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Online Appendix

A. Toeplitz Matrices

A.1. Discrete Fourier Transform

The 1-d discrete Fourier transform (DFT) of a (complex) vector $(f_m)_{m=0}^{M-1}$ is defined by:

$$\hat{f}_n = \sum_{m=0}^{M-1} e^{-2\pi imn/M} f_m, \quad n = 0, 1, \ldots, M-1.$$

where $i = \sqrt{-1}$. Let w be the primitive M-th root of unity, $w = e^{-2\pi i/M}$, and introduce the DFT matrix $\mathbb{F}_M$:

$$\mathbb{F}_M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w^1 & \cdots & w^{M-1} \\ 1 & w^2 & \cdots & w^{2(M-1)} \\ \vdots & \vdots & & \vdots \\ 1 & w^{M-1} & \cdots & w^{(M-1)(M-1)} \end{pmatrix}.$$

Then the 1-d DFT can be written in matrix form: $\hat{f} = \mathbb{F}_M f$. The inverse of the DFT matrix is given by

$$\mathbb{F}_M^{-1} = \frac{1}{M}\mathbb{F}_M^*,$$

where $\mathbb{F}_M^*$ is the conjugate transpose (or adjoint) of $\mathbb{F}_M$. Consequently, the inverse 1-d discrete Fourier transform (IDFT) is given by

$$f_m = \frac{1}{M}\sum_{n=0}^{M-1} e^{2\pi imn/M}\hat{f}_n, \quad m = 0, 1, \ldots, M-1,$$

The fast Fourier transform (FFT) is an efficient algorithm to compute the DFT (and IDFT) in $O(M\log_2 M)$ complex multiplications compared to M^2 complex multiplications for standard matrix-vector multiplication (e.g., Van Loan 1992).

Similarly, the 2-d DFT of a (complex) array $\{f_{nm}, 0 \leqslant n \leqslant N-1, 0 \leqslant m \leqslant M-1\}$ is defined by

$$\hat{f}_{kl} = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} e^{-2\pi ink/N} e^{-2\pi iml/M} f_{nm}, \quad 0 \leqslant k \leqslant N-1, \ \ 0 \leqslant l \leqslant M-1,$$

If we treat f as an $N \times M$ matrix, then the 2-d DFT can be expressed in matrix form as $\hat{f} = \mathbb{F}_N f \mathbb{F}_M$. The inverse 2-d DFT (IDFT) is given by $f = \mathbb{F}_N^{-1}\hat{f}\mathbb{F}_M^{-1}$, or

$$f_{nm} = \frac{1}{MN}\sum_{k=0}^{N-1}\sum_{l=0}^{M-1} e^{2\pi ink/N} e^{2\pi iml/M}\hat{f}_{kl}, \quad 0 \leqslant n \leqslant N-1, \ \ 0 \leqslant m \leqslant M-1.$$

The 2-d DFT (or IDFT) can be computed in $O(MN\log_2(MN))$ complex multiplications by applying the FFT to each of the M columns of f (or $\hat{f}$), and then to each of the N rows of the resulting matrix. The Kronecker tensor product $A\otimes B$ of an $n\times m$ matrix $A=(a_{ij})$ and an arbitrary matrix B is given by

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B\\ a_{21}B & a_{22}B & \cdots & a_{2m}B\\ \vdots & \vdots & & \vdots\\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}.$$

The operator **vec** maps any $n\times m$ matrix $A=(a_{ij})$ to a vector by stacking the columns of the matrix:

$$\mathbf{vec}(A)=(a_{11},\ldots,a_{n1},a_{12},\ldots,a_{n2},\ldots,a_{1m},\ldots,a_{nm})^{\top}.$$

Basic properties of the Kronecker product and the operator **vec** include (e.g., Horn and Johnson 1994, Chapter 4): $(A\otimes B)^*=A^*\otimes B^*$, $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$ if A and B are nonsingular, and $\mathbf{vec}(AXB)=(B^{\top}\otimes A)\mathbf{vec}(X)$. Then it is easy to see that

$$\mathbf{vec}(\hat{f})=(\mathbb{F}_M\otimes\mathbb{F}_N)\mathbf{vec}(f),\qquad \mathbf{vec}(f)=(\mathbb{F}_M\otimes\mathbb{F}_N)^{-1}\mathbf{vec}(\hat{f}).\tag{EC1}$$

A.2. Circulant Matrices

An $M\times M$ matrix is called *circulant* if it has the following form (e.g., Davis 1994):

$$C=\begin{pmatrix} c_0 & c_{M-1} & \cdots & c_1\\ c_1 & c_0 & \cdots & c_2\\ \vdots & \vdots & & \vdots\\ c_{M-1} & c_{M-2} & \cdots & c_0 \end{pmatrix}.$$

It is completely specified by its first column $c=(c_0,\ldots,c_{M-1})^{\top}$ and each column is obtained by doing a wrap-around downshift of the previous column. A circulant matrix is diagonalized by the DFT matrix (Davis 1994, Theorem 3.2.2; Vogel 2002, Corollary 5.16):

$$C=\mathbb{F}_M^{-1}\triangle\mathbb{F}_M,$$

where $\triangle$ is a diagonal matrix with the diagonal containing the eigenvalues of C:

$$\triangle=\mathrm{diag}(\mathbb{F}_M c).$$

This factorization can be used to perform efficient matrix-vector multiplication. Let $x=(x_j)_{j=0}^{M-1}$ be an M-dimensional vector and C an $M\times M$ circulant matrix. Then

$$Cx=\mathbb{F}^{-1}\triangle\mathbb{F}x=\mathbb{F}^{-1}(\mathbb{F}c\circ\mathbb{F}x),$$

where $\circ$ denotes the Hadamard element-wise vector multiplication. This can be computed efficiently using the FFT, by first computing the two DFTs $\mathbb{F}c$ and $\mathbb{F}x$ and then computing the IDFT $\mathbb{F}^{-1}(\mathbb{F}c\circ\mathbb{F}x)$. If the matrix-vector multiplication is performed repeatedly with the same circulant matrix and different vectors, the DFT $\mathbb{F}c$ needs to be computed only once at the beginning.

A.3. Toeplitz Matrices

An $m\times m$ matrix T is called *Toeplitz* if it has constant values along each (top-left to lower-right) diagonal. That is, a Toeplitz matrix has the form:

$$T=\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-(m-1)}\\ t_1 & t_0 & \cdots & t_{-(m-2)}\\ \vdots & \vdots & \vdots & \vdots\\ t_{m-1} & t_{m-2} & \cdots & t_0 \end{pmatrix}.$$

It is completely specified by its first row and its first column. An $m \times m$ Toeplitz matrix T can be embedded into an $M \times M$ circulant matrix C with the first column $c = (t_0, \ldots, t_{m-1}, 0, \ldots, 0, t_{-(m-1)}, \ldots, t_{-1})^\top$. Here $M = 2^l$ is the smallest power of two such that $M \geqslant 2m-1$. Note that $M-(2m-1)$ zeros are padded into the vector c. Using this embedding, the Toeplitz matrix-vector multiplication Tx can be computed as follows:

$$(Tx)_k = (Cx^*)_k = (\mathbb{F}^{-1}(\mathbb{F}c \circ \mathbb{F}x^*))_k, \quad k = 0, 1, \ldots, m-1,$$

where the M-dimensional vector x^* is an extension of the original m-dimensional vector x by appending $M-m$ zeros to x. Now the problem is reduced to computing the circulant matrix-vector multiplication, which can be computed efficiently using the FFT as described previously (we chose M to be the power of two in order to use the FFT of radix 2). Applications in finance of the Toeplitz matrix-vector multiplication were pioneered by Eydeland (1994).

A.4. Block Circulant Matrices with Circulant Blocks

An $MN \times MN$ block circulant matrix with circulant blocks (or BCCB) has the following form:

$$C = \begin{pmatrix} C_0 & C_{M-1} & \cdots & C_1 \\ C_1 & C_0 & \cdots & C_2 \\ \vdots & \vdots & & \vdots \\ C_{M-1} & C_{M-2} & \cdots & C_0 \end{pmatrix}.$$

where each C_j, $j = 0, \ldots, M-1$, is an $N \times N$ circulant matrix. A BCCB is completely specified by its first column. Let c_j, $j = 0, \ldots, M-1$, be the first column of C_j, and $c = (c_0, \ldots, c_{M-1})$ an $N \times M$ matrix. Denote the 2-d DFT of c by $\hat{c}$:

$$\mathbf{vec}(\hat{c}) = (\mathbb{F}_M \otimes \mathbb{F}_N)\mathbf{vec}(c)$$

Then C has the following diagonalization (Davis 1994, Theorem 5.8.1; Vogel 2002, Proposition 5.31):

$$C = (\mathbb{F}_M \otimes \mathbb{F}_N)^{-1} \Delta (\mathbb{F}_M \otimes \mathbb{F}_N)$$

where $\Delta = \text{diag}(\mathbf{vec}(\hat{c}))$. Hence the multiplication of a BCCB C by an MN-dimensional vector x can be computed efficiently as follows:

$$Cx = (\mathbb{F}_M \otimes \mathbb{F}_N)^{-1} \Delta (\mathbb{F}_M \otimes \mathbb{F}_N) x = (\mathbb{F}_M \otimes \mathbb{F}_N)^{-1} (\mathbf{vec}(\hat{c}) \circ (\mathbb{F}_M \otimes \mathbb{F}_N) x).$$

Recalling Equation (EC1), the above expression can be computed using two 2-d DFTs and one 2-d IDFT in $O(MN \log_2(MN))$ floating point operations. If the matrix-vector multiplication Cx is performed repeatedly with the same matrix C and different vectors x, the $\mathbf{vec}(\hat{c})$ needs to be computed only once at the beginning.

A.5. Block Toeplitz Matrices with Toeplitz Blocks

An $mn \times mn$ block Toeplitz matrix with Toeplitz blocks (BTTB) has the form:

$$T = \begin{pmatrix} T_0 & T_{-1} & \cdots & T_{-(m-1)} \\ T_1 & T_0 & \cdots & T_{-(m-2)} \\ \vdots & \vdots & & \vdots \\ T_{m-1} & T_{m-2} & \cdots & T_0 \end{pmatrix},$$

where each T_j, $j = -(m-1), \ldots, m-1$, is an $n \times n$ Toeplitz matrix. Each Toeplitz block T_j can be embedded into a $N \times N$ circulant block C_j, where N is the smallest power of 2 such that $N \geqslant 2n-1$. Then T can be embedded into an $MN \times MN$ BCCB C in a way similar to the embedding of a Toeplitz matrix into a circulant matrix, except that single elements are replaced by matrix blocks. Here M is the smallest power of two such that $M \geqslant 2m-1$. Then the multiplication of the BTTB T by an mn-dimensional vector x can be reduced to the multiplication of the BCCB C by an MN-dimensional vector x^*. Here x^* is the extension of x. If x is seen as a vector of m blocks with each block an n-dimension vector, then x^* is obtained by appending $N-n$ zeros to each blocks and appending additional $M-m$ zero vectors of dimension N. M and N are selected to be powers of 2 so that the 2-d DFTs can be computed using the FFT of radix 2. The total number of floating point operators is $O(mn \log_2(mn))$, as compared to $O(m^2 n^2)$ when direct matrix vector multiplication is used.