

# Structuring, pricing and hedging double-barrier step options

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This paper studies derivative contracts with payoffs contingent on the amount of time the underlying asset price spends outside of a pre-specified price range (occupation time). Proportional and simple double-barrier step options are gradual knockout options with the principal amortized based on the occupation time outside of the range. Delayed double-barrier options are extinguished when the occupation time outside of the range exceeds a prespecified knock-out window (delayed knockout). These contract designs are proposed as alternatives to the currently traded double-barrier options. They alleviate discrete “barrier event” risk and are easier to hedge.

## 1 Introduction

*Barrier options* are one of the oldest types of exotic options. Snyder (1969) describes down-and-out stock options as “limited risk special options”. Merton (1973) derives a closed-form pricing formula for down-and-out calls. A *down-and-out call* is identical to a European call with the additional provision that the contract is canceled (*knocked out*) if the underlying asset price hits a prespecified *lower* barrier level. An *up-and-out call* is the same, except the contract is canceled when the underlying asset price first reaches a prespecified *upper* barrier level. Down-and-out and up-and-out puts are similar modifications of European put options. *Knock-in options* are complementary to the knock-out options: they pay off at expiration if and only if the underlying asset price *does* reach the prespecified barrier prior to expiration. Rubinstein and Reiner (1991) derive closed-form pricing formulas for all eight types of single-barrier options.

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Barrier options are one of the most popular types of exotic options traded over-the-counter<sup>1</sup> on stocks, stock indices, foreign currencies, commodities and interest rates. Derman and Kani (1996) offer a detailed discussion of their investment, hedging and trading applications. There are three primary reasons to use barrier options rather than standard options. First, barrier options may more closely match investor beliefs about the future behavior of the asset. By buying a barrier option, one can eliminate paying for those scenarios one feels are unlikely. Second, barrier option premiums are generally lower than those of standard options since an additional condition has to be met for the option holder to receive the payoff (eg, the lower barrier not reached for down-and-out options). The premium discount afforded by the barrier provision can be substantial, especially when volatility is high. Third, barrier options may match hedging needs more closely in certain situations. One can envision a situation where the hedger does not need his option hedge any longer if the asset crosses a certain barrier level.

*Double-barrier (double knock-out)* options are canceled (knocked out) when the underlying asset first reaches *either* the upper *or* the lower barrier. Double-barrier options have been particularly popular in the OTC currency options markets over the past several years, owing in part to the significant volatility of exchange rates experienced during this period. In response to their popularity in the marketplace, there is a growing literature on double-barrier options. Kunitomo and Ikeda (1992) derive closed-form pricing formulas expressing the prices of double-barrier knock-out calls and puts through infinite series of normal probabilities. Geman and Yor (1996) analyze the problem by probabilistic methods and derive closed-form expressions for the Laplace transform of the double-barrier option price in maturity. Schroder (2000) inverts this Laplace transform analytically using the Cauchy Residue Theorem, expresses the resulting trigonometric series in terms of Theta functions, and studies its convergence and numerical properties. Pelsser (2000) considers several variations on the basic double-barrier knock-out options, including binary double-barrier options (rebate paid at the first exit time from the corridor) and double-barrier knock-in options, and expresses their pricing formulas in terms of trigonometric series. Hui (1997) prices partial double-barrier options, including front-end and rear-end barriers. Further analysis and extensions to various versions of double-barrier contracts traded in the marketplace are given by Douady (1998), Jamshidian (1997), Hui, Lo and Yuen (2000), Schroder (2000), Sidenius (1998) and Zhang (1997). Rogers and Zane (1997) develop numerical methods for double-barrier options with time-dependent barriers. Taleb (1997) discusses practical issues of trading and hedging double-barrier options. Taleb, Keirstead and Rebholz (1998) study double lookback options.

Despite their popularity, standard barrier option contract designs have a number of disadvantages affecting both option buyers and sellers. The barrier option buyer stands to lose his entire option position due to a short-term price spike through the barrier. Moreover, an obvious conflict of interest exists between

barrier option dealers and their clients, leading to the possibility of short-term market manipulation and increased volatility around popular barrier levels (see Hsu (1997) and Taleb (1997) for illuminating discussions of “barrier event” risk and “liquidity holes” in the currency markets). At the same time, the option seller has to cope with serious hedging difficulties near the barrier. The hedging problems with barrier options are well documented in the literature and are discussed in Section 5.1 of this paper.

To help alleviate some of the risk management problems with barrier options, several alternative contract designs have been proposed in the literature. Linetsky (1999) proposes to regularize knock-out (knock-in) options by introducing finite knock-out (knock-in) rates. A knock-out or knock-in contract is structured so that its principal is *gradually amortized* (knocked out) or *increased* (knocked in) based on the amount of time the underlying is beyond a pre-specified barrier level (*occupation time beyond the barrier*). These gradual knock-out (knock-in) contracts are called *step options*, after the Heaviside step function that enters into the definition of the option payoff. An example of a step option is a knock-out option that loses 10% of its *initial principal* per each trading day beyond a pre-specified barrier level (*simple or arithmetic step option*). Alternatively, a *proportional or geometric step option* loses 10% of its *then-current principal* per trading day beyond the barrier. The introduction of finite knock-out and knock-in rates solves the problems related to the discontinuity in structure at the barrier, such as the barrier event risk, possibilities for market manipulation, discontinuous and unbounded delta, and reduces model risk due to discrete trading, transaction costs and volatility misspecification. Several financial institutions started marketing step option-like contracts in the currency, interest rate and equity derivatives markets.

The analytical pricing methodology for claims contingent on occupation times beyond a given barrier level is developed in Linetsky (1999). In particular, single-barrier simple and proportional step options, as well as *delayed barrier options*,<sup>2</sup> are priced in closed form. The pricing is based on the closed-form expressions for the joint law of Brownian motion at time  $T > 0$  and the occupation time of a half-line up to time  $T$  (see Borodin and Salminen (1996) for a different representation of this joint law).<sup>3</sup>

In the present paper, we extend the results of Linetsky (1999) to *double-barrier options*. In particular, we investigate three alternatives to the currently traded standard double-barrier contracts: *simple* and *proportional double step options* and *delayed double-barrier options*. In this case we need a joint law of Brownian motion and the occupation time of an interval  $(l, u)$  (or a joint law of Brownian motion and two occupation times of the intervals  $(-\infty, l]$  and  $[u, \infty)$ ). Our methodology is close in spirit to Geman and Yor (1993) and Geman and Eydeland (1995) (Asian options), Geman and Yor (1996) (double-barrier options) and Chesney, Jeanblanc-Picque and Yor (1997) (Parisian options) and is based on the Feynman-Kac formula.<sup>4</sup> While Geman and Yor (1993) and Geman and Eydeland (1995) work with the average asset price, Geman and Yor (1995)

work with the exit time from an interval and Chesney, Jeanblanc-Pique and Yor (1997) work with the age of Brownian excursion, we work with occupation times. Similarly, we obtain closed-form expressions for the Laplace transform of the option price in maturity, and then invert this Laplace transform numerically via the Euler algorithm of Abate and Whitt (1995) and Choudhury, Lucantoni and Whitt (1994).<sup>5</sup>

This paper is organized as follows. In Section 2, we describe the structure of the contracts: simple and proportional double-barrier step options and delayed double-barrier options. In Section 3.1, a brief review of the known valuation results for double-barrier options is provided. In Sections 3.2–3.5, we develop the valuation methodology for claims contingent on occupation times beyond two barrier levels. Option prices and hedge ratios are expressed as inverse Laplace transforms. In Section 4.1, we numerically invert the Laplace transforms, compute option prices and hedge ratios and illustrate the properties of double-barrier step options and delayed double-barrier options with numerical examples. In Section 4.2, we discuss extensions to discrete monitoring of the barriers, state- and time-dependent volatility, and time-dependent interest rates and dividend yields. In Section 5, we study dynamic hedging of double-barrier and step options with particular focus on risks faced by option sellers. We show that the introduction of finite knock-out rates makes the option delta continuous and improves performance of dynamic delta-hedging schemes. Section 6 concludes the paper. Proofs are collected in Appendix A. Appendix B describes numerical Laplace transform inversion algorithms due to Abate and Whitt (1995) and Choudhury, Lucantoni and Whitt (1994).

## 2 The contracts

### 2.1 Proportional (geometric) double-barrier step options

Consider a standard call with strike price  $K$  and expiration  $T$ . A double knock-out provision renders the option worthless as soon as the underlying price exits a pre-specified price range  $(L, U)$ ,  $L < K < U$ , where  $L(U)$  is the lower (upper) barrier level. Accordingly, the payoff of a double-barrier call at expiration can be written as:<sup>6</sup>

$$\mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} \max(S_T - K, 0) \quad (1)$$

where  $S_T$  is the underlying price at expiration,  $\mathcal{T}_{(L,U)} = \inf\{t: S_t \notin (L, U)\}$  is the first exit time from the range  $(L, U)$  and  $\mathbf{1}_{\{A\}}$  is the indicator function of the event  $A$ .

We introduce *lower and upper knock-out rates*  $\rho^-$  and  $\rho^+$  and define the payoff of a *proportional double-barrier step call* by<sup>7</sup>

$$\exp(-\rho^- \tau_L^- - \rho^+ \tau_U^+) \max(S_T - K, 0) \quad (2)$$

where  $\tau_L^-$  and  $\tau_U^+$  are occupation times spent below the lower barrier  $L$  and above the upper barrier  $U$  until time  $T$ :

$$\tau_L^- := \int_0^T H(L - S_t) dt, \quad \tau_U^+ := \int_0^T H(S_t - U) dt \quad (3)$$

and  $H(\cdot)$  is the Heaviside step function,  $H(x) = 1(0)$  if  $x \geq 0$  ( $x < 0$ ).

The option (2) proportionally amortizes its principal based on the amount of time spent outside of the corridor  $(L, U)$ . A simpler version of the contract has equal knock-out rates  $\rho^- = \rho^+ = \rho$ . In the limit  $\rho \rightarrow \infty$ , the payoff tends to the payoff of an otherwise identical standard double-barrier option (1). In the limit  $\rho \rightarrow 0$ , we recover a plain vanilla European option.

Introduce a *daily knock-out factor*  $d$  (we assume 250 trading days per year)

$$d = \exp\left(-\frac{\rho}{250}\right) \quad (4)$$

It serves as a convenient practical measure of knock-out speed. The terminal option payoff is discounted for the time spent outside of the range:

$$d^n \max(S_T - K, 0), \quad n = n_L^- + n_U^+ \quad (5)$$

where  $n_L^-$  ( $n_U^+$ ) is the total number of trading days the underlying spent below the lower barrier  $L$  (above the upper barrier  $U$ ) during the option's life.<sup>8</sup> The option principal remaining after one day outside of the corridor is equal to the principal at the end of the previous day multiplied by  $d$  (*proportional principal amortization*). To illustrate, an option with the daily knock-out factor of 0.9 loses 10% of its then-current principal per trading day outside of the range. After ten trading days outside of the range, the remaining principal is  $(0.9)^{10} \approx 0.35$ , or 35% of the original principal. Proportional step options are also called *geometric* or *exponential step options*.

## 2.2 Simple (arithmetic) double-barrier step options

A practically interesting alternative is to consider simple (or linear) principal amortization based on the amount of time the underlying is outside of the range. A *simple (linear or arithmetic) double-barrier step call* is defined by its payoff at expiration:

$$\max(1 - R(\tau_L^- + \tau_U^+), 0) \max(S_T - K, 0) \quad (6)$$

The optionality on occupation time limits the option buyer's liability to not more than the premium paid for the option.<sup>9</sup>

A *knock-out window*  $\theta$  is defined as the minimum occupation time outside of the range required to reduce the option principal to zero. It follows from the definition of the payoff (6) that the knock-out window is the reciprocal of the knock-out rate,  $\theta = 1/R$ .

Introducing a *daily knock-out rate*  $R_d$ ,  $R_d = R/250$ , the payoff can be rewritten as (see note 8)

$$\max(1 - R_d n, 0) \max(S_T - K, 0), \quad n = n_L^- + n_U^+ \quad (7)$$

For example, a six-month contract may have a daily knock-out rate of 0.1 (or 10% per day). That means that the option will lose 10% of its initial principal per trading day outside of the range. There will be no principal left after ten days outside of the range.

Knock-in double-barrier step options are defined so that the sum of an in-option and the corresponding out-option is equal to the vanilla option:

$$\min(R(\tau_L^- + \tau_U^+), 1) \max(S_T - K, 0) \quad (8)$$

Alternatively, a knock-in option can be structured based on time spent *inside* of the corridor:

$$\min(R\tau_{(L,U)}, 1) \max(S_T - K, 0) \quad (9)$$

where  $\tau_{(L,U)}$  is the occupation time of the range  $(L, U)$ ,

$$\tau_{(L,U)} = T - \tau_L^- - \tau_U^+ \quad (10)$$

Such gradual knock-in options are popularly known as *expanding face options*, since their face value is expanding, up to a certain cap, based on the occupation time of the range. These options have been traded over-the-counter for some time.<sup>10</sup>

### 2.3 Delayed double-barrier options

Another interesting alternative to standard double-barrier options is a *delayed double-barrier option*

$$\mathbf{1}_{\{\tau_L^- + \tau_U^+ \leq \theta\}} \max(S_T - K, 0) \quad (11)$$

where  $\theta$  is a pre-specified *knock-out window*,  $0 < \theta < T$ . To illustrate, a six month option may have a knock-out window of ten days. That means that the option knocks out in its entirety after ten days outside of the corridor.<sup>11</sup> The limiting cases  $\theta = 0$  and  $\theta = T$  correspond to the standard double-barrier option and the plain vanilla option, respectively.

## 3 Pricing

### 3.1 Standard double-barrier options

To fix our notation, in this section we briefly review some material on the valuation of double-barrier options (Kunitomo and Ikeda, 1992; Geman and Yor, 1996; Pelsser, 2000; Schroder, 2000). Consider a standard double-barrier call with strike  $K$  and two knock-out barriers  $L$  and  $U$ ,  $0 < L < K < U$ . We assume we live in the Black–Scholes world with constant continuously compounded risk-free interest rate  $r$ , and the underlying asset follows a geometric Brownian motion with constant volatility  $\sigma$  and continuous dividend yield  $q$ . Then, accord-

ing to the standard option pricing theory (see, eg, Duffie, 1996), the double-barrier call's value at the contract inception  $t = 0$  is given by the discounted risk-neutral expectation of its payoff at expiration  $t = T$

$$C_{DB}(S; T, K, L, U) = e^{-rT} E_S \left[ \mathbf{1}_{\{\mathcal{T}_{(L,U)} > T\}} \max(S_T - K, 0) \right] \tag{12}$$

where  $\mathcal{T}_{(L,U)}$  is the first exit time from the range  $(L, U)$ , and  $E_S$  is the conditional expectation operator associated with the geometric Brownian motion  $S_t$  starting at  $S$  at time  $t = 0$  and solving the SDE  $dS_t = (r - q)S_t dt + \sigma S_t dB_t$  ( $B_t$  is a standard Brownian motion).

Introduce the following notation

$$u := \frac{1}{\sigma} \ln\left(\frac{U}{L}\right), \quad k := \frac{1}{\sigma} \ln\left(\frac{K}{L}\right), \quad x := \frac{1}{\sigma} \ln\left(\frac{S}{L}\right) \tag{13}$$

$$v := \frac{1}{\sigma} \left( r - q - \frac{\sigma^2}{2} \right), \quad \xi := r + \frac{v^2}{2} \tag{14}$$

To calculate the expectation, we note that the process  $S_t$  can be represented as follows

$$S_t = L e^{\sigma(W_t + vt)} \tag{15}$$

where  $W_t$  is a Brownian motion starting at  $x$  at time  $t = 0$ .<sup>12</sup> Then, due to the Cameron–Martin–Girsanov theorem, the expectation in Equation (12) takes the form

$$\begin{aligned} C_{DB}(S; T, K, L, U) &= e^{-rT} E_x \left[ e^{v(W_T - x) - \frac{v^2 T}{2}} \mathbf{1}_{\{\Sigma_{(0,u)} > T\}} \mathbf{1}_{\{W_T \geq k\}} (L e^{\sigma W_T} - K) \right] \\ &= e^{-\xi T - vx} \left[ L \Psi_{DB}(T, v + \sigma, u, k, x) - K \Psi_{DB}(T, v, u, k, x) \right] \end{aligned} \tag{16}$$

where  $\Sigma_{(0,u)} := \inf\{t : W_t \notin (0, u)\}$  is the first exit time from the range  $(0, u)$ ,  $E_x$  is the conditional expectation operator associated with  $W$ , and the function  $\Psi_{DB}$  is defined by ( $T \geq 0, \lambda \in \mathbb{R}, 0 \leq k \leq u, 0 \leq x \leq u$ )

$$\Psi_{DB}(T, \lambda, u, k, x) := E_x \left[ \mathbf{1}_{\{\Sigma_{(0,u)} > T\}} \mathbf{1}_{\{W_T \geq k\}} e^{\lambda W_T} \right] = \int_k^u e^{\lambda y} p_{DB}(T; x, y) dy \tag{17}$$

where  $p_{DB}(t; x, y)$  is the transition probability density for a Brownian motion with two absorbing barriers at 0 and  $u$  and starting at  $0 < x < u$ .

The problem of computing  $p_{DB}(t; x, y)$  is classic (see Feller, 1971, pp. 341–3 and p. 478, or Cox and Miller, 1965). First, introduce the resolvent kernel  $G_{DB}(s; x, y)$

$$G_{DB}(s; x, y) = \int_0^{\infty} e^{-st} p_{DB}(t; x, y) dt \quad (18)$$

It solves the ordinary differential equation ( $\delta(x)$  is the Dirac's delta function)

$$\frac{1}{2} G_{xx} - sG = -\delta(x - y) \quad (19)$$

subject to the absorbing boundary conditions at the lower and upper barriers

$$G_{DB}(s; 0, y) = G_{DB}(s; u, y) = 0 \quad (20)$$

The solution to this boundary value problem is given by (see Feller, 1971, p. 478)

$$G_{DB}(s; x, y) = \frac{e^{-\sqrt{2s}|x-y|} + e^{-\sqrt{2s}(2u-|x-y|)} - e^{-\sqrt{2s}(x+y)} - e^{-\sqrt{2s}(2u-y-x)}}{\sqrt{2s}(1 - e^{-2u\sqrt{2s}})} \quad (21)$$

It is also convenient to re-write the resolvent as follows

$$G_{DB}(s; x, y) = \frac{e^{-\sqrt{2s}|x-y|}}{\sqrt{2s}} + \frac{e^{\sqrt{2s}(x-y)} + e^{-\sqrt{2s}(x-y)} - e^{\sqrt{2s}(x+y)} - e^{-\sqrt{2s}(x+y-2u)}}{\sqrt{2s}(e^{2\sqrt{2s}u} - 1)} \quad (22)$$

where the first term is the standard resolvent kernel for Brownian motion on the real line starting at  $x$ . Expanding  $1/(1 - e^{-2\sqrt{2s}u})$  into a geometric series, one is led to the alternative representation (Feller, 1971, p. 478)

$$G_{DB}(s; x, y) = \frac{1}{\sqrt{2s}} \sum_{j=-\infty}^{\infty} \left[ e^{-\sqrt{2s}|x-y+2uj|} - e^{-\sqrt{2s}|x+y+2uj|} \right] \quad (23)$$

The Laplace transform (18) can be inverted analytically and one obtains the well-known representation for the transition probability density of Brownian motion with two absorbing barriers (Feller, 1971, p. 341):

$$p_{DB}(t; x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{j=-\infty}^{\infty} \left\{ e^{-\frac{(y-x+2ju)^2}{2t}} - e^{-\frac{(y+x+2ju)^2}{2t}} \right\} \quad (24)$$

An alternative representation is given by the classic Fourier series (eg, Cox and Miller, 1965)

$$p_{DB}(t; x, y) = \frac{2}{u} \sum_{n=1}^{\infty} e^{-\frac{\omega_n^2 t}{2}} \sin(\omega_n x) \sin(\omega_n y), \quad \omega_n = \frac{n\pi}{u} \quad (25)$$

Thus, we have three representations for the transition density  $p_{DB}(t; x, y)$ : as an inverse Laplace transform of the resolvent (22), as a series of normal densities (24) and as a Fourier series (25). They lead to three representations for the function  $\Psi_{DB}$  (17) entering the double-barrier valuation formula (16).



According to Equations (17) and (18), the Laplace transform of the function  $\Psi_{DB}$  in  $T$  is obtained by integrating the resolvent (22) (Geman and Yor, 1996) (for any complex number  $s$  with  $\text{Re}(s) > \lambda^2/2$ ):

$$\int_0^\infty e^{-sT} \Psi_{DB}(T, \lambda, u, k, x) dT = \int_k^u e^{\lambda y} G_{DB}(s; x, y) dy = F_1(s) + F_2(s) \quad (26)$$

where

$$F_1(s) = \begin{cases} \frac{1}{\sqrt{2s}(\sqrt{2s+\lambda})} \left( e^{\lambda x} - e^{\lambda k+(k-x)\sqrt{2s}} \right) + \\ \frac{1}{\sqrt{2s}(\sqrt{2s-\lambda})} \left( e^{\lambda x} - e^{\lambda u+(x-u)\sqrt{2s}} \right), & k \leq x \leq u \\ \frac{1}{\sqrt{2s}(\sqrt{2s-\lambda})} \left( e^{\lambda k+(x-k)\sqrt{2s}} - e^{\lambda u+(x-u)\sqrt{2s}} \right), & 0 \leq x \leq k \end{cases} \quad (27)$$

$$F_2(s) = \frac{1}{\sqrt{2s}(e^{2u\sqrt{2s}}-1)} \left[ \frac{1}{\sqrt{2s+\lambda}} \left( e^{\lambda u+(u-x)\sqrt{2s}} - e^{\lambda k+(k-x)\sqrt{2s}} - e^{\lambda u+(u+x)\sqrt{2s}} + e^{\lambda k+(k+x)\sqrt{2s}} \right) + \frac{1}{\sqrt{2s-\lambda}} \left( e^{\lambda k+(x-k)\sqrt{2s}} + e^{\lambda u+(u-x)\sqrt{2s}} - e^{\lambda u+(x-u)\sqrt{2s}} - e^{\lambda k+(2u-k-x)\sqrt{2s}} \right) \right] \quad (28)$$

Alternatively, substituting the expansion (24) in Equation (17) and performing the integration term-by-term leads to the representation as an infinite sum of normal probabilities (see Kunitomo and Ikeda, 1992)

$$\begin{aligned} & \Psi_{DB}(T, \lambda, u, k, x) \\ &= \sum_{j=-\infty}^\infty \left\{ e^{(x+2ju)\lambda + \frac{T\lambda^2}{2}} \left[ N\left(\frac{u-x-2ju-T\lambda}{\sqrt{T}}\right) - N\left(\frac{k-x-2ju-T\lambda}{\sqrt{T}}\right) \right] - e^{-(x+2ju)\lambda + \frac{T\lambda^2}{2}} \left[ N\left(\frac{u+x+2ju-T\lambda}{\sqrt{T}}\right) - N\left(\frac{k+x+2ju-T\lambda}{\sqrt{T}}\right) \right] \right\} \quad (29) \end{aligned}$$

where  $N(\cdot)$  is the cumulative standard normal distribution function.

An alternative representation is obtained by integrating the Fourier series (25) term-by-term (see Pelsser, 2000; Zhang, 1997):

$$\begin{aligned} \Psi_{DB}(T, \lambda, u, k, x) &= \frac{2}{u} \sum_{n=1}^\infty e^{-\frac{\omega_n^2 T}{2}} \frac{\sin(\omega_n x)}{\lambda^2 + \omega_n^2} \\ &\times \left[ \omega_n e^{\lambda k} \cos(\omega_n k) - \lambda e^{\lambda k} \sin(\omega_n k) - (-1)^n \omega_n e^{\lambda u} \right] \quad (30) \end{aligned}$$

The identity between (24) and (25) and, as a consequence, between (29) and (30), serves as a classic example of the Poisson summation formula (Feller, 1971). However, series (29) and (30) have very different numerical convergence properties. See Schroder (2000) for details.

### 3.2 Proportional (geometric) double-barrier step options

The price of a proportional double-barrier step call with the payoff (2) is given by the discounted risk-neutral expectation

$$C_{\rho^-, \rho^+}(S; T, K, L, U) = e^{-rT} E_S \left[ \exp(-\rho^- \tau_L^- - \rho^+ \tau_U^+) \max(S_T - K, 0) \right] \quad (31)$$

Similar to the calculation in Equation (16), the expectation can be re-written using the Cameron–Martin–Girsanov theorem ( $u, k, x, v$  and  $\xi$  are defined in Equations (13) and (14))

$$C_{\rho^-, \rho^+}(S; T, K, L, U) = e^{-\xi T} E_x \left[ e^{v(W_T - x) - \rho^- \Gamma_0^-(T) - \rho^+ \Gamma_u^+(T)} (L e^{\sigma W_T} - K) \mathbf{1}_{\{W_T \geq k\}} \right] \quad (32)$$

where  $\Gamma_0^-(T)$  ( $\Gamma_u^+(T)$ ) is the occupation time below zero (above the upper barrier at the level  $u$ ) until time  $T$ ,

$$\Gamma_0^-(T) := \int_0^T \mathbf{1}_{\{W_t \leq 0\}} dt, \quad \Gamma_u^+(T) := \int_0^T \mathbf{1}_{\{W_t \geq u\}} dt$$

Further, we can rewrite Equation (32) as

$$C_{\rho^-, \rho^+}(S; T, K, L, U) = e^{-\xi T - vx} \left[ L \Psi_{\rho^-, \rho^+}(T, v + \sigma, u, k, x) - K \Psi_{\rho^-, \rho^+}(T, v, u, k, x) \right] \quad (33)$$

where the function  $\Psi_{\rho^-, \rho^+}$  is defined by ( $\rho_- \geq 0, \rho_+ \geq 0, T \geq 0, \lambda \in \mathbb{R}, 0 \leq k \leq u, x \in \mathbb{R}$ )

$$\Psi_{\rho^-, \rho^+}(T, \lambda, u, k, x) := E_x \left[ e^{\lambda W_T - \rho^- \Gamma_0^-(T) - \rho^+ \Gamma_u^+(T)} \mathbf{1}_{\{W_T \geq k\}} \right] \quad (34)$$

It is convenient to express the proportional double-barrier step option price as a sum of the standard double-barrier option and a premium dependent on the knock-out rates  $\rho^-$  and  $\rho^+$

$$C_{\rho^-, \rho^+}(S; T, K, L, U) = C_{DB}(S; T, K, L, U) + e^{-\xi T - vx} \left[ L \tilde{\Psi}_{\rho^-, \rho^+}(T, v + \sigma, u, k, x) - K \tilde{\Psi}_{\rho^-, \rho^+}(T, v, u, k, x) \right] \quad (35)$$

where  $C_{DB}(S; T, K, L, U)$  is the standard double-barrier call (16), and the func-

tion  $\tilde{\Psi}$  is defined as the difference between the two functions  $\Psi_{\rho^-, \rho^+}$  and  $\Psi_{DB}$ :

$$\begin{aligned} \tilde{\Psi}_{\rho^-, \rho^+}(T, \lambda, u, k, x) &:= \Psi_{\rho^-, \rho^+}(T, \lambda, u, k, x) - \Psi_{DB}(T, \lambda, u, k, x) \\ &= E_x \left[ e^{\lambda W_T} \mathbf{1}_{\{W_T \geq k\}} \left( e^{-\rho^- \Gamma_0^-(T) - \rho^+ \Gamma_u^+(T)} - \mathbf{1}_{\{\Sigma_{(0, u)} > T\}} \right) \right] \end{aligned} \quad (36)$$

PROPOSITION 3.1 *The Laplace transform of the function  $\tilde{\Psi}_{\rho^-, \rho^+}(T, \lambda, u, k, x)$  is given by (for any complex number  $s$  with  $\text{Re}(s) > \lambda^2/2$ ):*

$$\begin{aligned} &\int_0^\infty e^{-sT} \tilde{\Psi}_{\rho^-, \rho^+}(T, \lambda, u, k, x) dT \\ &= \begin{cases} \frac{2e^{\alpha x}}{\Delta} \left[ (\beta - \gamma)C_+ - (\beta + \gamma)e^{2\beta u}C_- + \frac{2\beta e^{(\lambda + \beta)u}}{\gamma - \lambda} \right], & x \leq 0 \\ \frac{1}{\beta \Delta} D - F_2, & 0 \leq x \leq u \\ \frac{2e^{\lambda x}}{\gamma^2 - \lambda^2} - \frac{e^{\lambda u + \gamma(u-x)}}{\gamma(\gamma + \lambda)} \\ \quad + \frac{e^{-\gamma x}}{\Delta} \left[ 2e^{(\beta + \gamma)u}((\alpha + \beta)C_+ + (\alpha - \beta)C_-) - B \frac{e^{(\lambda + \gamma)u}}{\gamma(\gamma - \lambda)} \right], & x \geq u \end{cases} \end{aligned} \quad (37)$$

where<sup>13</sup>

$$\alpha := \sqrt{2(s + \rho^-)}, \quad \beta := \sqrt{2s}, \quad \gamma := \sqrt{2(s + \rho^+)} \quad (38)$$

$$\Delta := e^{2u\beta}(\alpha + \beta)(\beta + \gamma) + (\alpha - \beta)(\beta - \gamma) \quad (39)$$

$$A := e^{2u\beta}(\alpha - \beta)(\beta + \gamma) + (\alpha + \beta)(\beta - \gamma) \quad (40)$$

$$B := e^{2u\beta}(\alpha + \beta)(\beta - \gamma) + (\alpha - \beta)(\beta + \gamma) \quad (41)$$

$$C_+ := \frac{1}{\beta + \lambda} \left( e^{(\lambda + \beta)u} - e^{(\lambda + \beta)k} \right) \quad (42)$$

$$C_- := \frac{1}{\beta - \lambda} \left( e^{(\lambda - \beta)u} - e^{(\lambda - \beta)k} \right) \quad (43)$$

$$\begin{aligned} D &:= \left( (\alpha + \beta)e^{\beta x} - (\alpha - \beta)e^{-\beta x} \right) \left( (\beta - \gamma)C_+ + \frac{2\beta e^{(\lambda + \beta)u}}{\gamma - \lambda} \right) \\ &\quad + \left( (\beta - \gamma)e^{\beta x} + (\beta + \gamma)e^{\beta(2u-x)} \right) (\alpha - \beta)C_- \end{aligned} \quad (44)$$

and  $F_2$  is defined previously in Equation (28).

PROOF See Appendix A.

Now one can calculate the function  $\tilde{\Psi}$ , and hence the option price (33), by numerically inverting the Laplace transform (see Appendix B).

In the practically interesting special case of equal knock-out rates,  $\rho^- = \rho^+ = \rho$ , we have a simplification:

$$\alpha = \gamma = \sqrt{2(s + \rho)}, \quad \beta = \sqrt{2s} \tag{45}$$

$$\Delta = e^{2\beta u}(\alpha + \beta)^2 - (\alpha - \beta)^2, \quad A = (\alpha^2 - \beta^2)(e^{2\beta u} - 1), \quad B = -A \tag{46}$$

The price of the option is then

$$C_\rho(S; T, K, L, U) = C_{DB}(S; T, K, L, U) + e^{-\xi T - \nu x} \left[ L\tilde{\Psi}_\rho(T, \nu + \sigma, u, k, x) - K\tilde{\Psi}_\rho(T, \nu, u, k, x) \right] \tag{47}$$

where

$$\tilde{\Psi}_\rho(T, \lambda, u, k, x) \equiv \tilde{\Psi}_{\rho, \rho}(T, \lambda, u, k, x) = \Psi_{\rho, \rho}(T, \lambda, u, k, x) - \Psi_{DB}(T, \lambda, u, k, x) \tag{48}$$

### 3.3 Delayed double-barrier options

PROPOSITION 3.2 *The price of a delayed double-barrier call with the payoff  $\mathbf{1}_{\{\tau_L^- + \tau_U^+ \leq \theta\}} \max(S_T - K, 0)$  is given by ( $c > 0$ ):*

$$C_\theta^{\text{delayed}}(S; T, K, L, U) = \mathcal{L}_\theta^{-1} \left\{ \frac{1}{\rho} C_\rho(S; T, K, L, U) \right\} = C_{DB}(S; T, K, L, U) + e^{-\xi T - \nu x} \mathcal{L}_\theta^{-1} \left\{ \frac{1}{\rho} \left[ L\tilde{\Psi}_\rho(T, \nu + \sigma, u, k, x) - K\tilde{\Psi}_\rho(T, \nu, u, k, x) \right] \right\} = C_{DB}(S; T, K, L, U) + \frac{e^{-\xi T - \nu x}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\rho\theta}}{\rho} \left[ L\tilde{\Psi}_\rho(T, \nu + \sigma, u, k, x) - K\tilde{\Psi}_\rho(T, \nu, u, k, x) \right] d\rho \tag{49}$$

where  $C_\rho(S; T, K, L, U)$  is the proportional double-barrier step call price (47), and the inverse Laplace transform is taken with respect to the knock-out rate parameter  $\rho$  and calculated at the point  $\theta$  (knock-out window).

PROOF See Appendix A.

### 3.4 Simple (arithmetic) double-barrier step options

PROPOSITION 3.3 *The price of a simple double-barrier step call with the payoff  $\max(1 - R(\tau_L^- + \tau_U^+), 0) \max(S_T - K, 0)$  is given by ( $c > 0$ ):*

$$\begin{aligned}
C_R^{\text{simple}}(S; T, K, L, U) &= R \mathcal{L}_\theta^{-1} \left\{ \frac{1}{\rho^2} C_\rho(S; T, K, L, U) \right\} \\
&= C_{DB}(S; T, K, L, U) + \\
&R e^{-\xi T - \nu x} \mathcal{L}_\theta^{-1} \left\{ \frac{1}{\rho^2} \left[ L \tilde{\Psi}_\rho(T, \nu + \sigma, u, k, x) - K \tilde{\Psi}_\rho(T, \nu, u, k, x) \right] \right\} \\
&= C_{DB}(S; T, K, L, U) + \\
&\frac{R e^{-\xi T - \nu x}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{\rho\theta}}{\rho^2} \left[ L \tilde{\Psi}_\rho(T, \nu + \sigma, u, k, x) - K \tilde{\Psi}_\rho(T, \nu, u, k, x) \right] d\rho \quad (50)
\end{aligned}$$

where the inverse Laplace transform is taken with respect to the knock-out rate parameter  $\rho$  and calculated at the point  $\theta = 1/R$  (knock-out window).

PROOF See Appendix A.

## 4 Numerical implementation, examples and extensions

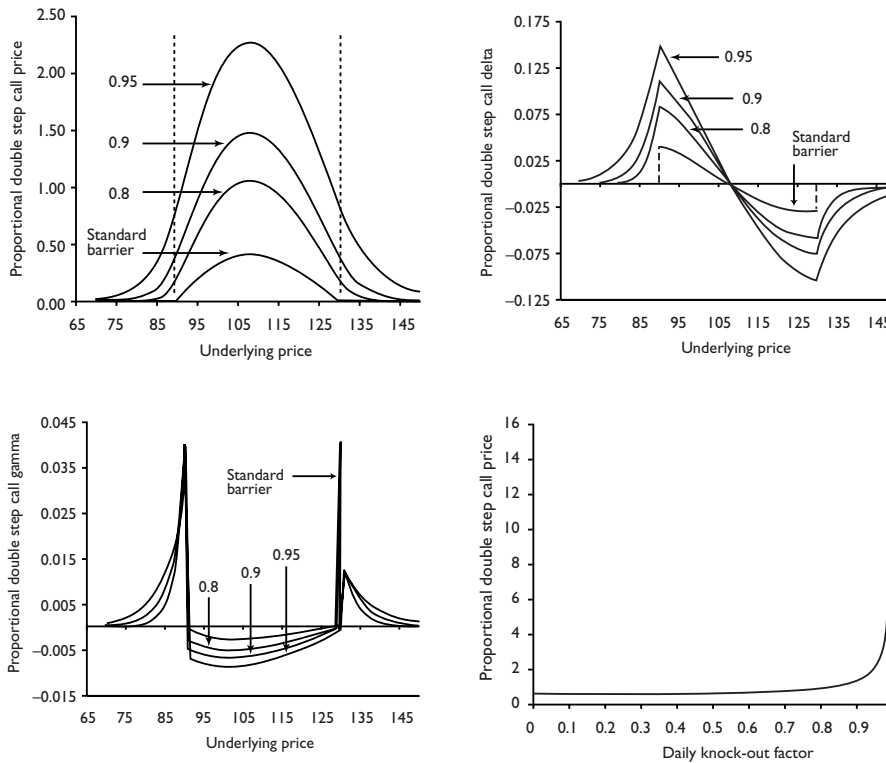
### 4.1 Numerical implementation of the pricing formulas

The pricing formulas (35), (47), (49) and (50) for proportional and simple double-barrier step options and delayed double-barrier options were implemented in C++. To invert Laplace transforms, we used the Euler algorithm described in Appendix B. On the Pentium II 300 MHz PC, computation speed ranged from several milliseconds for the easier case of proportional double-barrier step options that require one-dimensional Laplace transform inversions, to about half a second for the more involved cases of simple double-barrier step options and delayed double-barrier options that require two-dimensional Laplace transform inversions. Figures 1–3 illustrate our computation results.<sup>14</sup>

### 4.2 Extensions to discrete observations and state- and time-dependent volatility

In the previous sections we studied continuously monitored contracts. In practice, many barrier options are structured with discrete monitoring of the barriers, eg, by comparing daily closing prices to the barrier levels. The effect of discrete observations on barrier option prices is well documented in the literature (Cheuk and Vorst, 1996; Broadie, Glasserman and Kou, 1997). Similarly, in practice occupation time contracts are also structured with discrete (daily) monitoring. No analytical formulas are available for discretely sampled contracts. Vetzal and Forsyth (1999) develop accurate numerical finite-difference algorithms to price discretely-sampled single-barrier occupation time

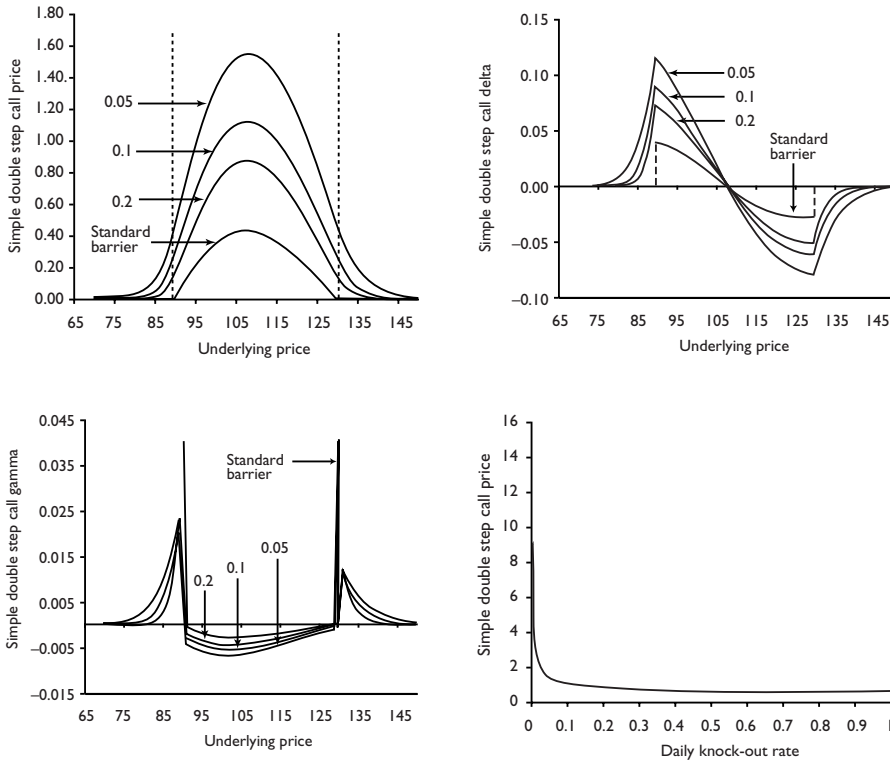
**FIGURE 1** Proportional double-barrier step call prices, deltas and gammas as functions of underlying asset price for daily knock-out factors  $d = 0.8, 0.9, 0.95$ . Standard double-barrier call is given for comparison



The lower right graph plots proportional double-barrier step call price as a function of daily knock-out factor  $d$  (for  $S = 100$ ). Parameters:  $K = 100, L = 90, U = 130, \sigma = 0.3, r = 0.05, q = 0, T = 1$  year.

derivatives, as well as Parisian options.<sup>15</sup> Their algorithms can be straightforwardly extended to the case of two barriers and can handle state- and time-dependent volatility and time-dependent interest rates and dividends. Heath Windcliff (University of Waterloo) implemented the algorithms and benchmarked them against the analytical formulas obtained in this paper. Although the analytical formulas are necessarily restrictive in their assumptions (continuous observations, constant volatility, interest rate and dividend yield), they facilitate the development and benchmarking of general-purpose numerical methods (and can be used as control variates). Table 4 shows convergence of discretely sampled double-barrier step option prices and deltas to the continuous versions of the contracts as the observation frequency increases. The results for standard double-barrier options are also given for comparison. One notes that the differences in prices between continuously and discretely sampled occupation time-dependent contracts are not nearly as large as the differences for continuous vs. discrete hard barrier options.

**FIGURE 2** Simple double-barrier step call prices, deltas and gammas as functions of underlying asset price for daily knock-out rates  $R_d = 5\%, 10\%, 20\%$  per day. Standard double-barrier call is given for comparison



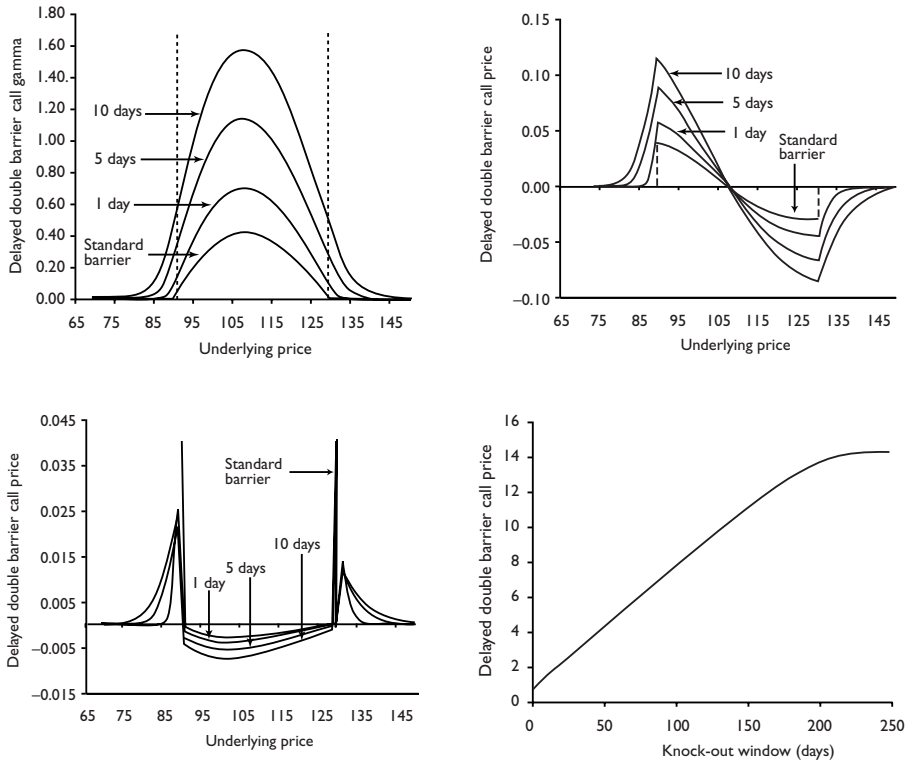
The lower right graph plots simple double-barrier step call price as a function of daily knock-out rate  $R_d$  (for  $S = 100$ ). Parameters:  $K = 100, L = 90, U = 130, \sigma = 0.3, r = 0.05, q = 0, T = 1$  year.

**TABLE I** Standard double-barrier call delta for different times to expiration and underlying asset prices

Asset price	Time to expiration (days)					
	10	5	1	0.1	0.01	0.001
	<b>Double barrier option delta</b>					
119.0	-3.200	-4.769	-8.614	0.075	1.000	1.000
119.4	-3.258	-4.994	-11.199	-10.100	1.000	1.000
119.8	-3.264	-5.072	-12.642	-36.989	-28.991	1.000
119.9	-3.257	-5.067	-12.756	-41.527	-94.382	-8.347
120.0	-3.246	-5.053	-12.763	-43.050	-138.890	-441.983

Option parameters:  $K = 100, L = 90, U = 120, \sigma = 0.15, r = 0.05, q = 0$ .

**FIGURE 3** Delayed double-barrier step call prices, deltas and gammas as functions of underlying asset price for knock-out windows  $\theta = 1, 5, 10$  days. Standard double-barrier call is given for comparison



The lower right graph plots delayed double-barrier step call price as a function of knock-out window  $\theta$  (for  $S = 100$ ). Parameters:  $K = 100, L = 90, U = 130, \sigma = 0.3, r = 0.05, q = 0, T = 1$  year.

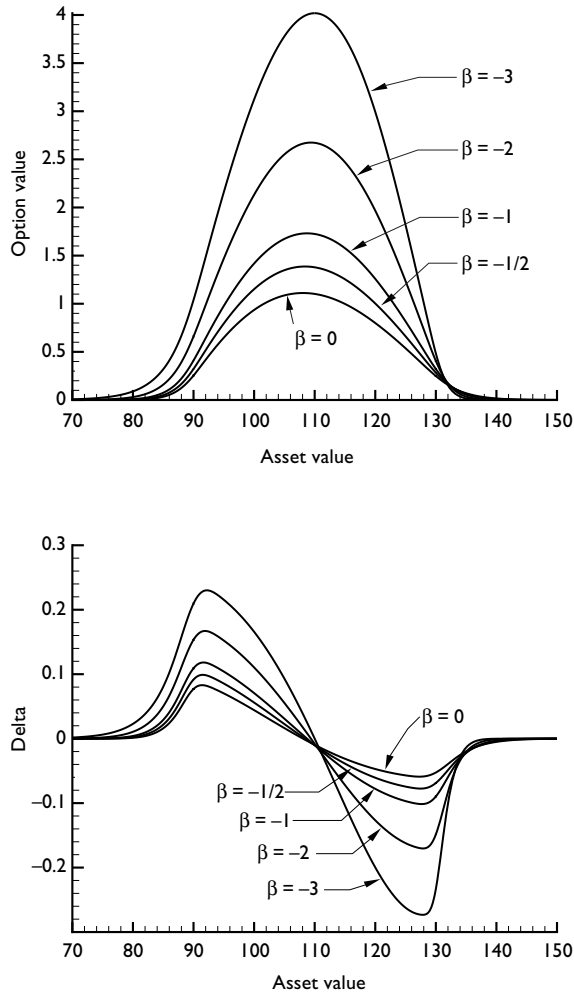
To investigate the impact of volatility smiles, the finite-difference algorithm was extended to handle state-dependent volatility. The constant elasticity of variance (CEV) model of Cox (1975) and Cox and Ross (1976) with negative elasticity of the local volatility function ( $\beta < 0$ )

$$dS_t = (r - q)S_t dt + \delta S_t^{1+\beta} dB_t$$

provides a simple example of a process that leads to downward-sloping volatility (half) smiles (skews) similar to the ones observed in the S&P 500 stock index options market. The parameter  $\delta$  fixes at-the-money volatility level, while  $\beta$  controls the slope of the volatility skew. Typical implied parameter estimates of  $\beta$  implicit in stock index option prices are in the  $-2$  to  $-4$  range (the Black-Scholes model with flat volatility corresponds to the case  $\beta = 0$ ). Boyle and Tian (1999) investigate the effect of the CEV elasticity  $\beta$  on prices of barrier and lookback options in the numerical trinomial lattice framework.



**FIGURE 4** Discretely sampled simple double-barrier step call prices and deltas under the CEV process as functions of the underlying asset price



Parameters:  $K = 100$ ,  $L = 90$ ,  $U = 130$ ,  $r = 0.05$ ,  $q = 0$ ,  $T = 1$  year, daily knock-out rate  $R_d = 0.1$  (10% per day), knock-out window 10 days. The CEV elasticity parameter  $\beta = 0, -0.5, -1, -2, -3$  ( $\beta = 0$  corresponds to the lognormal process). For each  $\beta$ , the parameter  $\delta$  is selected so that the local volatility level at the asset level  $S = 100$  is equal to 30%. The barriers are monitored once a day.

Davydov and Linetsky (2000; 2001) derive analytical expressions for barrier and lookback option prices under the CEV process, provide comparative statics analysis of option prices and hedge ratios, and conduct dynamic hedging simulation experiments to demonstrate that extrema-dependent exotic options such as barriers and lookbacks are extremely sensitive to the slope of the volatility skew. Figure 4 illustrates the effect of  $\beta$  on prices and deltas of double-barrier

step options. Similar to the extrema-dependent contracts, occupation time-dependent contracts are also very sensitive to the elasticity parameter  $\beta$  (and, hence, to the slope of the volatility skew). See Davydov and Linetsky (2001) for further details on the CEV process.

## 5 Hedging

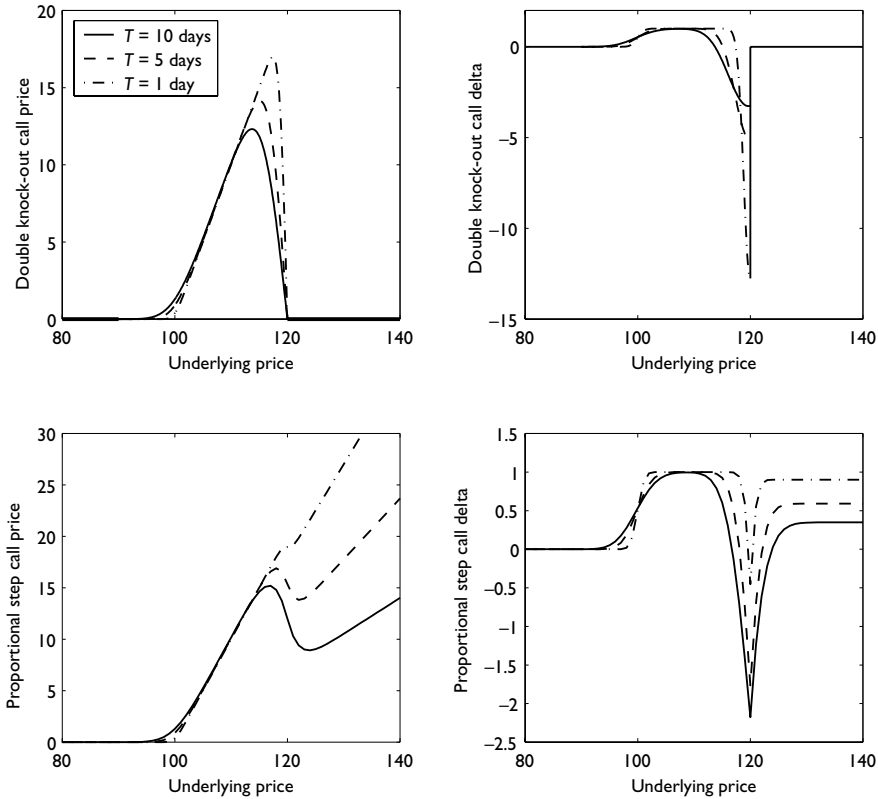
### 5.1 Problems with dynamic hedging of barrier options

Consider an option hedger who sold a standard European call. Assume the standard perfect markets assumptions hold and the underlying asset follows geometric Brownian motion. Then, according to the Black and Scholes (1973) argument, the hedger should execute a dynamic delta-hedging strategy by continuously trading in the underlying risky asset and the risk-free bond. The balance of the hedger's trading account will perfectly offset the liability on the short option position for all terminal values of the underlying price. For a standard European call, the hedge ratio or delta  $\Delta = \Delta(S, t)$  is a continuous function of  $S$  and always lies inside the interval  $[0, 1]$ .

The situation is more complicated for barrier options. First, the barrier option delta is discontinuous at the barrier for all times to maturity. For a double-barrier call, it is positive near the lower barrier and negative near the upper barrier (see Figure 1). To hedge close to the upper barrier, the hedger needs to take a short position in the underlying asset. As the underlying goes up and the barrier is crossed, the entire short hedge position has to be liquidated at once on a stop-loss order. The execution of a large stop-loss order has a risk of "slippage" (buying back the underlying asset at a price greater than the barrier level). This adds to the cost of hedging barrier options and is reflected in wider bid-ask spreads for these OTC contracts. Moreover, in the currency markets, it sometimes happens that many barrier option positions with closely placed barriers exist in the market ("densely mined market"). Consequently, all barrier option writers place their stop-loss buy orders around the same barrier levels to cover their short hedges. When the market rallies through the barrier, all stop-loss orders get triggered and, due to liquidity limitations, this results in a further rally ("liquidity hole") and poor execution ("slippage"). See Taleb (1997) for a detailed discussion of these phenomena of "mined markets" and "liquidity holes".

Second, a potentially more serious problem with barrier options that knock out in-the-money (*reverse knock-out options*), such as up-and-out calls, down-and-out puts and double-barrier options, is that their delta is *unbounded* as expiration approaches and the underlying price nears the barrier. For up-and-out and double-barrier calls, delta tends to  $-\infty$  as  $S \rightarrow U$  and  $t \rightarrow T$ , and the hedger is forced to take progressively larger short positions in the underlying. Figure 5 plots the double-barrier call delta as a function of the underlying price with 10, 5 and 1 day remaining to expiration. Table 1 gives the values of delta for various times to expiration and underlying prices in this dangerous area near the barrier close to expiration. In practice, arbitrarily large short positions are not acceptable, and the

**FIGURE 5** Standard double-barrier and proportional double-barrier step call prices and deltas as functions of the underlying asset price



Parameters:  $K = 100$ ,  $L = 90$ ,  $U = 120$ ,  $\sigma = 0.15$ ,  $r = 0.05$ ,  $q = 0$ ,  $T = 1, 5$  and 10 days to expiration, daily knock-out factor  $d = 0.9$ .

Black–Scholes hedging argument essentially breaks down for reverse knock-out options (but see the discussion in Section 5.3 on superhedging).

Furthermore, consider what happens if there are any imperfections, such as discrete rather than continuous trading, transaction costs, or volatility misspecification, that introduce hedging errors. It is clear from the previous analysis that hedging errors can explode for those sample paths that are near the barrier around expiration. For example, even small, but positive, proportional transaction costs applied to progressively larger positions will accumulate to increasingly large amounts.<sup>16</sup>

To illustrate what can happen in the final days of trading if the underlying is near the barrier, we conduct a simulation hedging experiment. We assume that a double knock-out call with the strike  $K = 100$  and upper and lower barriers  $U = 120$  and  $L = 90$  was previously sold for its theoretical Black–Scholes price and was successfully hedged up until the last day remaining to expiration. We assume that, under the physical measure  $P$ , the underlying follows a geometric

**TABLE 2** Dynamic delta hedging experiment. Standard double-barrier call and proportional double-step call are hedged during the last day prior to expiration

	Hedging frequency (times per day)							
	1	10	100	1,000	1	10	100	1,000
	<b>Double-barrier call hedging</b>				<b>Double-barrier call hedging with t.c.</b>			
Mean P&L	-1.229	-0.181	-0.052	-0.009	-3.279	-4.576	-10.309	-27.560
St. dev.	7.300	3.474	1.460	0.505	7.304	4.108	6.872	20.391
99%	-16.115	-12.701	-4.729	-1.295	-18.173	-20.646	-35.716	-99.064
99.9%	-16.935	-19.790	-12.080	-4.737	-18.994	-29.209	-52.510	-145.457
	<b>Step call hedging</b>				<b>Step call hedging with t.c.</b>			
Mean P&L	0.014	-0.001	-0.001	0.000	-0.131	-0.326	-0.671	-1.748
St. dev.	0.354	0.161	0.055	0.018	0.354	0.163	0.291	0.956
99%	-0.566	-0.379	-0.146	-0.049	-0.712	-0.769	-1.303	-3.841
99.9%	-0.640	-0.625	-0.226	-0.076	-0.786	-1.038	-1.412	-4.112

Hedging frequencies: 1, 10, 100 and 1,000 times per day. The cases without transaction costs and with proportional transaction costs of 0.1% are simulated. For each case, 10,000 Monte Carlo trials are simulated. The mean, standard deviation, and the 99th and 99.9th percentiles (100th and 10th worst outcomes) of the hedging error are given. Option parameters:  $K = 100$ ,  $L = 90$ ,  $U = 120$ ,  $\sigma = 0.15$ ,  $r = 0.05$ ,  $q = 0$ ,  $d = 0.9$ .

Brownian motion with the volatility  $\sigma = 15\%$  and drift  $m = 12\%$ , the risk-free rate is  $r = 5\%$ , and there are no dividends ( $q = 0$ ). We consider four hedging frequencies: 1, 10, 100 and 1,000 times per day. For each hedging frequency, two cases are considered: without transaction costs and with proportional transaction costs of 0.1%. For each case, 10,000 sample paths are simulated. Table 2 gives the summary statistics for dynamic hedging of the double-barrier call<sup>17</sup> over the last day prior to expiration when the underlying is just below the barrier at the beginning of the period,  $S_0 = 119$ . For each case, the mean, standard deviation and the 99th and 99.9th percentiles of the terminal profit and loss (P&L) distribution (hedging error) are reported. First, consider the case without transaction costs. As hedging frequency increases, the hedging error converges to zero extremely slowly. Even when hedging 1,000 times per day, the standard deviation is still sizable and equal to 50 cents. Moreover, the 99th and 99.9th percentiles are very large at  $-\$1.295$  and  $-\$4.734$ , respectively. Furthermore, when proportional transaction costs are introduced, hedging errors quickly explode.

A recently introduced method of *static hedging* (Carr, Ellis and Gupta, 1997; Derman, Ergener and Kani, 1995; Andersen, Andreasen and Eliezer, 2000) of barrier options with portfolios of vanilla calls and puts is an alternative to dynamic delta-hedging. Toft and Xuan (1998) study the performance of static hedging for up-and-out calls. They study static hedges proposed by Derman, Ergener and Kani (1995) that replicate an up-and-out call with a portfolio of vanilla calls of different maturities. They find that performance of static hedges, as with dynamic hedging schemes, is extremely sensitive to the discontinuity in

payoff on the barrier. They conclude: "In general, options with large discontinuous payoffs on the barrier are very difficult to delta hedge because the option's delta changes rapidly when the barrier is hit. It appears that a static hedge performs poorly in exactly the same situations as those where delta hedges are inadequate." Thus, reverse knock-out options with "hard" barriers are difficult to hedge not only dynamically, but statically as well.

We conclude this section with the quote from Tompkins (1997): "Therefore, it is not surprising that these products tend to trade in the Over the Counter market at prices which are approximately double the theoretical price. Clearly, the dynamic hedging of these products is problematic at best."

## 5.2 Dynamic hedging of step options

In contrast to "hard" barrier options, the step option delta is continuous for all times to maturity  $\tau > 0$  and remains bounded as  $\tau \rightarrow 0$ . Consider a proportional double-barrier step call with strike  $K = 100$ , upper and lower barriers  $U = 120$  and  $L = 90$  and daily knock-out factor  $d = 0.9$ . Figure 5 plots step option values and deltas with 10, 5, and 1 day remaining to expiration. The largest negative value of delta is attained at the upper barrier. For example, for  $d = 0.9$  and  $\sigma = 15\%$ , the largest negative value of delta,  $\Delta^* = -2.215$ , is attained at the upper barrier with  $\tau^* = 13$  days remaining to expiration. Table 3 gives the largest negative values of delta,  $\Delta^*$ , and respective times to expiration,  $\tau^*$ , for different values of daily knock-out factor and volatility.

Since the step option delta is bounded by a reasonable value, we expect a significant improvement in performance of the delta-hedging scheme over the "hard" double-barrier option. Table 2 gives the summary statistics for a dynamic hedging experiment similar to the one reported for the "hard" double-barrier call, where a step option is hedged over the last trading day before expiration when the initial underlying price is near the barrier,  $S_0 = 119$ . The hedging errors are orders-of-magnitude smaller than the errors for the "hard" double-barrier call and rapidly converge to zero as hedging frequency increases. The finite knock-out rate bounds the delta and essentially solves the problem of hedging in the dangerous area close to expiration and near the barrier.

**TABLE 3** Largest negative values of delta,  $\Delta^*$ , and respective times to maturity,  $\tau^*$ , for different values of daily knock-out factor and volatility

Volatility	Daily knock-out factor			
	0.95	0.9	0.8	
	<b>Minimum step option delta (time to maturity, days)</b>			
0.10	-2.329 (25)	-3.624 (12.0)	-5.592 (5.750)	-10.411 (1.80)
0.15	-1.356 (27)	-2.215 (13.0)	-3.524 (6.000)	-6.734 (1.85)
0.20	-0.864 (31)	-1.503 (13.5)	-2.483 (6.125)	-4.888 (1.90)

Proportional double-step call parameters:  $K = 100$ ,  $L = 90$ ,  $U = 120$ ,  $r = 0.05$ ,  $q = 0$ .

**TABLE 4** Convergence of discretely sampled simple double-barrier step call prices and deltas to continuously sampled values as the observation frequency increases

Underlying	Barrier monitoring frequency (times per day)			
	1	2	10	Continuous
<b>Standard double-barrier call price (delta)</b>				
90	0.066 (0.040)	0.043 (0.037)	0.017 (0.033)	0.000 (0.039)
100	0.492 (0.029)	0.440 (0.027)	0.376 (0.024)	0.329 (0.022)
130	0.065 (-0.027)	0.042 (-0.025)	0.016 (-0.022)	0.000 (-0.027)
<b>Simple double-barrier step call price (delta) <math>R_d = 20\%</math> per day</b>				
90	0.154 (0.060)	0.148 (0.063)	0.143 (0.069)	0.142 (0.074)
100	0.718 (0.037)	0.717 (0.037)	0.717 (0.037)	0.718 (0.037)
130	0.151 (-0.041)	0.145 (-0.043)	0.140 (-0.047)	0.140 (-0.050)
<b>Simple double-barrier step call price (delta) <math>R_d = 10\%</math> per day</b>				
90	0.254 (0.077)	0.249 (0.080)	0.245 (0.085)	0.244 (0.090)
100	0.935 (0.044)	0.937 (0.044)	0.938 (0.044)	0.939 (0.044)
130	0.249 (-0.053)	0.244 (-0.055)	0.240 (-0.058)	0.240 (-0.061)
<b>Simple double-barrier call price (delta) <math>R_d = 5\%</math> per day</b>				
90	0.451 (0.102)	0.447 (0.106)	0.444 (0.111)	0.444 (0.116)
100	1.315 (0.055)	1.319 (0.055)	1.322 (0.056)	1.323 (0.056)
130	0.444 (-0.070)	0.440 (-0.073)	0.437 (-0.076)	0.437 (-0.079)

Barrier monitoring frequencies: 1, 2, 10 times per day. Daily knock-out rates  $R_d = 5\%$ ,  $10\%$ , and  $20\%$  per day. Standard "hard" double-barrier call prices and deltas are given for comparison. Parameters:  $K = 100$ ,  $L = 90$ ,  $U = 130$ ,  $\sigma = 0.3$ ,  $r = 0.05$ ,  $q = 0$ ,  $T = 1$  year.

### 5.3 Interpretation in terms of rebates and superhedging

Let  $v(t, S)$ ,  $t \in [0, T]$ ,  $S \in [L, U]$ , be the value function of a (newly-written) proportional double-barrier step call. Define  $R_L(t) := v(t, L)$ ,  $R_U(t) := v(t, U)$ ,  $t \in [0, T]$ . Consider an auxiliary contingent claim  $V$  with the cash flows:

- An amount  $\max(S_T - K, 0)$  is paid at expiration  $T$  if the barriers were never reached during the claim's lifetime;
- A rebate  $R_L(\mathcal{T}_L)$  is paid at the first hitting time  $\mathcal{T}_L$  if the lower barrier is reached first;
- A rebate  $R_U(\mathcal{T}_U)$  is paid at the first hitting time  $\mathcal{T}_U$  if the upper barrier is reached first.

This is a double-barrier option with time-dependent rebates equal to the values of the step option on the barriers.

Let  $V(t, S)$  be the value function of the claim  $V$ . Since it is equal to the step option value  $v(t, S)$  on the barriers, has the same terminal condition at expiration, and satisfies the Black-Scholes PDE for  $S \in (L, U)$  and  $t \in [0, T]$ , it is equal to the step option value  $v(t, S)$  everywhere in the range  $S \in [L, U]$  for all  $t \in$

$[0, T]$ . Although it is not practical to market this claim with rebates as a stand-alone contract (note that the rebates  $R_L(t)$  and  $R_U(t)$  are equal to the step option prices on the barriers and, as such, are *model dependent* (if we assume a different process for the underlying asset price, eg, a CEV process, then the rebate amounts will be determined by solving the valuation problem for the step option and will differ from the Black–Scholes values)), it has an indirect hedging application. Obviously, the claim  $V$  dominates the barrier option with no rebates. Hence, the delta-hedging strategy for  $V$  can be used to superhedge the barrier option with no rebates. The P&L from the superhedging strategy will exactly offset the liability at expiration on the short barrier option position for those paths that never reach the barriers, and produce a profit (determined by the rebate amount) for those paths that reach either of the barriers. In contrast to the original delta-hedging strategy for the barrier option, the superhedging strategy has a bounded delta. This is one example of a possible superhedging strategy with bounded delta.

Superhedging strategies for barrier options and other path-dependent contracts are the focus of interesting recent work by Schmock, Shreve and Wystup (1999) and Wystup (1998), (1999). In particular, these authors impose a constraint on the possible values of the leverage or gearing ratio of the option (defined as the option delta times the underlying price divided by the option price) and formulate the pricing problem under the leverage constraint as a stochastic control problem. Remarkably, they are able to solve the problem in closed-form for single-barrier options, similar to the results of Broadie, Cvitanic and Soner (1998) for path-independent options. For double-barrier options, the solution can be obtained numerically. The solution provides an upper hedging price and an optimal superhedging strategy, given the hedger respects the leverage constraint. This solution can be interpreted in terms of an auxiliary claim that pays a rebate on the barrier. If the option trader's goal is to hedge an existing barrier option position under the leverage constraint in the most economical way, then he should follow this optimal superhedging strategy. However, the corresponding auxiliary claim with rebates is not marketable as a stand-alone contract, since the rebate amounts are endogenously determined in the model-dependent way by solving the pricing problem. In contrast, the step option is marketable as a stand-alone contract since its payoff has a simple model-independent form. In addition to possessing continuous delta and bounded gamma, the step option contract also alleviates the "barrier event" risk for the option buyer and reduces the incentives for market manipulation around popular barrier levels.

## 6 Conclusion

This paper has focused on structuring, pricing and hedging derivative contracts with payoffs contingent on the occupation time outside of a given price range (corridor). Double-barrier step options gradually amortize their principal based on the occupation time outside of the corridor. Delayed double-barrier options are

extinguished in their entirety as soon as the occupation time exceeds a pre-specified knock-out window.

From the option buyer’s perspective, these contracts serve as “no-regrets” alternatives to the currently traded double-barrier contracts. From the option seller’s perspective, these contracts solve the problem of discontinuous and unbounded delta that complicates hedging of double-barrier options. Occupation time-based contracts are easier to hedge than “hard” barrier options and, thus, smaller bid/ask spreads over the theoretical price would be required to compensate for the risks inherent in hedging the option sale.

### Appendix A – Proofs

PROOF OF PROPOSITION 3.1 For any complex number  $s$  with  $\text{Re}(s) > \lambda^2/2$ , consider the Laplace transform

$$g_{\rho^-, \rho^+}(s, \lambda, u, k, x) = \int_0^\infty e^{-sT} \Psi_{\rho^-, \rho^+}(T, \lambda, u, k, x) dt$$

According to the Feynman–Kac theorem (Karatzas and Shreve, 1991, Theorem 4.9, p. 271), the function  $g_{\rho^-, \rho^+}(s, \lambda, u, k, x)$  is the unique continuous and continuously differentiable solution of the ODE

$$\frac{1}{2} g_{xx} - (s + \rho^- \mathbf{1}_{\{x \leq 0\}} + \rho^+ \mathbf{1}_{\{x \geq u\}}) g = -f(x)$$

where  $f(x) = e^{\lambda x} \mathbf{1}_{\{x \geq k\}}$ . The function  $g$  and its first derivative  $g_x$  are continuous at the barriers 0 and  $u$ . The solution to this inhomogeneous ODE can be expressed in the form

$$g_{\rho^-, \rho^+}(s, \lambda, u, k, x) = \int_{-\infty}^\infty f(y) G_{\rho^-, \rho^+}(s; x, y) dy = \int_k^\infty e^{\lambda y} G_{\rho^-, \rho^+}(s; x, y) dy \quad (51)$$

where  $G_{\rho^-, \rho^+}(s; x, y)$  is the resolvent kernel that solves the ODE ( $\delta(x)$  is the Dirac’s delta function)

$$\frac{1}{2} G_{xx} - (s + \rho^- \mathbf{1}_{\{x \leq 0\}} + \rho^+ \mathbf{1}_{\{x \geq u\}}) G = -\delta(x - y)$$

subject to the continuity boundary conditions at the lower barrier at zero and the upper barrier at  $u$  (the continuity boundary conditions insure that both the resolvent and its first derivative are continuous at the barriers and, consequently, the function  $g$  is continuous and continuously differentiable at the barriers)

$$\lim_{\varepsilon \rightarrow 0^+} (G_{\rho^-, \rho^+}(s; \varepsilon, y) - G_{\rho^-, \rho^+}(s; -\varepsilon, y)) = 0$$

$$\lim_{\varepsilon \rightarrow 0^+} (G_{\rho^-, \rho^+}(s; u + \varepsilon, y) - G_{\rho^-, \rho^+}(s; u - \varepsilon, y)) = 0$$



$$\lim_{\varepsilon \rightarrow 0^+} \left( \frac{\partial G_{\rho^-, \rho^+}}{\partial x}(s; \varepsilon, y) - \frac{\partial G_{\rho^-, \rho^+}}{\partial x}(s; -\varepsilon, y) \right) = 0$$

$$\lim_{\varepsilon \rightarrow 0^+} \left( \frac{\partial G_{\rho^-, \rho^+}}{\partial x}(s; u + \varepsilon, y) - \frac{\partial G_{\rho^-, \rho^+}}{\partial x}(s; u - \varepsilon, y) \right) = 0$$

and the asymptotic boundary conditions at infinity

$$\lim_{x \rightarrow \pm\infty} G_{\rho^-, \rho^+}(s; x, y) = 0$$

In the notation of Proposition 3.1 (Equations (38)–(44)), the unique solution to this problem is given by:

□ Region (1,1):  $x \leq 0, y \leq 0$

$$G_{\rho^-, \rho^+}^{1,1}(s; x, y) = \frac{1}{\alpha} \left[ e^{-\alpha|x-y|} + \frac{A}{\Delta} e^{\alpha(x+y)} \right]$$

□ Region (1,2):  $x \leq 0, 0 \leq y \leq u$

$$G_{\rho^-, \rho^+}^{1,2}(s; x, y) = \frac{2}{\Delta} e^{\alpha x} [(\beta - \gamma)e^{\beta y} + (\beta + \gamma)e^{\beta(2u-y)}]$$

□ Region (1,3):  $x \leq 0, u \leq y$

$$G_{\rho^-, \rho^+}^{1,3}(s; x, y) = \frac{4\beta}{\Delta} e^{\alpha x + \beta u + \gamma(u-y)}$$

□ Region (2,2):  $0 \leq x \leq u, 0 \leq y \leq u$

$$G_{\rho^-, \rho^+}^{2,2}(s; x, y) = \frac{e^{-\beta|x-y|}}{\beta} + \frac{1}{\beta\Delta} \left[ (\alpha + \beta)(\beta - \gamma)e^{\beta(x+y)} - (\alpha - \beta)(\beta - \gamma)(e^{\beta(x-y)} + e^{-\beta(x-y)}) - e^{2\beta u}(\alpha - \beta)(\beta + \gamma)e^{-\beta(x+y)} \right]$$

□ Region (2,3):  $0 \leq x \leq u, u \leq y$

$$G_{\rho^-, \rho^+}^{2,3}(s; x, y) = \frac{2}{\Delta} e^{-\gamma(y-u)} [(\alpha + \beta)e^{\beta(x+u)} - (\alpha - \beta)e^{-\beta(x-u)}]$$

□ Region (3,3):  $u \leq x, u \leq y$

$$G_{\rho^-, \rho^+}^{3,3}(s; x, y) = \frac{1}{\gamma} \left[ e^{-\gamma|x-y|} - \frac{B}{\Delta} e^{\gamma(2u-x-y)} \right]$$

The solutions in the regions (2,1) ( $0 \leq x \leq u, y \leq 0$ ), (3,1) ( $u \leq x, y \leq 0$ ) and (3,2) ( $u \leq x, 0 \leq y \leq u$ ) are found from the symmetry of the resolvent:

$$G_{\rho^-, \rho^+}^{i,j}(s; x, y) = G_{\rho^-, \rho^+}^{j,i}(s; y, x)$$

As a function of  $y$ , the resolvent is a linear combination of exponentials of the form  $e^{cy}$ . For all complex  $s$  with  $\text{Re}(s) > \lambda^2/2$ , the integral in  $y$  on the right-hand side of Equation (51) exists and can be calculated in closed form. Finally, the Laplace transform of the function  $\tilde{\Psi}_{\rho^-, \rho^+}$  is obtained by subtracting the expression (26) for the Laplace transform of the function  $\Psi_{DB}$  from the expression for the Laplace transform of the function  $\Psi_{\rho^-, \rho^+}$ . After some tedious but straightforward algebra the final result can be represented in the form (37).  $\square$

**PROOF OF PROPOSITION 3.2** First, we note that for any non-negative  $t$  and positive  $\rho$

$$\int_0^\infty e^{-\rho\theta} \mathbf{1}_{\{t \leq \theta\}} d\theta = \frac{1}{\rho} e^{-\rho t}$$

This implies the relationship between the payoffs of delayed double-barrier options and proportional double-barrier step options

$$\int_0^\infty e^{-\rho\theta} \left( \mathbf{1}_{\{\tau_L^- + \tau_U^+ \leq \theta\}} \max(S_T - K, 0) \right) d\theta = \frac{1}{\rho} e^{-\rho(\tau_L^- + \tau_U^+)} \max(S_T - K, 0) \quad (52)$$

Then, taking present values of both sides of Equation (52) and appealing to Fubini's theorem to interchange the order of integration and expectation, we arrive at the relationship between the option prices

$$\int_0^\infty e^{-\rho\theta} C_\theta^{\text{delayed}}(S; T, K, L, U) d\theta = \frac{1}{\rho} C_\rho(S; T, K, L, U) \quad (53)$$

Finally, the first equality in Equation (49) follows from (53) by inverting the Laplace transform. The second equality in Equation (49) follows from Equation (47) (note that  $C_{DB}$  is independent of  $\rho$  and thus  $\mathcal{L}_\theta^{-1}\{(1/\rho)C_{DB}\} = \mathcal{L}_\theta^{-1}\{(1/\rho)\}C_{DB} = C_{DB}$ ).  $\square$

**PROOF OF PROPOSITION 3.2** First, we note that for any non-negative  $t$  and positive  $\rho$

$$\int_0^\infty \theta e^{-\rho\theta} \max\left(1 - \frac{t}{\theta}, 0\right) d\theta = \frac{1}{\rho^2} e^{-\rho t}$$

This implies the relationship between the payoffs of simple and proportional double-barrier step options

$$\int_0^\infty \theta e^{-\rho\theta} \left\{ \max \left[ 1 - \frac{1}{\theta} (\tau_L^- + \tau_U^+), 0 \right] \max (S_T - K, 0) \right\} d\theta = \frac{1}{\rho^2} e^{-\rho(\tau_L^- + \tau_U^+)} \max (S_T - K, 0) \tag{54}$$

Then, taking present values of both sides of Equation (54) and appealing to Fubini’s theorem to interchange the order of integration and expectation, we arrive at the relationship between the option prices

$$\int_0^\infty \theta e^{-\rho\theta} C_{\frac{1}{\theta}}^{\text{simple}} (S; T, K, L, U) d\theta = \frac{1}{\rho^2} C_\rho (S; T, K, L, U) \tag{55}$$

Finally, the first equality in Equation (50) follows from (55) by inverting the Laplace transform and recalling that  $\theta = 1/R$ . The second equality in Equation (50) follows from Equation (47) (note that  $C_{DB}$  is independent of  $\rho$  and thus  $\mathcal{L}_\theta^{-1} \{ (1/\rho^2) C_{DB} \} = \mathcal{L}_\theta^{-1} \{ (1/\rho^2) \} C_{DB} = \theta C_{DB} = (1/R) C_{DB}$ ).  $\square$

### Appendix B – Inverting Laplace transforms via the Euler method

If a given function  $F(s)$  is analytic in the complex region  $\text{Re}(s) > c_0$ , then the inverse Laplace transform of  $F(s)$  can be calculated as the Bromwich contour integral

$$f(t) = \mathcal{L}_t^{-1} \{ F(s) \} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds \tag{56}$$

where  $t > 0$ ,  $c > c_0$ , and  $i = \sqrt{-1}$ .

The functions on the right hand side of Equation (37) are analytic in the region  $\text{Re}(s) > c_0 = \lambda^2/2$ , and the inverse Laplace transform can be calculated numerically using a contour of integration to the right of  $\lambda^2/2$ . To invert Laplace transforms numerically we use the Euler method due to Abate and Whitt (1995). This method was previously applied to option pricing problems by Fu, Madan and Wang (1997) who used it to invert the Geman and Yor (1993) Laplace transform to compute Asian option prices. First, letting the contour of integration be a vertical line  $\text{Re}(s) = c$  such that  $F(s)$  has no singularities on or to the left of it ( $\text{Re}(s) > \lambda^2/2$  in our case), the Bromwich integral can be re-written in the form

$$f(t) = \mathcal{L}_t^{-1} \{ F(s) \} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds = \frac{2e^{ct}}{\pi} \int_0^\infty \cos ut \text{Re}[F(c + iu)] du$$

The integral is evaluated numerically by means of the trapezoidal rule. If we use a step size  $h = \pi/2t$  and let  $c = A/2t$ , we obtain the nearly alternating series:

$$f(t) \approx \frac{e^{A/2}}{2t} F\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{j=1}^{\infty} (-1)^j \operatorname{Re} \left[ F\left(\frac{A + 2j\pi i}{2t}\right) \right]$$

The Euler summation is used to numerically calculate the series. The function  $f(t)$  is approximated by the weighted average of partial sums

$$f(t) = \mathcal{L}_t^{-1}\{F(s)\} \approx 2^{-m} \sum_{k=0}^m \binom{m}{k} s_{n+k}(t)$$

where the partial sums are

$$s_{n+k}(t) = \frac{e^{A/2}}{2t} F\left(\frac{A}{2t}\right) + \frac{e^{A/2}}{t} \sum_{j=1}^{n+k} (-1)^j \operatorname{Re} \left[ F\left(\frac{A + 2j\pi i}{2t}\right) \right]$$

The choice of parameters  $A = 2ct$ ,  $n$  and  $m$  is dictated by the desired accuracy. The parameter  $A$  has to satisfy the condition  $A > 2c_0t$ , so that the contour of integration lies to the right of all the singularities of  $F(s)$ . To select  $A$  in that region, a discretization error can be estimated as (see Abate and Whitt, 1995, for details on this error estimate)

$$e_d = \sum_{k=1}^{\infty} e^{-kA} f((2k+1)t)$$

It is often straightforward to find an upper bound for  $f(t)$  in financial applications. For example, the step option price  $C_{\rho^-, \rho^+}(S; T, K, L, U)$  is always lower than the underlying price,  $C_{\rho^-, \rho^+}(S; T, K, L, U) \leq S$ , and  $e_d$  can be estimated as

$$|e_d| \leq S \frac{e^{-A}}{1 - e^{-A}}$$

The choice of  $A = \delta \ln 10$  produces a discretization error  $|e_d| \leq 10^{-\delta}S$ , and, in particular,  $A = 18.4$  corresponds to  $\delta = 8$ .

Further, parameters  $m$  and  $n$  control the truncation error associated with the Euler summation. Abate and Whitt (1995) suggest to start with  $m = 11$  and  $n = 15$  and then adjust  $n$  as needed. We found that  $n$  as small as 5 gives satisfactory results in our application.

To implement the pricing formulas (49) and (50), we need to invert double Laplace transforms numerically. For a function  $F(s_1, s_2)$  analytic in the complex domain where  $\operatorname{Re}(s_1) \geq c_1$  and  $\operatorname{Re}(s_2) \geq c_2$ , the inverse double Laplace transform can be represented as a bi-variate Bromwich integral

$$f(t_1, t_2) = \mathcal{L}_{t_1}^{-1}\{\mathcal{L}_{t_2}^{-1}\{F(s_1, s_2)\}\} = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} e^{s_1 t_1 + s_2 t_2} F(s_1, s_2) ds_2 ds_1 \tag{57}$$

where  $t_1, t_2 > 0$ .

When  $\rho^- = \rho^+ = \rho$ , the functions on the right-hand side of Equation (37) considered as functions of two complex variables  $s$  and  $\rho$  are analytic in the region where  $\text{Re}(s) > \lambda^2/2$  and  $\text{Re}(\rho) > 0$ .

The Euler algorithm for numerical inversion of Laplace transforms was extended to multiple dimensions by Choudhury, Lucantoni and Whitt (1994). First, the bi-variate Bromwich integral (57) is discretized similar to the one-dimensional case:

$$f(t_1, t_2) \approx \frac{e^{(A_1+A_2)/2}}{2t_1t_2} \left\{ \frac{1}{2} F\left(\frac{A_1}{2t_1}, \frac{A_2}{2t_2}\right) + \sum_{k=1}^{\infty} (-1)^k \text{Re} \left[ \sum_{j=0}^{\infty} (-1)^j \left( F\left(\frac{A_1 - 2j\pi i}{2t_1}, \frac{A_2 - 2k\pi i}{2t_2}\right) + F\left(\frac{A_1 - 2k\pi i}{2t_1}, \frac{A_2 - 2j\pi i}{2t_2}\right) \right) \right] \right\}$$

Then, to calculate the nearly alternating series, the Euler summation is used (we set  $l_1 = l_2 = 1$  in the general algorithm of Choudhury, Lucantoni and Whitt (1994) as this choice is enough to achieve the desired accuracy in our application):

$$f(t_1, t_2) \approx \frac{e^{(A_1+A_2)/2}}{2t_1t_2} \left\{ \frac{1}{2} F\left(\frac{A_1}{2t_1}, \frac{A_2}{2t_2}\right) + 2^{-m} \sum_{K=0}^m \binom{m}{K} s_{n+K} \right\}$$

where

$$s_{n+K} = \sum_{k=1}^{n+K} (-1)^k \text{Re} [\Phi_k]$$

$$\Phi_k = 2^{-m} \sum_{p=0}^m \sum_{q=0}^{n+p} (-1)^q \binom{m}{p} \times$$

$$\left[ F\left(\frac{A_1 - 2q\pi i}{2t_1}, \frac{A_2 - 2k\pi i}{2t_2}\right) + F\left(\frac{A_1 - 2k\pi i}{2t_1}, \frac{A_2 - 2q\pi i}{2t_2}\right) \right]$$

The parameters  $A_1, A_2, n$  and  $m$  control computation accuracy.

The discretization error is controlled by  $A_1$  and  $A_2$ . The truncation error is controlled by  $m$  and  $n$ . Choudhury, Lucantoni and Whitt (1994) suggest the choice of  $A_1 = A_2 = 19.1, m = 11$  and  $n = 38$ . As in the one-dimensional case,  $n$  can be adjusted as needed. We found that the choice of  $m = 11$  and  $n = 10$  produces satisfactory results for our application.

1. In addition to their popularity over-the-counter, several types of barrier options are traded on securities exchanges. Examples of exchange-traded barrier options are single- and double-barrier knock-out call and put warrants on the Australian stock index, the All Ordinaries Index, introduced in 1998 by the Australian Stock Exchange (ASE). These warrants are *Parisian-style*. They knock out after the underlying index stays beyond the barrier for three consecutive days. We thank Glen Kentwell for bringing these warrants to our attention.
2. Delayed barrier options knock in or out as soon as the occupation time beyond the barrier level exceeds a pre-specified *knock-in* or *knock-out window* (e.g., ten trading days). These contracts are also called *cumulative Parisian options* by Chesney, Jeanblanc-Pique and Yor (1997). Delayed barrier options and step options have different properties. The former knock in or out in their entirety as soon as the occupation time exceeds the knock-in or knock-out window, while the latter knock in or out gradually in time. Other interesting alternative barrier option contract designs proposed in the literature include *Parisian options* of Chesney, Jeanblanc-Picque and Yor (1997) and Chesney *et al.* (1997) and *soft barrier options* of Hart and Ross (1994).
3. See Hugonnier (1999) and Pechtl (1995) for more examples of occupation time-dependent contracts. See Akahori (1995), Dassios (1995) and Miura (1992) for a related class of path-dependent *quantile options*.
4. For details on the Feynman-Kac method and its applications to calculations of various Brownian functionals see Borodin and Salminen (1996), Fitzsimmons and Pitman (1998), Jeanblanc-Picque, Pitman and Yor (1997), Karatzas and Shreve (1991), and Revuz and Yor (1999).
5. Fu, Madan and Wang (1997) apply the Euler algorithm to compute the Geman-Yor Asian option formula.
6. All options in this paper are European style. In the interest of brevity only call options are discussed. Puts are treated similarly.
7. We assume that the strike is inside the range  $(L, U)$  for all options in this paper  $(L < K < U)$ .
8. In the present context of continuous monitoring,  $n_L^-$  and  $n_U^+$  do not have to be integers. They represent occupation times measured in days, rather than years. In Section 4.2 we discuss discretely (daily) monitored occupation time contracts where  $n_L^-$  and  $n_U^+$  become integers.
9. Note that if we naively define the payoff as  $(1 - R(\tau_L^- + \tau_U^+)) \max(S_T - K, 0)$ , it could become negative for sufficiently large times outside of the range.
10. After this paper was completed we received preprints Fusai (1999) and Fusai and Tagliani (2001). They focus on pricing a *corridor option* with the payoff  $(\tau_{(L, U)} - K)^+$  based on the law of the occupation time of a corridor. The problem in this paper is more general since our payoffs depend both on the occupation times and the terminal asset price, and the pricing relies on their joint law.
11. Note that the days do not have to be consecutive. This is in contrast with Parisian options that require  $n$  consecutive days beyond the barrier to knock out.
12. For our purposes it is more convenient to work with the Brownian motion  $W_t$  starting at  $x$  and use representation (15), rather than with the standard Brownian motion  $B_t$  starting at the origin and the conventional representation  $S_t = Se^{\sigma(B_t + vt)}$ .
13. Note that all of the quantities  $\alpha, \dots, D$  are functions of the transform variable  $s$ ,

$\alpha = \alpha(s), \dots, D = D(s)$ . To lighten notation we do not show this dependence explicitly.

14. We are grateful to Peter Forsyth, Ken Vetzal and Heath Windcliff for independently verifying some of our computational results via numerical finite-difference schemes.
15. Discretely monitored occupation time options are also studied by Fusai and Tagliani (2001).
16. A similar situation with European digital options is documented by Gallus (1997).
17. It is assumed that if the barrier is crossed, the hedge is liquidated at the price equal to the barrier (no slippage).

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