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THE SPECTRAL REPRESENTATION OF BESSEL PROCESSES WITH CONSTANT DRIFT: APPLICATIONS IN QUEUEING AND FINANCE

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Abstract

Bessel processes with constant negative drift have recently appeared as heavy-traffic limits in queueing theory. We derive a closed-form expression for the spectral representation of the transition density of the Bessel process of order $\nu > -1$ with constant drift $\mu \neq 0$. When $\nu > -\frac{1}{2}$ and $\mu < 0$, the first term of the spectral expansion is the steady-state gamma density corresponding to the zero principal eigenvalue $\lambda_0 = 0$, followed by an infinite series of terms corresponding to the higher eigenvalues $\lambda_n,\ n = 1, 2, \dots,$ as well as an integral over the continuous spectrum above $\mu^2/2$. When $-1 < \nu < -\frac{1}{2}$ and $\mu < 0$, there is only one eigenvalue $\lambda_0 = 0$ in addition to the continuous spectrum. As well as applications in queueing, Bessel processes with constant negative drift naturally lead to two new nonaffine analytically tractable specifications for short-term interest rates, credit spreads, and stochastic volatility in finance. The two processes serve as alternatives to the CIR process for modelling mean-reverting positive economic variables and have nonlinear infinitesimal drift and variance. On a historical note, the Sturm–Liouville equation associated with Bessel processes with constant negative drift is closely related to the celebrated Schrödinger equation with Coulomb potential used to describe the hydrogen atom in quantum mechanics. Another connection is with D. G. Kendall's pole-seeking Brownian motion.

Keywords: Bessel process; pole-seeking Brownian motion; Coulomb potential; spectral expansion; heavy traffic limit; CIR model; $\frac{3}{2}$ model; interest-rate model

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1. Introduction

In a recent paper, Coffman *et al.* (1998) showed that a Bessel diffusion $\{Z_t,\ t \geq 0\}$ with constant negative drift $c < 0$ and with the infinitesimal generator

$$(\mathcal{G} f)(z) = \tfrac{1}{2}\sigma^2 f''(z) + \left(\frac{d}{z} + c\right) f'(z)$$

arises naturally as a heavy-traffic limit in queueing theory. Assuming that $2d\sigma^{-2} > -1$, this process possesses a stationary distribution with gamma density (when $-1 < 2d\sigma^{-2} < 1$, the process can hit the origin, in which case it instantaneously reflects; when $2d\sigma^{-2} \geq 1$, the origin is an unattainable entrance boundary). The purpose of the present paper is (i) to give an explicit analytical expression for the spectral representation of the transition density of the Bessel process with constant drift and (ii) to introduce several new applications in finance.

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It is convenient to re-scale the process as $X_t = \sigma^{-1} Z_t$ and introduce standardized parameters $\mu := c/\sigma$ and $\nu := d\sigma^{-2} - \frac{1}{2}$. The re-scaled process X has the infinitesimal generator

$$(\mathcal{G} f)(x) = \tfrac{1}{2} f''(x) + \left(\frac{\nu + \frac{1}{2}}{x} + \mu \right) f'(x) = \frac{1}{\mathfrak{m}(x)} \left(\frac{f'(x)}{\mathfrak{s}(x)} \right)', \tag{1}$$

where the *scale* and *speed densities* are

$$\mathfrak{s}(x) = x^{-2\nu-1} \mathrm{e}^{-2\mu x} \quad \text{and} \quad \mathfrak{m}(x) = 2x^{2\nu+1} \mathrm{e}^{2\mu x}$$

respectively. According to Feller's classification of boundaries, infinity is a natural boundary and is non-attracting when $\mu < 0$ and attracting when $\mu > 0$. The boundary at the origin has the same classification as for the standard Bessel process of order ν with $\mu = 0$: the origin is entrance for $\nu \geq 0$, regular when $-1 < \nu < 0$ and exit for $\nu \leq -1$ (see e.g. Borodin and Salminen (1996, p. 114)). In this paper we assume that $\nu > -1$ and impose a reflecting boundary condition at the origin in the regular case $-1 < \nu < 0$. When $\nu = -\frac{1}{2}$, the process reduces to the well-studied reflected Brownian motion with drift (see e.g. Abate and Whitt (1987a), (1987b)) and in this paper we always assume that $\nu \neq -\frac{1}{2}$.

Let $\mathrm{C_b}([0, \infty))$ be the Banach space of continuous and bounded functions on $[0, \infty)$. Conditional expectation operators

$$(\mathcal{P}_t f)(x) := \mathrm{E}_x[f(X_t)]$$

form a Feller semigroup in $\mathrm{C_b}([0, \infty))$ with the infinitesimal generator (1) with domain

$$D(\mathcal{G}) = \{ f \in \mathrm{C_b}([0, \infty)) : \mathcal{G} f \in \mathrm{C_b}([0, \infty)), \text{boundary condition at } 0 \}$$

with the boundary condition at the origin:

$$\lim_{x \downarrow 0} \frac{f'(x)}{\mathfrak{s}(x)} = 0. \tag{2}$$

When $\mu < 0$, the process possesses a stationary distribution with gamma density

$$\pi(x) = \frac{2|\mu|}{\Gamma(2\nu + 2)} (2|\mu| x)^{2\nu+1} \mathrm{e}^{2\mu x}. \tag{3}$$

When $\mu > 0$, the process is transient.

In Section 2 we obtain an explicit analytical expression for the spectral representation of the transition density. In Section 3 we illustrate this with numerical examples. In Section 4 we show that Bessel processes with constant drift naturally lead to two new analytically tractable nonaffine specifications for short-term interest rates and other positive mean-reverting financial variables. The two processes serve as nonaffine alternatives to the CIR (Cox–Ingersoll–Ross) process and have nonlinear infinitesimal drift and variance. In Section 5 we state a useful absolute continuity relationship between the distributions of Bessel processes with different order and drift. This relationship is then used in Section 6 to obtain explicit expressions for the state-price densities of the two nonaffine interest-rate models.

2. The spectral representation of the transition density

It is classical (see McKean (1956) and Itô and McKean (1974, Chapter 4.11)) that the transition density $p(t; x, y)$ admits a spectral representation (termed the eigendifferential expansion by Itô and McKean) associated with the spectral decomposition of a self-adjoint operator in the Hilbert space $L^2([0, \infty), \mathfrak{m})$ associated with the infinitesimal generator $\mathcal{G}$. More precisely, let $L^2([0, \infty), \mathfrak{m})$ be the Hilbert space of square-integrable functions with the speed density $\mathfrak{m}$ and endowed with the inner product $(f, g) = \int_0^\infty f(x)g(x)\mathfrak{m}(x)\,\mathrm{d}x$. The Feller semigroup $\{\mathcal{P}_t,\ t \geq 0\}$ restricted to $\mathrm{C_b}([0, \infty)) \cap L^2([0, \infty), \mathfrak{m})$ extends uniquely to a strongly continuous semigroup of self-adjoint contractions in $L^2([0, \infty), \mathfrak{m})$ with the infinitesimal generator $\mathcal{G}$, an unbounded self-adjoint, nonpositive operator in $L^2([0, \infty), \mathfrak{m})$ (see McKean (1956) and also Langer and Schenk (1990)). We use the same notation for the infinitesimal generator in $\mathrm{C_b}([0, \infty))$ and in $L^2([0, \infty))$; it should not create any confusion. The domain of $\mathcal{G}$ in $L^2([0, \infty), \mathfrak{m})$ is

$$D(\mathcal{G}) = \left\{ f \in L^2([0, \infty), \mathfrak{m}) : f, f' \in \mathrm{AC}([0, \infty)), \mathcal{G}f \in L^2([0, \infty), \mathfrak{m}), \lim_{x \downarrow 0} \frac{f'(x)}{\mathfrak{s}(x)} = 0 \right\},$$

where $\mathrm{AC}([0, \infty))$ is the space of absolutely continuous functions. A number of general properties of the spectral representation of $p(t; x, y)$ associated with the spectral decomposition of $\mathcal{G}$ were proved by McKean (1956). Here we are interested in an explicit analytical expression for the spectral representation of the Bessel process with constant drift.

Proposition 1. (Spectral representation.) *Define*

$$\beta := -\mu(\nu + \tfrac{1}{2}). \tag{4}$$

For $\nu > -1$, $\nu \neq -\frac{1}{2}$, $\mu \in \mathbb{R}$, $\mu \neq 0$, and $t > 0$, the transition density has the following spectral representation:

$$p(t; x, y) = p_\mathrm{d}(t; x, y) + p_\mathrm{c}(t; x, y), \tag{5}$$

where $p_\mathrm{d}(t; x, y)$ and $p_\mathrm{c}(t; x, y)$ are, respectively, the discrete and continuous parts of the spectral representation. For $x > 0$,

$$p_\mathrm{c}(t; x, y) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{-(1/2)(\mu^2+\rho^2)t} \left(\frac{y}{x}\right)^{\nu+1/2} \mathrm{e}^{\mu(y-x)+\pi\beta/\rho} \times M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho x) M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho y) \left| \frac{\Gamma(\frac{1}{2} + \nu + \mathrm{i}\beta/\rho)}{\Gamma(1 + 2\nu)} \right|^2 \mathrm{d}\rho, \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function and $M_{\varkappa,\nu}(\cdot)$ is the Whittaker function related to the Kummer confluent hypergeometric function (see Slater (1960) and Buchholz (1969) and also (14) below). When $x = 0$,

$$p_\mathrm{c}(t; 0, y) = \frac{1}{2\pi} \int_0^\infty \mathrm{e}^{-(1/2)(\mu^2+\rho^2)t} y^{\nu+1/2} \mathrm{e}^{\mu y+\pi\beta/\rho} \times M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho y) \mathrm{e}^{\mathrm{i}\pi(\nu/2+1/4)} \left| \frac{\Gamma(\frac{1}{2} + \nu + \mathrm{i}\beta/\rho)}{\Gamma(1 + 2\nu)} \right|^2 (2\rho)^{\nu+1/2}\, \mathrm{d}\rho. \tag{7}$$

The discrete part has the following form:

(i) *When* $\nu > -\frac{1}{2}$, $\mu < 0$, *and* $x > 0$,

$$p_{\mathrm{d}}(t; x, y) = \pi(y) + \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} \frac{n!\, \beta (2\beta y)^{2\nu+1} \mathrm{e}^{2\mu y}}{(n + \nu + \frac{1}{2})^{2\nu+3} \Gamma(2\nu + n + 1)} \times \exp\left(-\frac{\mu n}{n + \nu + \frac{1}{2}}(x + y)\right) L_n^{(2\nu)}\left(\frac{2\beta x}{n + \nu + \frac{1}{2}}\right) L_n^{(2\nu)}\left(\frac{2\beta y}{n + \nu + \frac{1}{2}}\right), \tag{8}$$

where $L_n^{(\alpha)}(\cdot)$ *is the generalized Laguerre polynomial,* $\pi(\cdot)$ *is the stationary density (3), and*

$$\lambda_n = \frac{\mu^2}{2}\left\{1 - \frac{(\nu + \frac{1}{2})^2}{(n + \nu + \frac{1}{2})^2}\right\}, \qquad n = 1, 2, \ldots. \tag{9}$$

When $\nu > -\frac{1}{2}$, $\mu < 0$, *and* $x = 0$,

$$p_{\mathrm{d}}(t; 0, y) = \pi(y) + \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} \frac{\beta (2\beta y)^{2\nu+1} \mathrm{e}^{2\mu y}}{(n + \nu + \frac{1}{2})^{2\nu+3} \Gamma(2\nu + 1)} \times \exp\left(-\frac{\mu n}{n + \nu + \frac{1}{2}} y\right) L_n^{(2\nu)}\left(\frac{2\beta y}{n + \nu + \frac{1}{2}}\right). \tag{10}$$

(ii) *When* $-1 < \nu < -\frac{1}{2}$, $\mu < 0$, *and* $x \geq 0$,

$$p_{\mathrm{d}}(t; x, y) = \pi(y). \tag{11}$$

(iii) *When* $\nu > -\frac{1}{2}$, $\mu > 0$, *and* $x \geq 0$,

$$p_{\mathrm{d}}(t; x, y) \equiv 0. \tag{12}$$

(iv) *When* $-1 < \nu < -\frac{1}{2}$, $\mu > 0$, *and* $x > 0$,

$$p_{\mathrm{d}}(t; x, y) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} \frac{n!\, \beta (2\beta y)^{2\nu+1} \mathrm{e}^{2\mu y}}{(n + \nu + \frac{1}{2})^{2\nu+3} \Gamma(2\nu + n + 1)} \times \exp\left(-\frac{\mu n}{n + \nu + \frac{1}{2}}(x + y)\right) L_n^{(2\nu)}\left(\frac{2\beta x}{n + \nu + \frac{1}{2}}\right) L_n^{(2\nu)}\left(\frac{2\beta y}{n + \nu + \frac{1}{2}}\right), \tag{13}$$

where the λ_n *are as given by (9). When* $-1 < \nu < -\frac{1}{2}$, $\mu > 0$, *and* $x = 0$,

$$p_{\mathrm{d}}(t; 0, y) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} \frac{\beta (2\beta y)^{2\nu+1} \mathrm{e}^{2\mu y}}{(n + \nu + \frac{1}{2})^{2\nu+3} \Gamma(2\nu + 1)} \times \exp\left(-\frac{\mu n}{n + \nu + \frac{1}{2}} y\right) L_n^{(2\nu)}\left(\frac{2\beta y}{n + \nu + \frac{1}{2}}\right).$$

Remark 1. (*Symmetry of the integrand.*) In (6), the Whittaker function $M_{\varkappa,\nu}(\cdot)$ appears twice with an imaginary argument. This function can be defined as

$$M_{\varkappa,\nu}(z) = z^{\nu+1/2}\mathrm{e}^{-z/2}\,{}_1F_1(\nu - \varkappa + \tfrac{1}{2}; 1 + 2\nu; z), \tag{14}$$

where

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

is the Kummer confluent hypergeometric function and $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 \equiv 1$ is the Pochhammer symbol. Due to the presence of the factor $z^{\nu+1/2}$ in the definition, $M_{\varkappa,\nu}(\cdot)$ is a multivalued function of a complex variable. We take as its principal branch that which lies in the complex plane cut along the negative real axis. In this case, $-\pi < \arg(z) \le \pi$ (see Slater (1960) and Buchholz (1969) for more details). The expression $M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho x)M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho y)$ in the integrand in (6) is both real and symmetric in x and y:

$$\begin{aligned} M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho x)M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho y) &= M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho x)M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho y) \\ &= M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho x)M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho y)\mathrm{e}^{\mathrm{i}\pi(\nu+1/2)} \\ &= M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho x)M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho y)\mathrm{e}^{-\mathrm{i}\pi(\nu+1/2)}. \end{aligned} \tag{15}$$

This stems from the symmetry property of the Whittaker function (Slater (1960, p. 11)):

$$M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho x) = \mathrm{e}^{-\mathrm{i}\pi(\nu+1/2)}M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho x).$$

Similarly, the expression $M_{\mathrm{i}\beta/\rho,\nu}(-2\mathrm{i}\rho y)\mathrm{e}^{\mathrm{i}\pi(\nu/2+1/4)}$ appearing in (7) is real.

For $s > 0$, let

$$G_s(x, y) = \int_0^\infty \mathrm{e}^{-st}\,p(t; x, y)\,\mathrm{d}t \tag{16}$$

be the *resolvent kernel* or *Green's function*. To prove Proposition 1, we start with the following lemma.

Lemma 1. (Resolvent kernel.) *Suppose that* $\nu > -1$, $\nu \neq -\frac{1}{2}$, *and* $\mu \in \mathbb{R}$, $\mu \neq 0$. *For* $s > 0$, *define*

$$\varkappa(s) := \frac{\beta}{\sqrt{2s + \mu^2}} \tag{17}$$

(where β *is defined in (4)). Then*

$$\begin{aligned} G_s(x, y) = {}& \frac{\Gamma(\frac{1}{2} + \nu - \varkappa(s))}{\sqrt{2s+\mu^2}\,\Gamma(1+2\nu)}\left(\frac{y}{x}\right)^{\nu+1/2}\mathrm{e}^{\mu(y-x)} \\ & \times M_{\varkappa(s),\nu}(2(x \wedge y)\sqrt{2s+\mu^2}\,)W_{\varkappa(s),\nu}(2(x \vee y)\sqrt{2s+\mu^2}\,), \end{aligned} \tag{18}$$

where $x \wedge y \equiv \min\{x, y\}$, $x \vee y \equiv \max\{x, y\}$, *and* $M_{\varkappa,\nu}(\cdot)$ *and* $W_{\varkappa,\nu}(\cdot)$ *are the Whittaker functions related to the Kummer and Tricomi confluent hypergeometric functions respectively (see Slater (1960) and Buchholz (1969)). When* $x = 0$, *this reduces to*

$$G_s(0, y) = \frac{\Gamma(\frac{1}{2} + \nu - \varkappa(s))}{\sqrt{2s+\mu^2}\,\Gamma(1+2\nu)}y^{\nu+1/2}\mathrm{e}^{\mu y}W_{\varkappa(s),\nu}(2y\sqrt{2s+\mu^2}\,). \tag{19}$$

Proof. It is classical (see Itô and McKean (1974)) that the resolvent kernel can be given in the form

$$G_s(x, y) = w_s^{-1}\mathfrak{m}(y)\psi_s(x \wedge y)\phi_s(x \vee y), \tag{20}$$

where the functions $\psi_s(\cdot)$ and $\phi_s(\cdot)$ can be characterized as the unique (up to a multiple independent of x) solutions of the ODE

$$(\mathcal{G}u)(x) \equiv \tfrac{1}{2}u''(x) + \left(\frac{\nu + \frac{1}{2}}{x} + \mu\right)u(x) = su(x) \tag{21}$$

(see e.g. Borodin and Salminen (1996, p. 18)) by first demanding that ψ_s is increasing and ϕ_s is decreasing and, secondly, demanding that ψ_s satisfies the boundary condition (2) at the origin in the regular case $-1 < \nu < 0$ (in the singular case $\nu \geq 0$, this boundary condition is satisfied automatically). The Wronskian w_s in (20) is defined by

$$\phi_s(x)\psi_s'(x) - \psi_s(x)\phi_s'(x) = \mathfrak{s}(x)w_s. \tag{22}$$

Substituting the functional form

$$u(x) = x^{-\nu-1/2}\mathrm{e}^{-\mu x}w(2x\sqrt{2s + \mu^2})$$

into (21), we arrive at Whittaker's form for the confluent hypergeometric equation for $w = w(z)$ (see e.g. Slater (1960) and Buchholz (1969); $\varkappa$ is defined in (17)):

$$w''(z) + \left(-\frac{1}{4} + \frac{\frac{1}{4} - \nu^2}{z^2} + \frac{\varkappa(s)}{z}\right)w(z) = 0$$

with the Whittaker functions $M_{\varkappa,\nu}(\cdot)$ and $W_{\varkappa,\nu}(\cdot)$ as solutions. Using the properties of the Whittaker functions, we can verify that

$$\psi_s(x) = x^{-\nu-1/2}\mathrm{e}^{-\mu x}M_{\varkappa(s),\nu}(2x\sqrt{2s + \mu^2}), \tag{23}$$

$$\phi_s(x) = x^{-\nu-1/2}\mathrm{e}^{-\mu x}W_{\varkappa(s),\nu}(2x\sqrt{2s + \mu^2}) \tag{24}$$

are the desired solutions, and their Wronskian (22) is

$$w_s = \frac{2\sqrt{2s + \mu^2}\,\Gamma(1 + 2\nu)}{\Gamma(\frac{1}{2} + \nu - \varkappa(s))} \tag{25}$$

(Slater (1960, p. 26, Equation (2.4.27))). Substituting (23)–(25) into (20) we arrive at the result (18). Taking the limit as $x \to 0$ and using the asymptotics

$$M_{\varkappa,\nu}(z) \sim z^{\nu+1/2} \quad \text{as } z \to 0, \tag{26}$$

we arrive at (19).

Remark 2. (*Connection with the Schrödinger equation with Coulomb potential.*) For $s \in \mathbb{C}$ the second-order ODE (21) has the form of the Sturm–Liouville equation

$$\mathcal{A}u = \lambda u \tag{27}$$

with the Sturm–Liouville operator

$$\mathcal{A} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(\frac{\nu + \frac{1}{2}}{x} + \mu\right)\frac{\mathrm{d}}{\mathrm{d}x} \tag{28}$$

and the spectral parameter $\lambda = -s$. The Liouville transformation

$$y = \sqrt{2}x,$$
$$v(y) = 2^{1/4}\left(\frac{y}{\sqrt{2}}\right)^{\nu+1/2} \mathrm{e}^{\mu y/\sqrt{2}} u\left(\frac{y}{\sqrt{2}}\right)$$

reduces (27) and (28) to the Liouville normal form

$$-v''(y) + \left(\frac{\mu^2}{2} - \frac{\sqrt{2}\beta}{y} + \frac{\nu^2 - \frac{1}{4}}{y^2}\right)v(y) = \lambda v(y)$$

(see e.g. Linetsky (2002b) for a discussion of the Liouville transform). Setting $E = \lambda - \mu^2/2$, $l = \nu - \frac{1}{2}$, and $a = \sqrt{2}\beta$, this normal form becomes

$$v''(y) + \left(E + \frac{a}{y} - \frac{l(l+1)}{y^2}\right)v(y) = 0. \tag{29}$$

When l is zero or a positive integer and $a > 0$, this is the celebrated *Schrödinger equation with Coulomb potential* in the theory of the hydrogen atom in quantum physics (see e.g. Titchmarsh (1962, p. 99) or Morse and Feshbach (1953, p. 1663)). For half-integers $\nu = l + \frac{1}{2}$, the resolvent kernel of the Bessel process with constant negative drift is thus very closely related to Green's function of the hydrogen atom. In this case, the eigenvalues (9) are closely related to the discrete energy levels of the hydrogen atom.

Remark 3. (*The spectrum.*) In this paper, when we talk about the spectrum we refer to the spectrum of the nonnegative Sturm–Liouville operator $\mathcal{A} = -\mathcal{G}$, the negative of the infinitesimal generator $\mathcal{G}$ in $L^2([0, \infty), \mathfrak{m})$. We prefer to talk about the nonnegative spectrum of $\mathcal{A}$ rather than the nonpositive spectrum of $\mathcal{G}$.

Remark 4. (*Connection with D. G. Kendall's pole-seeking Brownian motion.*) When $\nu = d/2 - 1$ for some integer $d \geq 2$, a Bessel process with constant drift arises as the radial part of the d-dimensional diffusion with the infinitesimal generator

$$\tfrac{1}{2}\Delta + \mu\frac{x}{|x|} \cdot \nabla,$$

where Δ is the standard d-dimensional Laplacian. When $d = 2$ and $\mu < 0$, this process was studied by Kendall (1974) in connection with his study of bird navigation, where it was called *pole-seeking Brownian motion* (see Pitman and Yor (1981, pp. 362–364) and Yor (1984) for further developments).

Remark 5. (*Bessel process terminology.*) Yor (1984, p. 104) called the process with the infinitesimal generator

$$\mathcal{G} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\nu+\frac{1}{2}}{x} - \delta\right)\frac{\mathrm{d}}{\mathrm{d}x}$$

with $\delta > 0$ a Bessel process with 'naive' drift δ in order to avoid confusion with the diffusion obtained by taking the radial part of an $\mathbb{R}^d$-valued Brownian motion started at the origin and with some drift vector $\vec{\delta}$ and with the infinitesimal generator

$$\tfrac{1}{2}\Delta + \vec{\delta}\cdot\nabla.$$

This latter diffusion has the infinitesimal generator

$$\mathcal{G} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\nu+\frac{1}{2}}{x} + \frac{\delta I_{\nu+1}(\delta x)}{I_\nu(\delta x)}\right)\frac{\mathrm{d}}{\mathrm{d}x},$$

where $\nu = d/2 - 1$, $\delta = |\vec{\delta}|$, and $I_\nu(z)$ is the Bessel function of order ν, and is usually called a Bessel process with drift (see Watanabe (1975) and Pitman and Yor (1981, p. 310)). Here we call the former process a Bessel process with *constant* drift to emphasize that the additional drift term is a constant, and also to avoid confusion with the diffusion with infinitesimal generator

$$\mathcal{G} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\nu+\frac{1}{2}}{x} - \delta x\right)\frac{\mathrm{d}}{\mathrm{d}x},$$

which we call a Bessel process with *linear* drift.

Remark 6. (*Connection of Lemma 1 with Proposition 12.2 of Pitman and Yor (1981).*) The result of Lemma 1 is closely related to Proposition 12.2 of Pitman and Yor (1981, p. 363) (also Yor (1984, p. 106)) and can, in fact, be directly extracted from it by applying the absolute continuity relationship in Lemma 2 below. For completeness, here we give an elementary proof of the result in a form convenient for the purpose of proving Proposition 1. For related results on Bessel processes with drift, the closely related topic of first hitting times of square root boundaries for Bessel processes, and their applications to calculations of various functionals of interest in mathematical finance, see Yor (1984), Matsumoto and Yor (2000, pp. 148–149), Delbaen and Yor (2002), Kyprianou and Pistorius (2003) and Linetsky (2002a), (2003).

Proof of Proposition 1. Following Titchmarsh's (1962) complex-variable approach to spectral expansions, we analytically invert the Laplace transform (16) with the resolvent kernel given by Lemma 1. We only sketch the proof here as it is similar to the proof given by Titchmarsh (1962, pp. 98–100) for the spectral expansion associated with the Schrödinger equation with Coulomb potential (29) when l is zero or a positive integer.

Regarded as a function of the complex variable s, Green's function (18) has a branch cut from $s = -\mu^2/2$ to $s \to -\infty$ on the negative real axis. It is convenient to parameterize the branch cut as $s = -\frac{1}{2}(\rho^2 + \mu^2)$. The jump in G_s across the branch cut is

$$
\begin{aligned}
&G_{(1/2)(\mu^2+\rho^2)e^{i\pi}}(x,y) - G_{(1/2)(\mu^2+\rho^2)e^{-i\pi}}(x,y) \\
&\quad = -\frac{i}{\rho}\frac{|\Gamma(\frac{1}{2}+\nu+i\beta/\rho)|^2}{\Gamma(1+2\nu)}\left(\frac{y}{x}\right)^{\nu+1/2} e^{\mu(y-x)} \\
&\qquad \times \left\{ \frac{M_{i\beta/\rho,\nu}(-2i\rho(x\wedge y))W_{i\beta/\rho,\nu}(-2i\rho(x\vee y))}{\Gamma(\frac{1}{2}+\nu+i\beta/\rho)} \right. \\
&\qquad\qquad \left. + \frac{M_{-i\beta/\rho,\nu}(2i\rho(x\wedge y))W_{-i\beta/\rho,\nu}(2i\rho(x\vee y))}{\Gamma(\frac{1}{2}+\nu-i\beta/\rho)} \right\} \\
&\quad = -\frac{i}{\rho}\frac{|\Gamma(\frac{1}{2}+\nu+i\beta/\rho)|^2}{\Gamma(1+2\nu)}\left(\frac{y}{x}\right)^{\nu+1/2} e^{\mu(y-x)} \\
&\qquad \times M_{i\beta/\rho,\nu}(-2i\rho(x\wedge y)) \left\{ \frac{W_{i\beta/\rho,\nu}(-2i\rho(x\vee y))}{\Gamma(\frac{1}{2}+\nu+i\beta/\rho)} + \frac{e^{i\pi(\nu+1/2)}W_{-i\beta/\rho,\nu}(2i\rho(x\vee y))}{\Gamma(\frac{1}{2}+\nu-i\beta/\rho)} \right\} \\
&\quad = -\frac{i}{\rho}\left|\frac{\Gamma(\frac{1}{2}+\nu+i\beta/\rho)}{\Gamma(1+2\nu)}\right|^2 \left(\frac{y}{x}\right)^{\nu+1/2} e^{\mu(y-x)+\pi\beta/\rho} M_{i\beta/\rho,\nu}(-2i\rho x) M_{-i\beta/\rho,\nu}(2i\rho y).
\end{aligned}
\tag{30}
$$

In the second equality, we used the symmetry property (15). In the third equality we used the identity

$$
\frac{W_{-\varkappa,\nu}(-z)}{\Gamma(\frac{1}{2}+\nu-\varkappa)} + \frac{e^{i\pi(\nu+1/2)}W_{\varkappa,\nu}(z)}{\Gamma(\frac{1}{2}+\nu+\varkappa)} = \frac{e^{i\pi\varkappa}M_{\varkappa,\nu}(z)}{\Gamma(1+2\nu)}
$$

(see Slater (1960, p. 14, Equation (1.9.9))).

(i) When $\nu > -\frac{1}{2}$ and $\mu < 0$, Green's function (18) also has simple poles $\{s_n, n = 0, 1, \dots\}$ at

$$
\tfrac{1}{2} + \nu - \varkappa(s_n) = -n, \qquad n = 0, 1, \dots
$$

(poles of the gamma function $\Gamma(\frac{1}{2}+\nu-\varkappa(s))$), that is,

$$
s_0 = 0 \quad \text{and} \quad s_n = -\frac{\mu^2}{2}\left\{1 - \frac{(\nu+\frac{1}{2})^2}{(n+\nu+\frac{1}{2})^2}\right\} = -\lambda_n, \quad n = 1, 2, \dots . \tag{31}
$$

Green's function has the residues

$$
\begin{aligned}
\operatorname{Res}_{s=s_n} G_s(x,y) &= \frac{(-1)^n\beta}{n!\,(n+\nu+\frac{1}{2})^2\Gamma(1+2\nu)}\left(\frac{y}{x}\right)^{\nu+1/2} e^{\mu(y-x)} \\
&\quad \times M_{n+\nu+1/2,\nu}\left(\frac{2\beta(x\wedge y)}{n+\nu+\frac{1}{2}}\right) W_{n+\nu+1/2,\nu}\left(\frac{2\beta(x\vee y)}{n+\nu+\frac{1}{2}}\right) \\
&= \frac{n!\,\beta(2\beta y)^{2\nu+1}e^{2\mu y}}{(n+\nu+\frac{1}{2})^{2\nu+3}\Gamma(2\nu+n+1)} \exp\left(-\frac{\mu n}{n+\nu+\frac{1}{2}}(x+y)\right) \\
&\quad \times L_n^{(2\nu)}\left(\frac{2\beta x}{n+\nu+\frac{1}{2}}\right) L_n^{(2\nu)}\left(\frac{2\beta y}{n+\nu+\frac{1}{2}}\right).
\end{aligned}
\tag{32}
$$

The second equality follows from the reduction of the Whittaker functions to the generalized Laguerre polynomials when the difference between the two indexes is a positive half-integer. When $\varkappa = \nu + n + \frac{1}{2}$ for $n \in \mathbb{N}$, the functions $M_{\varkappa,\nu}(\cdot)$ and $W_{\varkappa,\nu}(\cdot)$ become linearly dependent and reduce to generalized Laguerre polynomials (see Abramowitz and Stegun (1972, pp. 505, 509–510)):

$$M_{\nu+n+1/2,\nu}(z) = \frac{n!\,\Gamma(1+2\nu)}{\Gamma(2\nu+n+1)} \mathrm{e}^{-z/2} z^{\nu+1/2} L_n^{(2\nu)}(z),$$
$$W_{\nu+n+1/2,\nu}(z) = (-1)^n n!\,\mathrm{e}^{-z/2} z^{\nu+1/2} L_n^{(2\nu)}(z).$$

For $n = 0$, $\lambda_0 = 0$, and $L_0^{(2\nu)}(z) = 1$ and recalling that $\beta = -\mu(\nu + \frac{1}{2})$, the residue (32) with $n = 0$ reduces to the stationary density (3).

From the complex inversion formula we have

$$\begin{aligned} p(t; x, y) &= p_{\mathrm{d}}(t; x, y) + p_{\mathrm{c}}(t; x, y) \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{s_n t} \mathrm{Res}_{s=s_n} G_s(x, y) \\ &\quad - \frac{1}{2\pi \mathrm{i}} \int_0^{\infty} \mathrm{e}^{-(1/2)(\mu^2+\rho^2)t} \{G_{(1/2)(\mu^2+\rho^2)\mathrm{e}^{\mathrm{i}\pi}}(x, y) - G_{(1/2)(\mu^2+\rho^2)\mathrm{e}^{-\mathrm{i}\pi}}(x, y)\} \rho \,\mathrm{d}\rho \end{aligned}$$

(see Titchmarsh (1962)). Substituting the results (30), (31), and (32), we arrive at the spectral representation (5), (6), and (8). The branch cut gives the continuous spectrum and the poles give the discrete spectrum.

Since the origin is an entrance boundary when $\nu \geq 0$ and the regular reflecting boundary when $-1 < \nu < 0$, the limit $\lim_{x\to 0} p(t; x, y) = p(t; 0, y)$ exists (the process can be started at the origin). From the asymptotics (26) we have the limit

$$\lim_{x\to 0} (x^{-\nu-1/2} \mathrm{e}^{-\mu x} M_{-\mathrm{i}\beta/\rho,\nu}(2\mathrm{i}\rho x)) = (2\rho)^{\nu+1/2} \mathrm{e}^{\mathrm{i}\pi(\nu/2+1/4)}$$

and, recalling that

$$L_n^{(2\nu)}(0) = \frac{\Gamma(n+2\nu+1)}{n!\,\Gamma(2\nu+1)},$$

the limit $p(t; 0, y)$ is given by (5), (7), and (10) for $\mu < 0$.

(ii) When $-1 < \nu < -\frac{1}{2}$ and $\mu < 0$, Green's function has only one simple pole, $s_0 = 0$, with the residue equal to the stationary density $\pi(\cdot)$, as in (11).

(iii) When $\nu > -\frac{1}{2}$ and $\mu > 0$, Green's function has no poles and, hence, there are no eigenvalues (as in (12)).

(iv) When $-1 < \nu < -\frac{1}{2}$ and $\mu > 0$, Green's function has simple negative poles $\{s_n = -\lambda_n, n = 1, 2, \dots\}$ with λ_n given by (9) and residues (32), proving (13).

Remark 7. (*Real-variable approach to Sturm–Liouville expansions.*) An alternative approach to the proof of Proposition 1 is based on the real-variable approach to Sturm–Liouville expansions (see McKean (1956) and Levitan and Sargsjan (1975)). This approach is entirely real. We first consider the Sturm–Liouville problem (27) on the finite interval $[0, u]$ with a boundary condition (e.g. reflection) at the regular boundary u and then take the limit $u \to \infty$. The problem on the finite interval has a purely discrete spectrum. As u increases to infinity, the eigenvalues above $\mu^2/2$ are distributed more and more densely in $[\mu^2/2, \infty)$ and in the limit merge into the continuous spectrum (see e.g. Levitan and Sargsjan (1975, pp. 266–273)). A calculation of this type can be found in Linetsky (2001).

3. Computation and applications in queueing

The transition density given by Proposition 1 is easy to compute using MATHEMATICA® and MAPLE® as these software packages include all the required special functions as built-in functions. MATHEMATICA 4.1 running on a Pentium® III PC was used for all calculations in this paper. For an example calculation, take $\nu = \frac{1}{2}$ and $\mu = -1$ ($\beta = 1$) and start the process at the origin, $x = 0$. We need to compute the expressions (7) and (10). The Whittaker function $M_{\varkappa,\nu}(\cdot)$ is expressed in terms of the confluent hypergeometric function ${}_1F_1(a; b; z)$ by (14). The function ${}_1F_1(a; b; z)$ is available in MATHEMATICA with the command `Hypergeometric1F1[a, b, z]`. The generalized Laguerre polynomials are available with the command `LaguerreL[n, a, z]`. Due to the presence of the factor $(n + \nu + \frac{1}{2})^{2\nu+3}$ in the denominator in (10), the terms in the series decrease as $n^{-2\nu-3}$ (in our example as n^{-4}) and the series converges rapidly. For example, the series needed to calculate the value $p_{\mathrm{d}}(1; 0, 1)$ requires 12 terms to converge to three decimals and 48 terms to converge to four decimals (with a computation time of a fraction of a second). To compute the integral (7), we truncate the integration region at some finite value R and use the built-in numerical integration routine in MATHEMATICA. The integrand in (7) includes an exponential factor $\mathrm{e}^{-(1/2)\rho^2 t}$ that ensures rapid convergence of the integral. For example, the integral needed to calculate the value $p(1; 0, 1)$ converges to five decimals on the interval $[0, 5]$ (with a computation time of approximately a second).

Figure 1 plots transition densities with $t = \frac{1}{2}, 1, 2, 5$, as well as the stationary density (3) (it was produced with the plot function in MATHEMATICA). As t increases, the density approaches the stationary density, with the $t = 5$ density being already very close to the stationary density. This is intuitive from the analytical structure of the spectral expansion. The stationary density is the first term in the spectral expansion corresponding to the principal eigenvalue $\lambda_0 = 0$. The terms in the series corresponding to the higher eigenvalues λ_n are suppressed with the factor $\mathrm{e}^{-\lambda_n t}$ (the λ_n are eigenvalues of the Sturm–Liouville operator,

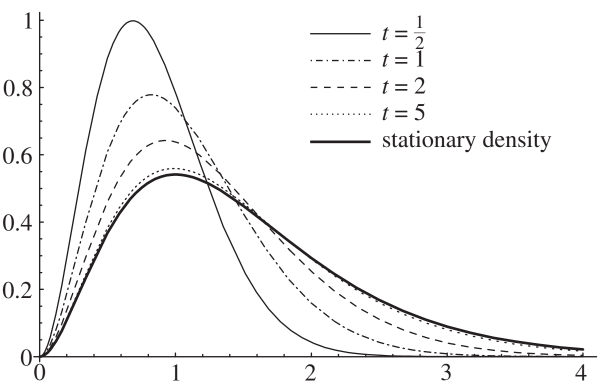

FIGURE 1: Transition densities $p(t; 0, y)$ with $t = \frac{1}{2}, 1, 2, 5$ and the stationary density $\pi(y)$. The Bessel process parameters are $\nu = \frac{1}{2}$, $\mu = -1$, and $x = 0$.

the negative of the infinitesimal generator $\mathcal{G}$). The integrand in (7) corresponding to the continuous spectrum is suppressed with the factor $\mathrm{e}^{-(1/2)(\mu^2+\rho^2)t}$. Intuitively, the spectral expansion provides a large-time expansion for the transition density, with the first term being the stationary density. Traditionally in heavy-traffic studies of queueing systems, researchers obtained explicit closed-form expressions for the stationary density and studied systems in steady state (see e.g. Coffman *et al.* (1998)). In this paper, we demonstrate that the spectral-expansion method produces the transition density in a form especially convenient for the study of *how the system approaches its steady state*. Figure 1 clearly demonstrates it for the Bessel process with constant drift $\mu < 0$.

To evaluate the convergence rate to the steady state, the decay parameter is defined by

$$\delta := \sup\{\alpha \geq 0 : p(t; x, y) - \pi(y) = O(\mathrm{e}^{-\alpha t}) \text{ for all } x \in S\},$$

where $\pi(\cdot)$ is the steady-state density and S is the state space (see Kijima (1997) for Markov chains and Kou and Kou (2002) for diffusions; the author thanks Steven Kou for pointing this out). For diffusions, if there are nonzero eigenvalues in addition to the principal eigenvalue $\lambda_0 = 0$, then $\delta = \lambda_1$. If the remaining spectrum is continuous, then δ is equal to the lowest point of the continuous spectrum. For Bessel processes with constant drift $\mu < 0$,

$$\delta = \begin{cases} \dfrac{\mu^2}{2}\left\{1 - \left(\dfrac{\nu + \frac{1}{2}}{\nu + \frac{3}{2}}\right)^2\right\}, & \nu > -\frac{1}{2}, \\[2ex] \dfrac{\mu^2}{2}, & -1 < \nu < -\frac{1}{2}. \end{cases}$$

In both cases, the drift parameter $\mu < 0$ regulates the convergence rate to the steady state: the larger the absolute value of the drift, the larger the decay parameter and the faster the system converges to its steady state.

4. Applications in finance: two analytically tractable nonaffine models

Feller's (1951) square-root diffusion solving the SDE

$$\mathrm{d}X_t = \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}B_t, \qquad X_0 = x > 0,$$

(where B is a standard Brownian motion) is one of the most widely used processes in finance, where it is known as a CIR (Cox–Ingersoll–Ross) process (see Cox *et al.* (1985), Pitman and Yor (1982), and Göing-Jaeschke and Yor (2003) for detailed studies and references). It is used to model financial variables that are restricted to staying positive and are empirically known to exhibit mean reversion, including the instantaneous default-free interest rate (Cox *et al.* (1985)), instantaneous credit spreads (see e.g. Duffie and Singleton (1999)) and stochastic volatility (see e.g. Heston (1993)). The model parameters θ, κ, and σ are interpreted as the long-run level, the rate of mean reversion, and the volatility parameter respectively. When the Feller condition is satisfied, $2\kappa\theta\sigma^{-2} \geq 1$, the process has an unattainable entrance boundary at the origin and possesses a stationary distribution with gamma density. The CIR process is analytically tractable, admitting explicit closed-form expressions for the transition density, as well as the state-price density, zero-coupon bonds and bond options in the CIR interest-rate model. The availability of the closed-form expression for the transition density greatly facilitates statistical estimation of the process parameters from empirical time-series data. The availability of the closed-form expression for the state-price density greatly facilitates pricing of interest-rate derivatives in the CIR interest-rate model. For the spectral representation of the CIR process and its applications in finance, see Davydov and Linetsky (2003), Gorovoi and Linetsky (2004), Linetsky (2002b), (2003) (the latter paper deals with spectral expansions for CIR and OU diffusions; related results on the OU diffusion can be found in Alili *et al.* (2003)).

A crucial feature of the CIR model is that both the infinitesimal drift $\kappa(\theta - x)$ and variance $\sigma^2 x$ are affine functions of the state variable. However, accumulated empirical evidence estimating interest-rate models with the infinitesimal variance of the form $\sigma^2 x^{2\gamma}$ suggests that empirically $\gamma \sim \frac{3}{2}$ rather than $\gamma \sim \frac{1}{2}$ (see e.g. Chan *et al.* (1992)), thus contradicting the CIR specification. Furthermore, recent empirical studies suggest that the short-rate drift is substantially nonlinear in the short rate as well (see e.g. Ahn and Gao (1999)). It is thus desirable to find some nonaffine specifications that are also analytically tractable. We shall show that the Bessel processes with constant drift studied in this paper lead to two such specifications.

Let $\{X_t,\ t \geq 0\}$ be a Bessel process with $\nu > 1$ and drift $\mu < 0$ solving the SDE

$$\mathrm{d}X_t = \left(\frac{\nu + \frac{1}{2}}{X_t} + \mu\right)\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x > 0$$

(recall that for $\nu \geq 0$ the origin is an unattainable entrance boundary). For some $\sigma > 0$, define a new process $\{r_t := 4(\sigma X_t)^{-2},\ t \geq 0\}$. From Itô's lemma, it is a diffusion on $(0, \infty)$ solving the SDE

$$\mathrm{d}r_t = \kappa(\theta - r_t^{1/2})r_t^{3/2}\,\mathrm{d}t + \sigma r_t^{3/2}\,\mathrm{d}\tilde{B}_t, \qquad r_0 = 4(\sigma x)^{-2} > 0, \qquad \tilde{B} = -B, \tag{33}$$

$$\kappa = \frac{\sigma^2}{2}(\nu - 1) > 0, \qquad \theta = -\frac{2\mu}{\sigma(\nu - 1)} > 0, \tag{34}$$

with the infinitesimal variance $\sigma^2 r^3$ and nonlinear drift $\kappa(\theta - r^{1/2})r^{3/2}$. An explicit expression for the transition density follows immediately from that for the Bessel process with constant drift.

An alternative specification is obtained as follows. Let $\{X_t,\ t \geq 0\}$ be a Bessel process with $\nu > \frac{1}{2}$ and drift $\mu < 0$. For some $\sigma > 0$, define a new process $\{r_t := (\sigma X_t)^{-1},\ t \geq 0\}$. From Itô's lemma, it is a diffusion on $(0, \infty)$ solving the SDE

$$\mathrm{d}r_t = \kappa(\theta - r_t)r_t^2\,\mathrm{d}t + \sigma r_t^2\,\mathrm{d}\tilde{B}_t, \qquad r_0 = (\sigma x)^{-1} > 0, \qquad \tilde{B} = -B, \tag{35}$$

$$\kappa = \sigma^2(\nu - \tfrac{1}{2}) > 0, \qquad \theta = -\frac{\mu}{\sigma(\nu - \frac{1}{2})} > 0, \tag{36}$$

with the infinitesimal variance $\sigma^2 r^4$ and nonlinear drift $\kappa(\theta - r)r^2$. An explicit expression for the transition density follows immediately from that for the Bessel process with constant drift. A model with interest-rate volatility σr^2 but a different drift specification has been recently studied by Lewis (1998).

The availability of closed-form expressions for the transition density (under the physical measure) greatly facilitates statistical estimation of the process parameters from empirical time-series data. Given the nonlinear nature of the infinitesimal drift and variance, we can expect these specifications to fit the empirical interest-rate data better than the CIR process. In addition to modelling interest rates, the processes of (33) and (34) and of (35) and (36) can be applied for the modelling of other positive financial variables with mean reversion, such as instantaneous credit spreads and stochastic volatility.

For the pricing of interest-rate derivatives, we need the state-price density. Let $\{r_t,\ t \geq 0\}$ be the instantaneous interest-rate (short-rate) process (under the risk-neutral probability measure).

Any European-style interest-rate derivative with some payoff function $f(r_t)$ at time $t > 0$ can be valued by integrating the payoff function against the state-price density:

$$\mathrm{E}\left[\exp\left(-\int_0^t r_u \mathrm{d}u\right) f(r_t) \,\middle|\, r_0 = x\right] = \int_0^\infty f(y)\pi(t; x, y)\,\mathrm{d}y. \tag{37}$$

The state-price density can be interpreted as the transition density of the diffusion r *killed at the linear rate*. We shall now show that the state-price density $\pi(t; x, y)$ for the short-rate models of (33) and (34) and of (35) and (36) can be easily obtained in closed form from the transition density of the Bessel process with constant drift.

Remark 8. ($\frac{3}{2}$ *models.*) The process (33) is an example of a so-called $\frac{3}{2}$ model. Another example that has recently been extensively considered in the literature can be constructed as follows. Let $\{X_t,\ t \geq 0\}$ be a Bessel process with $\nu > 1$ and with *linear* drift with $\mu < 0$ such that

$$\mathrm{d}X_t = \left(\frac{\nu + \frac{1}{2}}{X_t} + \mu X_t\right)\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x > 0.$$

For $\sigma > 0$, from Itô's lemma, the *reciprocal* squared process $\{r_t := (4/\sigma^2)X_t^{-2},\ t \geq 0\}$ is a diffusion on $(0, \infty)$ solving the SDE

$$\mathrm{d}r_t = \kappa(\theta - r_t)r_t\,\mathrm{d}t + \sigma r_t^{3/2}\,\mathrm{d}\tilde{B}_t, \qquad \tilde{B} = -B, \qquad r_0 = \frac{4}{\sigma^2}x^{-2},$$

$$\kappa = \frac{\sigma^2}{2}(\nu - 1) > 0, \qquad \theta = -\frac{4\mu}{\sigma^2(\nu - 1)} > 0.$$

This diffusion with nonlinear drift and infinitesimal variance $\sigma^2 x^3$ is the reciprocal of the square-root CIR process. This process was proposed by Cox *et al.* (1985, p. 402, Equation 50) as a model for the inflation rate in their three-factor inflation model. They were able to solve the three-factor valuation PDE for the real value of a nominal bond (see Equations (53) and (54) of Cox *et al.* (1985)). More recently this diffusion appeared in Lewis (1994), Heston (1997), Ahn and Gao (1999), and Lewis (2000) in different contexts. Heston (1997) and Lewis (2000) applied this process in the context of stochastic volatility models. Lewis (1994) and Ahn and Gao (1999) proposed this process as a model for the nominal short rate. Lewis (1994) (see also Ahn and Gao (1999)) obtained an analytical solution for the zero-coupon bond price by directly solving the PDE. This solution is, in fact, contained in a more general solution given in Equation (54) of Cox *et al.* (1985) for their three-factor inflation model. Linetsky (2002b) gives an alternative derivation based on the spectral expansion. Here, the process (33) has a different nonlinear drift structure.

5. A useful absolute-continuity relationship for Bessel processes with different indexes and drifts

We will need the following useful absolute continuity relationship for Bessel processes with different indexes and drifts. Let $\{R_t,\ t \geq 0\}$ be a Bessel process with index $\nu \geq 0$ and constant drift $\mu \in \mathbb{R}$ starting at $x \geq 0$. Consider on the space $\Omega = \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ the process of coordinates $\{R_t(\omega) = \omega(t),\ t \geq 0\}$ and its natural filtration $\{\mathcal{F}_t = \sigma\{R_s,\ s \leq t\},\ 0 \leq t \leq \infty\}$ and let $\mathrm{P}_x^{(\nu,\mu)}$ denote the associated probability distribution on $(\Omega, \mathcal{F}_\infty)$.

Lemma 2. *Fix $x > 0$, $\nu, \alpha, \beta \geq 0$, and $\mu \leq 0$. For any $\{\mathcal{F}_{t+}\}$ stopping time T,*

$$\frac{\mathrm{d}\,\mathrm{P}_x^{(\zeta,\eta)}}{\mathrm{d}\,\mathrm{P}_x^{(\nu,\mu)}} = \left(\frac{x}{R_T}\right)^{\nu-\zeta} \exp\left\{\frac{1}{2}(\mu^2 - \eta^2)T + (\mu - \eta)(x - R_T) - \beta \int_0^T \frac{\mathrm{d}u}{R_u} - \frac{\alpha^2}{2}\int_0^T \frac{\mathrm{d}u}{R_u^2}\right\}$$

on $\mathcal{F}_{T+} \cap \{T < \infty\}$, where

$$\zeta = \sqrt{\nu^2 + \alpha^2}, \qquad \eta = \frac{\mu(\nu + \frac{1}{2}) + \beta}{\zeta + \frac{1}{2}}. \tag{38}$$

Proof. The result follows immediately from combining the absolute-continuity relationship for standard Bessel processes without drift (Proposition 2.1 and Equation (2.f) of Pitman and Yor (1981, p. 293); see also Yor (1984, p. 101)):

$$\frac{\mathrm{d}\,\mathrm{P}_x^{(\zeta,0)}}{\mathrm{d}\,\mathrm{P}_x^{(\nu,0)}} = \left(\frac{x}{R_T}\right)^{\nu-\zeta} \exp\left\{-\frac{\alpha^2}{2}\int_0^T \frac{\mathrm{d}u}{R_u^2}\right\} \quad \text{on } \mathcal{F}_{T+} \cap \{T < \infty\}, \text{ where } \zeta = \sqrt{\nu^2 + \alpha^2};$$

and the absolute-continuity relationship for Bessel processes with and without constant drift (Pitman and Yor (1981, p. 362)):

$$\frac{\mathrm{d}\,\mathrm{P}_x^{(\nu,\mu)}}{\mathrm{d}\,\mathrm{P}_x^{(\nu,0)}} = \exp\left\{-\frac{\mu^2}{2}T + \mu\left(R_T - x - (\nu + \tfrac{1}{2})\int_0^T \frac{\mathrm{d}u}{R_u}\right)\right\} \quad \text{on } \mathcal{F}_{T+} \cap \{T < \infty\}.$$

In particular, for any fixed $t > 0$,

$$\begin{aligned} &\mathrm{E}_x^{(\nu,\mu)}\left[\exp\left\{-\beta\int_0^t \frac{\mathrm{d}u}{R_u} - \frac{\alpha^2}{2}\int_0^t \frac{\mathrm{d}u}{R_u^2}\right\} f(R_t)\right] \\ &\quad = \mathrm{e}^{(1/2)(\eta^2-\mu^2)t}\,\mathrm{E}_x^{(\zeta,\eta)}\left[\left(\frac{x}{R_t}\right)^{\zeta-\nu} \mathrm{e}^{(\eta-\mu)(x-R_t)} f(R_t)\right], \end{aligned} \tag{39}$$

where $\mathrm{E}_x^{(\nu,\mu)}$ is the expectation with respect to $\mathrm{P}_x^{(\nu,\mu)}$ and ζ and η are given by (38).

Remark 9. For $f \in C^2$, (39) can be seen as follows. Define the function

$$u(t, x) := \mathrm{E}_x^{(\nu,\mu)}\left[\exp\left\{-\beta\int_0^t \frac{\mathrm{d}u}{R_u} - \frac{\alpha^2}{2}\int_0^t \frac{\mathrm{d}u}{R_u^2}\right\} f(R_t)\right]. \tag{40}$$

By the Feynman–Kac theorem, it solves the PDE

$$\tfrac{1}{2}u_{xx} + \left(\frac{\nu + \frac{1}{2}}{x} + \mu\right)u_x - \left(\frac{\beta}{x} + \frac{\alpha^2}{2x^2}\right)u = u_t \tag{41}$$

with $u(0, x) = f(x)$. Define ζ and η as in (38). The substitution

$$u(t, x) = \mathrm{e}^{(1/2)(\eta^2-\mu^2)t} x^{\zeta-\nu} \mathrm{e}^{(\eta-\mu)x} v(t, x) \tag{42}$$

reduces (41) to

$$\tfrac{1}{2}v_{xx} + \left(\frac{\zeta + \frac{1}{2}}{x} + \eta\right)v_x = v_t$$

with $v(0, x) = x^{-(\zeta-\nu)}\mathrm{e}^{-(\eta-\mu)x} f(x)$. By the Feynman–Kac theorem, the solution is

$$v(t, x) = \mathrm{E}_x^{(\zeta,\eta)}[R_t^{-(\zeta-\nu)}\mathrm{e}^{-(\eta-\mu)R_t} f(R_t)]. \tag{43}$$

Substituting (40) and (43) into (42) we arrive at (39).

6. The state-price densities of the nonaffine term structure models

Let $\{r_t,\ t \geq 0\}$ be the short-rate process (33) with some $\sigma > 0$, $\theta > 0$, and $\kappa > 0$ and starting at $r_0 = x$. Then, on application of Itô's lemma, the process $\{R_t := 2\sigma^{-1} r_t^{-1/2}\}$ is a Bessel diffusion with index ν and constant drift μ,

$$\nu = \frac{2\kappa}{\sigma^2} + 1 > 1, \qquad \mu = -\frac{\kappa\theta}{\sigma} < 0,$$

and starting at $R_0 = (2/\sigma)x^{-1/2}$. From (39), for the expectation (37) we have

$$\begin{aligned}
&\mathrm{E}\left[\exp\left(-\int_0^t r_u\,\mathrm{d}u f(r_t)\right)\,\middle|\, r_0 = x\right]\\
&\quad = \mathrm{E}^{(\nu,\mu)}_{(2/\sigma)x^{-1/2}}\left[\exp\left\{-\frac{4}{\sigma^2}\int_0^t \frac{\mathrm{d}u}{R_u^2}\right\} f\left(\frac{4}{\sigma^2 R_t^2}\right)\right]\\
&\quad = \mathrm{e}^{(1/2)(\eta^2-\mu^2)t}\\
&\qquad \times \mathrm{E}^{(\zeta,\eta)}_{(2/\sigma)x^{-1/2}}\left[\left(\frac{2\sigma^{-1}x^{-1/2}}{R_t}\right)^{\zeta-\nu} \exp\left((\eta-\mu)\left(\frac{2}{\sigma}x^{-1/2} - R_t\right)\right) f\left(\frac{4}{\sigma^2 R_t^2}\right)\right],
\end{aligned}$$

where

$$\zeta = \sqrt{\nu^2 + 8\sigma^{-2}} > 1, \qquad \eta = \mu\left(\frac{\nu+\frac{1}{2}}{\zeta+\frac{1}{2}}\right) < 0,$$

and for the state-price density of the interest-rate model (33) we have

$$\begin{aligned}
\pi(t; x, y) = {}& \mathrm{e}^{(1/2)(\eta^2-\mu^2)t}\left(\frac{y}{x}\right)^{(1/2)(\zeta-\nu)} \exp\left(\frac{2}{\sigma}(\eta-\mu)(x^{-1/2} - y^{-1/2})\right)\\
&\times p^{(\zeta,\eta)}(t; 2\sigma^{-1}x^{-1/2}, 2\sigma^{-1}x^{-1/2})\sigma^{-1}y^{-3/2},
\end{aligned} \tag{44}$$

where $p^{(\nu,\mu)}(t; x, y)$ is the transition density of the Bessel process with index ν and constant drift μ given explicitly in Proposition 1.

Next, let $\{r_t,\ t \geq 0\}$ be the short-rate process (35) with some $\sigma > 0$, $\theta > 0$, and $\kappa > 0$ and starting at $r_0 = x$. Then, on application of Itô's lemma, the process $\{R_t := \sigma^{-1} r_t^{-1}\}$ is a Bessel diffusion with index ν and drift μ,

$$\nu = \frac{\kappa}{\sigma^2} + \frac{1}{2} > \frac{1}{2}, \qquad \mu = -\frac{\kappa\theta}{\sigma} < 0,$$

and starting at $R_0 = 1/\sigma x$. From (39), for the expectation (37) we have

$$\begin{aligned}
\mathrm{E}\left[\exp\left(-\int_0^t r_u\mathrm{d}u\right) f(r_t)\,\middle|\, r_0 = x\right] &= \mathrm{E}^{(\nu,\mu)}_{1/\sigma x}\left[\exp\left\{-\frac{1}{\sigma}\int_0^t \frac{\mathrm{d}u}{R_u}\right\} f\left(\frac{1}{\sigma R_t}\right)\right]\\
&= \mathrm{e}^{(1/2)(\eta^2-\mu^2)t}\,\mathrm{E}^{(\nu,\eta)}_{1/\sigma x}\left[\mathrm{e}^{(\eta-\mu)(1/\sigma x - R_t)} f\left(\frac{1}{\sigma R_t}\right)\right],
\end{aligned}$$

where

$$\eta = \mu + \frac{1}{\sigma(\nu+\frac{1}{2})},$$

and for the state-price density of the interest-rate model (35) we have

$$\pi(t; x, y) = \exp\{\tfrac{1}{2}(\eta^2 - \mu^2)t + \sigma^{-1}(\eta - \mu)(x^{-1} - y^{-1})\} p^{(\nu,\eta)}(t; \sigma^{-1}x^{-1}, \sigma^{-1}y^{-1})\sigma^{-1}y^{-2}. \tag{45}$$

Having explicit analytical expressions for the state-price density (44) and (45) allows us to price interest-rate derivatives in the short-rate models (33) and (35).

7. Conclusion

The main results of the present paper are (i) an explicit analytical expression for the spectral representation of the transition density of the Bessel process with constant drift recently appearing as a heavy-traffic limit in queueing theory and (ii) two new nonaffine positive and mean-reverting diffusion specifications suited for modelling interest rates, credit spreads, and stochastic volatility in finance. In heavy-traffic studies in queueing theory, researchers often are limited to the steady-state analysis because no explicit solutions for the transition density are available. Here we demonstrate that the spectral expansion method produces the transition density in the form especially convenient for studying how the system approaches its steady state, the stationary density being the first term in the spectral expansion. In finance, having an explicit expression for the transition density greatly facilitates statistical estimation of model parameters from historical time-series data, while having the state-price density (that includes discounting) allows us to price derivatives by integrating the payoff against the state-price density. We have obtained both the transition density and the state-price density for the two new nonaffine models by relating them to the transition density of the Bessel process with constant drift.

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