N	Rebate $T = 1$ Month				
1	15.3320	12.5486	6.9059	2.0199	−2.2197
10	−1.3140	0.1604	0.0695	0.0307	0.0097
20	0.2389	0.1556	0.0755	0.0304	0.0097
50	0.2121	0.1561	0.0755	0.0304	0.0097
Laplace	0.2121	0.1561	0.0755	0.0304	0.0097
	Capped Call $T = 1$ Month				
	3.2928	3.3001	3.3026	3.3061	3.3108
N	Up-and-Out Call $T = 12$ Months				
1	0.0405	0.2006	0.7417	1.3290	1.8765
5	0.6591	0.8709	1.1246	1.4752	1.9061
10	0.7710	0.8709	1.1246	1.4752	1.9061
Laplace	0.7711	0.8709	1.1246	1.4752	1.9061
N	Rebate $T = 12$ Months				
1	15.5820	13.8024	11.7924	11.1851	10.9988
5	11.0576	10.8384	10.9107	10.9596	10.9617
10	10.7934	10.8384	10.9107	10.9596	10.9617
Laplace	10.7934	10.8384	10.9107	10.9596	10.9617
	Capped Call $T = 12$ Months				
	11.5645	11.7093	12.0353	12.4348	12.8678

Notes. For $T = 1$ month, for each up-and-out call five values are given: Partial sums of the first N terms of the eigenfunction expansion Equation (81) ($N = 1, 10, 20, 50$) and the value obtained by the numerical Laplace inversion. For each rebate five values are given: Partial sums including the first N terms of the series in Equation (102) ($N = 1, 10, 20, 50$) times the rebate amount ($U - K$) and the value obtained by the numerical Laplace inversion. For $T = 12$ months, for each up-and-out call four values are given: Partial sums of the first N terms of the expansion ($N = 1, 5, 10$) and the value obtained by the numerical Laplace inversion. For each rebate four values are given: Partial sums including the first N terms of the series in Equation (102) ($N = 1, 5, 10$) times the rebate amount ($U - K$) and the value obtained by the numerical Laplace inversion. Capped call prices are calculated according to Equation (101) by adding the rebate to the up-and-out call. All numerical Laplace inversion values are taken from Davydov and Linetsky (2001). Parameters: $S = K = 100$, $U = 120$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

because the eigenvalues (96) grow linearly with n, in contrast to the n^2 growth for double-barrier and up-and-out.

Table 4 illustrates convergence of the eigenfunction expansion for down-and-out call prices. The values obtained by the numerical Laplace transform inversion in Davydov and Linetsky (2001) are provided for comparison.

5.4. Vanilla Options

Table 5 illustrates convergence of eigenfunction expansions for vanilla calls. The values obtained by computing Cox's (1975) formula (112) are provided for comparison (we use the algorithm provided by Schroder 1989). The convergence for vanilla options is slower than for double-barrier and up-and-out options because the eigenvalues in (105) grow linearly with n.

5.5. Interest Rate Barrier Options

Table 6 illustrates convergence of the series (129) for knock-out bonds with $T = 0.5$, 1, 3, 5, and 10 years to maturity. The CIR process parameters are $\theta = 0.07$, $\kappa = 0.2$, $\sigma = 0.1$. The initial short rate is $r = 0.06$, and the lower and upper barriers are $\underline{L} = 0.02$ and $\overline{L} = 0.11$. The series converges rapidly. For 5 and 10 years the first term is enough to achieve the accuracy of five significant digits. For shorter maturities more terms are needed. For comparison, Table 6 also gives vanilla zero-coupon CIR bonds prices and yields. The spread compensates for the risk of knock-out.

Table 4. Convergence of eigenfunction expansions for down-and-out put and call prices under the CEV processes with $\beta = -0.5, -1, -2, -3, -4$ and $T = 3$ and 12 months.

	$\beta = -0.5$	-1	-2	-3	-4
N	Down-and-Out Put $T = 3$ Months				
10	0.0754	0.1388	0.2028	0.2139	0.2013
50	0.3179	0.3769	0.3330	0.2788	0.2336
150	0.4352	0.4006	0.3338	0.2788	0.2336
250	0.4392	0.4006	0.3338	0.2788	0.2336
Laplace	0.4391	0.4005	0.3337	0.2788	0.2336
N	Down-and-In Forward $T = 3$ Months				
1	9.7751	7.2079	4.2073	2.9261	2.3094
5	9.7938	7.4046	4.7494	3.6183	3.0442
10	9.7938	7.4047	4.7496	3.6185	3.0446
	Down-and-Out Call $T = 3$ Months				
$N = 250$	5.9609	5.9336	5.8791	5.8246	5.7704
Laplace	5.9608	5.9336	5.8790	5.8246	5.7704
N	Down-and-Out Put $T = 12$ Months				
10	0.0381	0.0474	0.0411	0.0316	0.0241
50	0.0642	0.0558	0.0419	0.0317	0.0241
75	0.0646	0.0558	0.0419	0.0317	0.0241
Laplace	0.0646	0.0558	0.0419	0.0317	0.0241
N	Down-and-In Forward $T = 12$ Months				
1	4.7053	2.8181	1.0643	0.5838	0.4707
5	4.7185	2.9328	1.2893	0.7919	0.6319
10	4.7185	2.9328	1.2893	0.7919	0.6319
	Down-and-Out Call $T = 12$ Months				
$N = 75$	11.2354	11.1540	11.0086	10.8821	10.7713
Laplace	11.2354	11.1540	11.0086	10.8821	10.7713

Notes. For $T = 3$ months, for each down-and-out put five values are given: Partial sums of the first N terms of the eigenfunction expansion Equation (89) ($N = 10, 50, 150, 250$) and the value obtained by the numerical Laplace inversion. For each down-and-in forward three values are given: Partial sums including the first N terms in the series in Equation (99) ($N = 1, 5, 10$). For $T = 12$ months, for each down-and-out put four values are given: Partial sums of the first N terms of the series Equation (89) ($N = 10, 50, 75$) and the value obtained by the numerical Laplace inversion. For each down-and-in forward three values are given: partial sums including the first N terms of the series in Equation (99) ($N = 1, 5, 10$). Down-and-out call prices are calculated according to the put-call parity relationship for the down-and-out options Equation (98). N indicates the number of terms taken in the eigenfunction expansion for the down-and-out put. All numerical Laplace inversion values are taken from Davydov and Linetsky (2001). Parameters: $S = K = 100$, $L = 90$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

5.6. Computational Performance of Eigenfunction Expansions

Here we compare computational performance of the eigenfunction expansions developed in this paper with numerical finite-difference schemes to integrate the option pricing PDE. From the previous analysis, the computational performance of eigenfunction expansions improves as the time to expiration T increases because the higher terms are suppressed by the factors $e^{-\lambda_n T}$. In contrast, the numerical PDE schemes slow down as the time to expiration increases because the PDE needs to be integrated further in time. Thus, the two methods have opposite convergence behavior and are complementary. As a test, we consider a double-barrier call option with the same parameters as in Table 2, but with times to expiration ranging from one month to three years.

Eigenfunction expansions in this paper have been implemented in C++. To implement the Whittaker functions appearing in the formulas, we used representations given in Slater (1960). We have also checked our C++ implementation against the results obtained using *Mathematica 4.0*. Whittaker functions are closely related to the Kummer and Tricomi confluent hypergeometric functions available in *Mathematica 4.0* as built-in functions. The computation of eigenfunction expansions proceeds in two steps. First, we find the eigenvalues. Note that we need to do it only once, as the same eigenvalues are used to price options of all strikes and maturities. Thus, we do not include the time needed to compute the eigenvalues in the reported computation time for each option price.

To numerically integrate the CEV PDE with the two Dirichlet boundary conditions at the lower and upper bar-

Table 5. Convergence of eigenfunction expansions for vanilla call prices under the CEV processes with $\beta = -0.5, -1, -2, -3, -4$, and $T = 1$ and 12 months.

	$\beta = -0.5$	-1	-2	-3	-4
N	Vanilla Call $T = 1$ Month				
10	10.1991	3.9635	2.6486	4.1398	4.5958
100	4.6459	3.1336	3.2952	3.3072	3.3105
200	2.9938	3.3218	3.3027	3.3063	3.3109
500	3.3261	3.3011	3.3031	3.3063	3.3109
1,000	3.3005	3.3012	3.3031	3.3063	3.3109
Cox	3.3005	3.3012	3.3031	3.3063	3.3109
N	Vanilla Call $T = 12$ Months				
2	17.9452	20.2703	16.1075	15.2465	15.3931
10	17.0308	14.7942	15.0904	15.2647	15.4770
50	14.9762	15.0022	15.0892	15.2619	15.4767
100	14.9824	15.0022	15.0892	15.2619	15.4767
Cox	14.9824	15.0022	15.0892	15.2619	15.4767

Notes. For $T = 1$ month, for each price six values are given: Partial sums of the first N terms of the expansion in Equation (107) ($N = 10, 100, 200, 500, 1000$) and the value obtained by Cox's formula (112). For $T = 12$ months, for each price five values are given: Partial sums of the first N terms of the expansion ($N = 2, 10, 50, 100$) and the value obtained by Cox's formula. Parameters: $S = K = 100$, $r = 0.1$, $q = 0$, $\sigma(100) = 0.25$.

Table 6. Convergence of eigenfunction expansions for zero-coupon double knock-out bonds with maturities $T = 0.5, 1, 3, 5, 10$ years in the CIR term structure model.

	$T = 0.5$	1	3	5	10
N	Knock-Out Bond Price				
1	1.0012	0.8671	0.4878	0.2745	0.0652
2	1.0145	0.8741	0.4884	0.2745	0.0652
3	0.9573	0.8608	0.4883	0.2745	0.0652
4	0.9561	0.8607	0.4883	0.2745	0.0652
5	0.9578	0.8608	0.4883	0.2745	0.0652
6	0.9579	0.8608	0.4883	0.2745	0.0652
	Knock-Out Bond Yield				
	0.0861	0.1499	0.2389	0.2586	0.2731
	Vanilla Bond Price				
	0.9702	0.9410	0.8306	0.7320	0.5334
	Vanilla Bond Yield				
	0.0605	0.0608	0.0619	0.0624	0.0628

Notes. Each bond pays one dollar at maturity T if the three-month LIBOR rate never leaves the corridor (0.02, 0.11) during the lifetime of the bond, and zero otherwise. For each bond, six values are given: Partial sums of the first N terms in the eigenfunction expansion ($N = 1, 2, \ldots, 6$). Parameters: $\kappa = 0.2$, $\sigma = 0.1$, $\theta = 0.07$, $r_0 = 0.06$. Corresponding vanilla bond prices and yields are given for comparison.

riers L and U and the terminal condition at expiration $t = T$, we implemented the Crank-Nicholson finite-difference scheme that is widely used for numerical options pricing.

Table 7 gives computational results for double-barrier calls with times to expiration of one and three months and one, two and three years. We run the eigenfunction expansion until adding more terms does not change the first seven significant digits. Table 7 reports option prices computed at this level of precision, as well as the number of terms needed to obtain the result and the computation time in sec-

Table 7. Comparison of eigenfunction expansions and finite-difference PDE schemes for double-barrier calls under the CEV process with $\beta = -4$ and $T = 1$ and 3 months and 1, 2, and 3 years.

	Time to Expiration T				
	1 month	3 months	1 year	2 years	3 years
	Eigenfunction Expansion				
Price	3.279322	4.295898	0.5822268	0.02934524	0.001477778
N	8	4	2	2	1
Time	1.8	0.85	0.5	0.5	0.25
$X \times (YT)$	Finite-Difference PDE				
$1200 \times (600T)$	3.26643	4.30115	0.582491	0.0287646	0.00120688
Time	0.01	0.03	0.12	0.25	0.3
$2400 \times (1200T)$	3.27289	4.29852	0.582358	0.0290551	0.00134233
Time	0.05	0.13	0.5	1	1.5
$12000 \times (6000T)$	3.27804	4.29642	0.582253	0.0292874	0.00145069
Time	1.6	5	18	37	57
$24000 \times (12000T)$	3.27868	4.29616	0.582240	0.0293165	0.00146423
Time	8	24	95	190	285

Notes. For the eigenfunction expansions, for each option three values are given: Price with precision of seven significant digits, number of terms in the expansion needed to achieve this level of precision, and the computation time in seconds. For the finite-difference PDE scheme, we consider four discretizations of the problem domain $[L, U] \times [0, T]$, $X \times (YT)$, where X, $X = 1{,}200$, $2{,}400$, $12{,}000$, $24{,}000$, is the number of state (price) steps in the interval $[L, U]$ and YT, $Y = 600$, $1{,}200$, $6{,}000$, $12{,}000$, is the number of time steps in the interval $[0, T]$. Time to expiration T is measured in years and Y is the number of time steps in one year. Parameters: $S = K = 100$, $L = 90$, $U = 120$, $r = 0.1$, $q = 0$, $\beta = -4$, $\sigma(100) = 0.25$.

onds. One term is enough to achieve this level of precision for three years to expiration, while eight terms are needed for the case of one month. Accordingly, the computation times decrease as T increases. Most CPU time is spent on computing Whittaker functions.

For the finite-difference PDE scheme, we consider four discretizations of the problem domain $[L, U] \times [0, T]$, $X \times (YT)$, where X, $X = 1{,}200$, 2,400, 12,000, 24,000, is the number of state (price) steps in the interval $[L, U]$, and YT, $Y = 600$, 1,200, 6,000, 12,000, is the corresponding number of time steps in the interval $[0, T]$. Time to expiration T is measured in years and Y is the number of time steps in one year. For each discretization two numbers are given: the option price and the corresponding computation time. The data confirm the observation that the finite-difference scheme and the eigenfunction expansion have opposite convergence behavior. The finite-difference scheme slows down for longer times to expiration, while convergence of the eigenfunction expansion accelerates. Thus the eigenfunction expansion has a particular advantage in pricing longer-dated contracts. The eigenfunction expansion is an analytical formula that can be computed to an arbitrary level of precision and, thus, can be used to benchmark numerical methods such as finite-difference schemes and simulation algorithms. In addition, the hedge ratios (delta and gamma) can be calculated by taking analytical derivatives. The strengh of numerical PDE methods is in their flexibility. Many market realities such as discrete dividends, day count conventions, discrete sampling of barriers, early exercise, can be incorporated with relative ease.

6. CONCLUSION

This paper develops an eigenfunction expansion approach to pricing options on scalar diffusion processes. All European-style contingent claims with payoffs square-integrable with the speed measure of the diffusion are unbundled into portfolios of primitive securities called *eigensecurities*. The eigensecurities are eigenvectors of the pricing operator and are fundamental building blocks in our approach. All other European-style contingent claims are represented as portfolios of eigensecurities. In particular, Arrow-Debreu securities themselves are unbundled into portfolios of eigensecurities. This produces an eigenfunction expansion of the state-price density (spectral representation of the state-price density).

In this paper, we show that the eigenfunction expansion method is a powerful computational tool for derivatives pricing. While the state-price density solves the initial- and boundary-value problem for the pricing PDE, the eigensecurities are solutions to the *static* pricing equation without the time derivative term. This static pricing equation can be interpreted as a second-order Sturm-Liouville ODE. The Sturm-Liouville theory can then be applied to derivatives pricing.

To illustrate the computational power of the method, this paper develops two specific applications: pricing vanilla, single- and double-barrier options under the CEV process and interest rate knock-out options in the CIR term structure model. For the CEV process, our main result is the analytical inversion of the Laplace transforms in maturity for single- and double-barrier options obtained in Davydov and Linetsky (2001). For the CIR process, we derive analytical expressions for the prices of knock-out bonds. In both applications, the eigenfunction expansions converge rapidly.

Further applications of eigenfunction expansions to problems in financial engineering will be explored in future research. Linetsky (2001, 2002a) obtains analytical solutions for arithmetic Asian options under the geometric Brownian motion assumption as an application of the singular Sturm-Liouville theory. Gorovoi and Linetsky (2001) obtain pricing formulas for *step options* introduced by Linetsky (1998, 1999) (see also Davydov and Linetsky 2001/2002) under the CEV process. Gorovoi and Linetsky (2003) obtain analytical solutions to Black's (1995) model of interest rates as options. Linetsky (2002b) obtains analytical solutions for diffusion hitting times and lookback options in terms of spectral expansions.

APPENDIX PROOFS

Proof of Proposition 1. The function $V(x, T)$ defined by Equation (2) for any payoff $f \in \mathscr{H}$ is a unique continuous solution of the PDE (the operator $\mathscr{A}$ is defined in Equation (9))

$$\mathscr{A}V + \frac{\partial V}{\partial T} = 0, \quad x \in (L, U), \quad T \in (0, \infty),$$

with the Dirichlet boundary conditions $V(L, T) = 0$, $V(U, T) = 0$, $T \in [0, \infty)$, and the initial condition $V(x, 0) = f(x)$, $x \in (L, U)$. If a payoff $\varphi(x_T)$ has the eigenvector property (6) for some λ, then its price at time zero is $V(x, T) = e^{-\lambda T}\varphi(x)$. Substituting this formula into the PDE, we find that $\varphi(x)$ must solve the Sturm-Liouville problem (10). It is classical that the spectrum of a regular Sturm-Liouville problem on the interval $[L, U]$ with $a(x) > 0$ and $r(x) \geqslant 0$ on $[L, U]$ and Dirichlet boundary conditions at both end-points is simple, purely discrete and positive: $0 < \lambda_1 < \lambda_2 < \cdots \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$ (Dunford and Schwartz 1963, Levitan and Sargsjan 1975, Stakgold 1998, Zwillinger 1998). The eigenfunctions φ_n of the regular Sturm-Liouville problem form a complete orthonormal basis in the Hilbert space $\mathscr{H}$. Then any payoff in $\mathscr{H}$ is in the span of φ_n, and the coefficients c_n in (7) are determined by calculating the inner products of the payoff function with the eigenpayoffs (8). The convergence of the eigenfunction expansion (7) is in the norm of the Hilbert space. This proves parts (i) and (ii). Finally, the pricing formula (11) follows from the eigenfunction expansion of the payoff (7), the linearity of the pricing operator, and the eigenvector property of φ_n (6). □

PROOF OF PROPOSITION 2. To show that the functions (31) satisfy the eigenfunction property (32) with the eigenvalues (33), it is enough to show that they solve the Sturm-Liouville problem

$$-\frac{1}{2}\sigma^2 x^2 u'' - (r-q)xu' + ru = \lambda u, \quad x \in (L, U),$$
$$u(L) = 0, \quad u(U) = 0. \tag{139}$$

The transformation

$$y = \frac{\sqrt{2}}{\sigma}\ln\left(\frac{x}{L}\right), \quad u(x) = 2^{-\frac{1}{4}}\sqrt{\sigma}x^{-\nu/\sigma}v(y(x)), \tag{140}$$

reduces the Sturm-Liouville problem (139) to the Liouville normal form with the constant potential

$$-v'' + Qv = \lambda v, \ \ y \in (0, B), \ \ v(0) = 0, \ \ v(B) = 0, \tag{141}$$
$$B = \frac{\sqrt{2}}{\sigma}\ln\left(\frac{U}{L}\right), \quad Q = r + \frac{\nu^2}{2}. \tag{142}$$

It is classical that the functions

$$\left\{\sqrt{\frac{2}{B}}\sin\left(\frac{n\pi y}{B}\right), n = 1, 2, \ldots\right\}$$

form a complete set of normalized eigenfunctions for the problem (141) with $Q = 0$ and eigenvalues $n^2\pi^2/B^2$. Then the same functions are eigenfunctions of the problem with constant Q, but with the eigenvalues $\lambda_n = Q + n^2\pi^2/B^2$. Inverting the Liouville transformation yields the functions (31). They form a complete set of eigenfunctions of the original problem (139). Then the rest of Proposition 2 follows from the general results of Proposition 1. For the double-barrier call payoff we have

$$c_n = \int_K^U (x-K)\varphi_n(x)\mathfrak{m}(x)\,dx$$
$$= \frac{2}{\sigma\sqrt{\sigma u}}\int_K^U (x-K)x^{(\nu/\sigma)-1}\sin\left(\frac{\pi n}{\sigma u}\ln(x/L)\right)dx.$$

Finally, the result (35) follows from the identity ($\omega_n = (\pi n/u)$):

$$\int_k^u e^{az}\sin(\omega_n z)\,dz$$
$$= \frac{1}{\omega_n^2 + a^2}\Big[e^{ak}(\omega_n\cos(\omega_n k) - a\sin(\omega_n k)) - (-1)^n\omega_n e^{au}\Big]. \quad \square$$

PROOF OF PROPOSITION 3. The coefficients c_n are given by the inner product of the call payoff with the eigenfunctions (66). From Equation (69) we have

$$c_n = \int_K^U (Y-K)\varphi_n(Y)\mathfrak{m}(Y)\,dY$$
$$= N_n\frac{\Gamma(\frac{1}{2}+m-k)}{\Gamma(1+2m)}\left[W_{k_n,m}(l)I_n - M_{k_n,m}(l)J_n\right],$$

where ($y := (\mu/(\delta^2|\beta|))Y^{-2\beta}$)

$$I_n = \int_K^U (Y-K)Y^{\frac{1}{2}+\beta}e^{-y(Y)/2}M_{k_n,m}(y(Y))\mathfrak{m}(Y)\,dY, \tag{143}$$
$$J_n = \int_K^U (Y-K)Y^{\frac{1}{2}+\beta}e^{-y(Y)/2}W_{k_n,m}(y(Y))\mathfrak{m}(Y)\,dY. \tag{144}$$

Substituting the expression (52) for $\mathfrak{m}(Y)$ and using the indefinite integrals in Slater (1960, pp. 23–25), the integrals I_n and J_n are calculated in closed form yielding Equations (71)–(74). $\square$

PROOF OF PROPOSITION 4. The end-point U is regular and the solution $\eta_\lambda(S)$ with the initial conditions (18) is the same as in the double-barrier case and is given by Equations (58), (61). The end-point 0 is regular for $\beta < -\frac{1}{2}$ and singular limit-point nonoscillatory for $-\frac{1}{2} \leqslant \beta < 0$. In the regular case, we impose the initial conditions (17) at zero. The solution is

$$\xi_\lambda(S) = S^{\frac{1}{2}+\beta}e^{-x(S)/2}M_{k(\lambda),m}(x(S)). \tag{145}$$

Using the Wronskian $W_x(W_{k,m}, M_{k,m}) = (\Gamma(1+2m))/(\Gamma(\frac{1}{2}+m-k))$, the Wronskian of η and ξ is given by Equation (19) with

$$C(\lambda) = U^{\frac{1}{2}+\beta}e^{u/2}M_{k(\lambda),m}(u). \tag{146}$$

Let k_n be a zero of the Whittaker function $M_{k,m}(u)$ (as noted previously, all zeros $\{k_n\}_{n=1}^\infty$ of the $M_{k,m}(u)$ are simple and concentrated along the positive real half-line). Then the functions $\xi_{\lambda_n}(S)$ and $\eta_{\lambda_n}(S)$ are linearly dependent (λ_n is related to k_n by Equation (64)):

$$\eta_{\lambda_n}(S) = -\frac{\delta^2\Gamma(m-k_n+\frac{1}{2})}{2\mu\Gamma(1+2m)}U^{\frac{1}{2}+\beta}e^{\frac{u}{2}}W_{k_n,m}(u)\xi_{\lambda_n}(S). \tag{147}$$

Then the normalized eigenfunction (24) takes the form (83)–(84). Finally, the up-and-out call payoff is square-integrable with the speed density on the interval $(0, U]$ and, thus, its price is given by the eigenfunction expansion (81) with the coefficients $c_n = \langle f, \varphi_n\rangle = N_n I_n$, where I_n is the integral (143). It was calculated in closed form in Proposition 3 and is given by Equation (72).

In the singular limit-point case $-\frac{1}{2} \leqslant \beta < 0$, only one solution (up to a multiplicative factor independent of S) is in $L^2((0, U], \mathfrak{m})$. The square-integrability criterion singles out this solution in the form (145). Moreover, it is an entire function of λ for fixed S. The Green's function takes the same form as in the regular case, and the above analysis for $\beta < -\frac{1}{2}$ goes through verbatim for the singular limit-point case $-\frac{1}{2} \leqslant \beta < 0$. $\square$

PROOF OF PROPOSITION 5. The domain of the problem is $[L, \infty)$. The end-point L is regular and the solution $\xi_\lambda(S)$ with the initial conditions (17) at $S = L$ is the same as in the double-barrier case and is given by Equations (58) and (61). Infinity is limit-point and nonoscillatory, and only one solution is square-integrable with the speed density near infinity. This solution can be taken in the form

$$\eta_\lambda(S) = S^{\frac{1}{2}+\beta}e^{-x(S)/2}W_{k(\lambda),m}(x(S)). \tag{148}$$

Moreover, it is entire in λ for fixed S. The Wronskian of η and ξ is given by Equation (19) with

$$C(\lambda) = L^{\frac{1}{2}+\beta} e^{\frac{l}{2}} W_{k(\lambda),m}(l). \tag{149}$$

Let k_n be a zero of the Whittaker function $W_{k,m}(l)$. Then the functions $\xi_{\lambda_n}(S)$ and $\eta_{\lambda_n}(S)$ are linearly dependent (λ_n is related to k_n by Equation (64)):

$$\xi_{\lambda_n}(S) = -\frac{\delta^2\Gamma(m-k_n+\frac{1}{2})}{2\mu\Gamma(1+2m)} L^{\frac{1}{2}+\beta} e^{\frac{l}{2}} M_{k_n,m}(l)\eta_{\lambda_n}(S), \tag{150}$$

and the normalized eigenfunction (24) can be taken in the form (91)–(92).

Finally, the down-and-out put payoff is square-integrable with the speed density on the interval $[L,\infty)$ and, thus, its price is given by the eigenfunction expansion (89) with the coefficients $c_n = \langle f, \varphi_n\rangle = N_n \int_L^K (K-Y) Y^{\frac{1}{2}+\beta} e^{-y/2} W_{k_n,m}(y)\mathfrak{m}(Y)\,dY$. The integral is calculated in closed form similar to the integrals (143), (144) by using the indefinite integrals in Slater (1960, pp. 23–25). □

Proof of Proposition 6. For any $\alpha \geqslant 0$ and $0 < L < S < \infty$, introduce the following notation

$$\Psi_\alpha^-(T; S, L) := E_S[e^{-\alpha T_L}\mathbf{1}_{\{T_L\leqslant T\}}]. \tag{151}$$

We need to compute the down-and-in forward price

$$\begin{aligned} f_{DI} &= e^{-rT} E_S\left[(S_T-K)\mathbf{1}_{\{T_L\leqslant T\}}\right] \\ &= e^{-rT} E_S\left[S_T\mathbf{1}_{\{T_L\leqslant T\}}\right] - e^{-rT} K E_S\left[\mathbf{1}_{\{T_L\leqslant T\}}\right]. \end{aligned}$$

The expectation in the first term simplifies as follows

$$\begin{aligned} E_S\left[S_T\mathbf{1}_{\{T_L\leqslant T\}}\right] &= E_S\left[E[S_T|\mathcal{F}_{T_L}]\mathbf{1}_{\{T_L\leqslant T\}}\right] \\ &= E_S[e^{\mu(T-T_L)} L\,\mathbf{1}_{\{T_L\leqslant T\}}] \\ &= e^{\mu T} L\, E_S[e^{-\mu T_L}\mathbf{1}_{\{T_L\leqslant T\}}]. \end{aligned}$$

Then the down-and-in forward price takes the form

$$f_{DI} = e^{-qT} L\,\Psi_\mu^-(T; S, L) - e^{-rT} K\,\Psi_0^-(T; S, L). \tag{152}$$

From Davydov and Linetsky (2001, Propositions 1, 2, and 5), for any $s>0$ the Laplace transform of the function $\Psi_\alpha^-(T; S, L)$ in time to maturity T can be expressed in the form

$$\int_0^\infty e^{-sT}\Psi_\alpha^-(T; S, L)\,dT = \frac{S^{\frac{1}{2}+\beta}e^{-x/2}}{L^{\frac{1}{2}+\beta}e^{-l/2}}\left(\frac{W_{m-\frac{1}{2}-((s+\alpha)/(2\mu|\beta|)),\,m}(x)}{sW_{m-\frac{1}{2}-((s+\alpha)/(2\mu|\beta|)),\,m}(l)}\right), \tag{153}$$

for $0 < L \leqslant S < \infty$ $(0 < l \leqslant x < \infty)$. We invert this Laplace transform by means of the Cauchy Residue Theorem along the lines of Doetsch (1974, pp. 169–173). As a function of the complex variable s, the right-hand side of Equation (153) (in what follows denoted by $F(s)$) is a ratio of two entire functions. The only (simple) zeros of the denominator are $s=0$ and $s=-(\alpha+\lambda_n)$, $n=1,2,\ldots$, where λ_n are the eigenvalues of the down-and-out problem estimated in Equation (96). For large $|k|$, the Whittaker function W is approximated by Equation (94). Then the expression in parenthesis on the right-hand side of Equation (153) is approximated by

$$\frac{x^{\frac{1}{4}}\cos(\pi k(s) - 2\sqrt{k(s)x} - \pi/4)}{s\,l^{\frac{1}{4}}\cos(\pi k(s) - 2\sqrt{k(s)l} - \pi/4)},$$

$$k(s) = m - \frac{1}{4} - \frac{\alpha+s}{2\mu|\beta|}. \tag{154}$$

We select a family of circular arcs $\mathfrak{C}_n$ in the s-plane centered at the origin and with the radii R_n such that the absolute value of the cosine in the denominator of (154) is maximized and is equal to one when $k=k(-R_n)$. Then the ratio of the two cosines in (154) is bounded on $\mathfrak{C}_n$ as $n\to\infty$ $(R_n\to\infty)$, and the function $F(s)$ vanishes on $\mathfrak{C}_n$ as $n\to\infty$ $(R_n\to\infty)$ due to the presence of the factor s in the denominator in (154). The Hypotheses H1 and H2 in Doetsch (1974, p. 171) needed to apply the Cauchy Residue Theorem are then satisfied, and the inverse Laplace transform is expressed as a sum of residues of the function $e^{sT}F(s)$ at $s=0$ and $s=-(\alpha+\lambda_n)$, $n=1,2,\ldots$ (Doetsch 1974, p. 171, Equations (3) and (5)):

$$\Psi_\alpha^-(T; S, L) = \frac{S^{\frac{1}{2}+\beta}e^{-x/2}}{L^{\frac{1}{2}+\beta}e^{-l/2}}\left\{\frac{W_{m-\frac{1}{2}-(\alpha/(2\mu|\beta|)),m}(x)}{W_{m-\frac{1}{2}-(\alpha/(2\mu|\beta|)),m}(l)} + \sum_{n=1}^\infty \frac{2\mu|\beta|e^{-(\alpha+\lambda_n)T}W_{k_n,m}(x)}{(\alpha+\lambda_n)\left[\partial W_{k,m}(l)/\partial k\right]\big|_{k=k_n}}\right\}, \tag{155}$$

where k_n are the roots of Equation (90). Substituting this result into Equation (152) and observing that (see Abramowitz and Stegun 1972, p. 510; $\Gamma(\nu,x)=\Gamma(\nu)G(\nu,x)$ is the incomplete Gamma function)

$$W_{m-\frac{1}{2},m}(x) = x^{-m+\frac{1}{2}}e^{x/2}\Gamma(2m,x),$$

$$W_{-m-\frac{1}{2},m}(x) = x^{m+\frac{1}{2}}e^{x/2}\Gamma(-2m,x),$$

after some algebra we arrive at the final result (99) for the present value of the down-and-in forward contract. □

Proof of Proposition 7. The proof is similar to the proof of Proposition 6. From Davydov and Linetsky (2001, Propositions 1, 2, and 5), for any $s>0$ the Laplace transform of $E_S[e^{-rT_U}\mathbf{1}_{\{T_U\leqslant T\}}]$ in time to maturity T is $(0 < S \leqslant U < \infty)$

$$\int_0^\infty e^{-sT}E_S[e^{-rT_U}\mathbf{1}_{\{T_U\leqslant T\}}]\,dT = \frac{S^{\frac{1}{2}+\beta}e^{-x/2}}{U^{\frac{1}{2}+\beta}e^{-u/2}}\left(\frac{M_{m-\frac{1}{2}-((s+r)/(2\mu|\beta|)),m}(x)}{sM_{m-\frac{1}{2}-((s+r)/(2\mu|\beta|)),m}(u)}\right). \tag{156}$$

We invert this Laplace transform similar to (153) by means of the Cauchy Residue Theorem. As a function of $s\in\mathbb{C}$, the right-hand side of Equation (156) is a ratio of two entire functions. The (simple) zeros of the denominator are $s=0$ and $s=-(r+\lambda_n)$, $n=1,2,\ldots$, where λ_n are the eigenvalues of the up-and-out problem estimated in Equation (88). For large $|k|$, the Whittaker function M is approximated by Equation (86). Then the expression in parenthesis on the right-hand side of Equation (156) is approximated by

$$\frac{x^{\frac{1}{4}}\cos(2\sqrt{k(s)x}-\pi m-\pi/4)}{s\,u^{\frac{1}{4}}\cos(2\sqrt{k(s)u}-\pi m-\pi/4)},$$
$$k(s)=m-\frac{1}{4}-\frac{\alpha+s}{2\mu|\beta|}. \tag{157}$$

We select a family of circular arcs $\mathfrak{C}_n$ in the s-plane centered at the origin and with the radii R_n such that the absolute value of the cosine in the denominator is maximized and is equal to one when $k=k(-R_n)$. Then the ratio of the two cosines is bounded on $\mathfrak{C}_n$ as $n\to\infty$, and the function (157) vanishes on $\mathfrak{C}_n$ as $n\to\infty$ due to the presence of the factor s in the denominator. The Hypotheses H1 and H2 in Doetsch (1974, p. 171) needed to apply the Cauchy Residue Theorem are then satisfied, and the inverse Laplace transform is expressed as a sum of residues at $s=0$ and $s=-(r+\lambda_n)$, $n=1,2,\ldots$, yielding the result (102). □

Proof of Proposition 8. (i) The solution ξ_λ is taken in the form (145) as in the Proof of Proposition 4. The solution η_λ is taken in the form (148) as in the Proof of Proposition 5. The Wronskian is given by Equation (19) with

$$C(\lambda)=\frac{2\mu\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-k)}=\frac{2\mu\Gamma(1+2m)}{\Gamma\left(1-(\lambda/(2\mu|\beta|))\right)}.$$

The Green's function is given by

$$g(S,Y;\lambda)=\mathfrak{m}(Y)\frac{\delta^2}{2\mu}(SY)^{\frac{1}{2}+\beta}e^{-(x+y)/2}\times\frac{\Gamma\left(1-(\lambda/(2\mu|\beta|))\right)}{\Gamma(1+2m)}M_{k,m}(x\wedge y)W_{k,m}(x\vee y). \tag{158}$$

Because the problem is nonoscillatory at both end-points and the spectrum is simple, purely discrete, and bounded below, at an eigenvalue the Green's function has a simple pole. Both $W_{k,m}(a)$ and $M_{k,m}(a)$ are entire functions of k for all fixed $m>0$ and $a>0$. The Gamma function $\Gamma(1-(\lambda/(2\mu|\beta|)))$ in the numerator of (158) has simple poles at $\lambda=2\mu|\beta|n$, $n=1,2,\ldots$ with the corresponding residues $((-1)^n 2\mu|\beta|)/(n-1)!$. Thus the eigenvalues are $\lambda_n=2\mu|\beta|n$, $n=1,2,\ldots$. The residues of the Green's function at $\lambda=\lambda_n$ are

$$\mathrm{Res}_{\lambda=\lambda_n}g(S,Y;\lambda)=\mathfrak{m}(Y)\frac{(-1)^n\delta^2|\beta|(SY)^{\frac{1}{2}+\beta}e^{-(x+y)/2}}{(n-1)!\Gamma(1+2m)}\times M_{m+n-\frac{1}{2},m}(x\wedge y)W_{m+n-\frac{1}{2},m}(x\vee y).$$

When $k=m+n-\frac{1}{2}$, $n=1,2,\ldots$, the functions $M_{k,m}(x)$ and $W_{k,m}(x)$ become linearly dependent and reduce to generalized Laguerre polynomials (Abramowitz and Stegun 1972, p. 505 and pp. 509–510, Buchholz 1969, p. 214)

$$M_{m+n-\frac{1}{2},m}(x)=\frac{(n-1)!\Gamma(1+2m)}{\Gamma(2m+n)}e^{-x/2}x^{m+\frac{1}{2}}L_{n-1}^{(2m)}(x),$$
$$W_{m+n-\frac{1}{2},m}(x)=(-1)^{n-1}(n-1)!e^{-x/2}x^{m+\frac{1}{2}}L_{n-1}^{(2m)}(x).$$

Then the residues can be rewritten in the form

$$\mathrm{Res}_{\lambda=\lambda_n}g(S,Y;\lambda)=-\mathfrak{m}(Y)\frac{\delta^2|\beta|(n-1)!}{\Gamma(n+2m)}\left(\frac{\mu}{\delta^2|\beta|}\right)^{1+2m}\times(SY)e^{-(x+y)/2}L_{n-1}^{(2m)}(x)L_{n-1}^{(2m)}(y).$$

On the other hand, $\mathrm{Res}_{\lambda=\lambda_n}g(S,Y;\lambda)=-\mathfrak{m}(Y)\varphi_n(S)\varphi_n(Y)$, and we recognize the eigenfunctions (105).

(ii) The origin is a killing boundary and the continuous density $p(T;S,S_T)$ is defective. Integrating the representation (111) produces the hitting probability (106) via the relationship: $\int_0^\infty p(T;S,S_T)\,dS_T=1-\Pr(\mathcal{T}_0\leqslant T\mid S_0=S)$.

(iii) Similar to the down-and-out call, the vanilla call is not in $L^2([0,\infty),\mathfrak{m})$. We price the vanilla put first. To price the put, we decompose the put payoff into two parts: $(K-S_T)^+=K\mathbf{1}_{\{T_0\leqslant T\}}+(K-S_T)^+\mathbf{1}_{\{T_0>T\}}$. The first part is the "bankruptcy claim" that pays off the strike price K in the case of killing at zero ("bankruptcy") prior to and including maturity T. The price of the bankruptcy claim contributes the first term in Equation (107). The second part can be interpreted as a down-and-out put with the barrier placed at zero. Its terminal payoff is in $L^2([0,\infty),\mathfrak{m})$ and its price is given by the eigenfunction expansion in (107). The coefficients of the expansion (108) are calculated in closed form using the integrals in Prudnikov et al. (1986), p. 51, 463. Finally, the vanilla call price is recovered from the put-call parity. □

Proof of Proposition 9. (i) The proof is similar to (i) of Proposition 8. (ii) The zero-coupon bond payoff $f(r_T)=1$ is in the span of the eigensecurities ($f=1$ is square-integrable on $(0,\infty)$ with the weight (121)). The coefficients $c_n=\langle 1,\varphi_n\rangle$ are calculated in closed form using the integral (Gradshteyn and Ryzhik 1994, p. 850; $\nu>-1$, $s>0$, $n=0,1,2,\ldots$)

$$\int_0^\infty e^{-sx}x^\nu L_n^{(\nu)}(x)dx=\frac{\Gamma(\nu+n+1)(s-1)^n}{n!\,s^{\nu+n+1}}.\quad\square$$

ENDNOTES

1. The foundations of semigroup pricing theory in a general Markov context are developed by Duffie (1985), Duffie and Garman (1985), and Garman (1985). See Dynkin (1965) and Ethier and Kurtz (1986) for the semigroup approach to Markov processes. See Dunford and Schwartz (1963) for spectral theory of self-adjoint operators in Hilbert space.
2. Feller's boundary classification for one-dimensional diffusions is given in Karlin and Taylor (1981, chapter 15) and Borodin and Salminen (1996, chapter 2).

3. See Karlin and Taylor (1981, p. 194), Karatzas and Shreve (1991, p. 343), and Borodin and Salminen (1996, p. 17) for discussions of scale and speed densities. Our definition of the speed density coincides with that of Karatzas and Shreve (1991) and Borodin and Salminen (1996) and differs from Karlin and Taylor (1981), who do not include 2 in the definition.

4. See Dunford and Schwartz (1963), Fulton and Pruess (1994), Fulton et al. (1996), Levitan and Sargsjan (1975), Pryce (1993), Stakgold (1998, pp. 435–490), Titchmarsh (1962), and Zwillinger (1998, pp. 94–99) for the account of the theory of Sturm-Liouville boundary-value problems.

5. The limit-point/limit-circle and oscillatory/nonoscillatory classifications remain invariant under the conversion to the Liouville normal form. The Liouville transformation can transform a regular end-point of the original equation into a singular end-point of the equation in the Liouville normal form. However, since regular end-points are limit circle and nonoscillatory, it follows that the corresponding singular end-point will be limit circle and nonoscillatory.

6. Recall that the LIBOR $L(t, t+\delta)$ is a simple interest rate for the period $[t, t+\delta]$, and $L(t, t+\delta) = 1/\delta\, (1/(P(t, t+\delta))) - 1$, where $P(t, t+\delta)$ is the time-t price of a zero-coupon bond with unit face and maturity $t+\delta$.

ACKNOWLEDGMENTS

The authors thank the two anonymous referees for many constructive comments and suggestions. This research was supported in part by the National Science Foundation under grant DMI-0200429.

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