Asymptotic Duality in Stochastic Integer Programming

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Outline

• Duality
• Integer Problems in Duality
• Stochastic Programs
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Duality

Primal Problem:

\[ P^* = \min_{x \in X} f(x) \]
\[ \text{s.t. } g(x) \leq 0 \]

Typically, \( f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m \)

Dual Problem:

\[ D^* = \max_{\lambda \geq 0} \min_{x \in X} \left[ f(x) + \lambda^T g(x) \right] \]

STRONG DUALITY RESULT: Under good conditions, \( P^* = D^* \).
Dual Uses

Easier to Solve: Often, $g(x) \leq 0$ causes complications. In objective, less trouble.

Bounds available (weak duality):

$$P^* \geq D^*.$$ 

Suppose $x^*, \lambda^* \geq 0$ optimal:

$$P^* = f(x^*) \geq f(x^*) + \lambda^T g(x^*)$$

$$\geq \min_{x \in X} f(x) + \lambda^T g(x)$$

$$= D^*$$

DUALITY GAP: $P^* > D^*$. Possible for $X = Z^+_n$. 

Duality Gap

Example:

Primal Problem:

\[ P^* = \min_{x_1, x_2 \in \mathbb{Z}^+} 1.5x_1 + x_2 \]
\[ s.t. \quad 5.5 - 6x_1 - 5x_2 \leq 0 \]

Solution: \( x^* = (1,0) \). \( P^* = 1.5 \).

Dual Problem:

\[ D^* = \max_{\lambda \geq 0} \min_{x_1, x_2 \in \mathbb{Z}^+} 1.5x_1 + x_2 + \lambda(5.5 - 6x_1 - 5x_2) \]
\[ = \max_{\lambda \geq 0} \min_{x_1, x_2 \in \mathbb{Z}^+} (1.5 - 6\lambda)x_1 + (1 - 5\lambda)x_2 + 5.5\lambda \]

Solution: \( \lambda^* = 0.2 \). \( D^* = 1.15 \).
Closing the Duality Gap in Integer Programs

Many variables:

Suppose: $x$ has many possibilities all with different weights:

Primal Problem: $P^* =$

$$
\min_{x_i \in \mathbb{Z}_n^+} \sum_i c_i x_i \\
\text{s.t.} \quad 5.5 - \sum_i w_i x_i \leq 0
$$

Result: With random $c_i$ and $w_i$ or $x_i$ binary, can close the gap as $n$ increases. (Turnpike Result.)

Our problem:

$x : \Omega \mapsto \mathbb{Z}_n^+$ where $\Omega$ is a sample space.

Now, $x$ is a random variable onto integers (stochastic program).

What can we say now?
Stochastic Integer Program

Two-Stage Primal Problem:

\[ P^* = \min_{x \in X} f(x) \]
\[ s.t. \quad x(\omega) - E[x(\omega)] = 0, \text{a.s.,} \]

where \( X \) represents the integer mapping, \( f(x) = E[g(x(\omega))] \) is an expectation with a probability measure defined on \( \Omega \). The constraint forces nonanticipativity.

Dual Problem:

\[ D^* = \max_{\lambda(\omega) \geq 0} \min_{x \in X} \left[ f(x) + E[\lambda \cdot [x(\omega) - E[x(\omega)]]] \right] \]

NOTE: The dual problem separates into separate problems for each \( \omega \).

STRONG DUALITY RESULT:

In the limit, \( |\Omega| \to \infty \), \( P^* = D^* \), i.e., no duality gap as sample size increases.
Key Insights

Make $x$ a binary mapping:

Note: $x(\omega) - E[x(\omega)] = 0$ for all $\omega$ is the same as $x(\omega') - E[x(\omega)] = 0$ for one $\omega' \in \Omega$.

If one $x(\omega)$ is 1, then all must be.

Implication: Just one constraint but $|\Omega|$ can increase.

Let $N = |\Omega| < \infty$. The problems are then:

$$P^* = \min P = \sum_{i=1}^{N} p^i F^i(x^i)$$

s. t. $x^i \in X^i$, $i = 1, \ldots, N$,

$$\sum_{i=1}^{N} H^i x^i \leq b.$$

$$D^* = \max D(\lambda) = \min \sum_{i=1}^{N} [p^i F^i(x^i) + \lambda^T H^i x^i] - \lambda^T b$$

s. t. $x^i \in X^i$, $i = 1, \ldots, N$,

$\lambda \geq 0$,
Key Insights (cont.)

Theorem. If $P$ has a solution, for every $i$, the set $\{(x^i, F^i(x^i))|x^i \in X^i\}$ is compact, and, for every $\hat{x} \in \text{conv}(X^i)$, there exists $x \in X^i$ such that $H^i x \leq H^i \hat{x}$, then

$$ \inf P - \sup D \leq (q + 1) \rho, $$

where $\rho = \max_{i=1,\ldots,N} \sup(p^i F^j(x^i)|x^i \in X^i) - \inf(p^i F^j(x^i)|x^i \in X^i)$ and $q$ is the number of constraints.

Main Result

**Theorem.** Suppose the conditions of previous theorem, that \(X^i = Y^i \times S^i\) where \(Y^i\) is convex and \(S^i\) is intersection of convex and integer,

\[
\inf P - \sup D \leq (2n + 1) \rho,
\]

where \(\rho = \max_{i=1,...,N} \sup(p^i F^i(x^i)|x^i \in X^i) - \inf(p^i F^i(x^i)|x^i \in X^i)\).

**PROOF OUTLINE:**

Following Bertsekas, consider:

\(H^i\) split into \(L^i\) and \(G^i\) so that \(H^i x^i \leq b\) is equivalent to \(\sum_{i=1}^{N} L^i y^i \leq l\) and \(\sum_{i=1}^{N} G^i s^i \leq g\) and \(F^i\) split so that \(F^i(x^i) = C^i(y^i) + D^i(s^i)\).

Then, let

\[
W^i = \{w^i|w^i = [L^i y^i, C^i(y^i)], y^i \in Y^i\}
\]

and

\[
Z^i = \{z^i|z^i = [G^i s^i, D^i(s^i)], s^i \in S^i\}.
\]

Consider \(W = \sum_{i=1}^{N} W^i\) and \(Z = \sum_{i=1}^{N} Z^i\), then we have:

\[
\inf P = \min\{u + v|\exists ((w, u), (z, v)) \in W \times Z,
\]

such that \(w \leq g, z \leq l\).
PROOF OUTLINE (cont.):

From duality theory, we have that
\[ \sup D = \min \{ u + v | \exists ((w, u), (z, v)) \in \text{conv}(W \times Z) \] such that \( w \leq g, z \leq l \}, \]
where \( \text{conv} \) denotes the convex hull.

Note \( \text{conv}(W \times Z) = W \times \text{conv}(Z) \), since \( Y^i \)
is convex.

Now use the Shapley-Folkman theorem to write every \( z \in \text{conv}(Z) \) using a subset \( I(z) \in \{1, \ldots, N\} \)
with at most \( 2n + 1 \) indices such that
\[ z \in [ \sum_{i \not\in I(z)} Z^i + \sum_{i \in I(z)} \text{conv}(Z^i) ]. \]

Now, suppose \( ((\bar{w}, \bar{u}), (\bar{z}, \bar{v})) \in W \times \text{conv}(Z) \)
with \( \bar{u} + \bar{v} = \sup D \) and \( \bar{w} \leq g, \bar{z} \leq l. \) Then we have \( \bar{y}^i \in Y^i \) such that \( \Sigma_{i=1}^N L_i \bar{y}^i \leq g \) and \( \Sigma_{i=1}^N C_i(\bar{y}^i) = \bar{u}. \)
PROOF OUTLINE (cont.):

From Shapley-Folkman, we also have some $\bar{I} \subset \{1, \ldots, N\}$ with $|\bar{I}| \leq 2n+1$ with $(\bar{l}^i, \bar{v}^i) \in \text{conv}(Z^i)$ and $\bar{s}^i \in S^i$, $i \not\in \bar{I}$, such that

$$\sum_{i \not\in \bar{I}} G^i(s^i) + \sum_{i \in \bar{I}} l^i = \bar{z} \leq l,$$

and

$$\sum_{i=1}^{N} C^i(\bar{y}^i) + \sum_{i \not\in \bar{I}} D^i \bar{s}^i + \sum_{i \in \bar{I}} v^i = \sup(D).$$

Now, we can obtain for every $i \in \bar{I}$, some $\bar{s}^i$ such that $G^i \bar{s}^i \leq l^i$ and $f_i(\bar{s}^i) \leq \bar{v}^i + \rho^i + \epsilon$ for any $\epsilon > 0$. Thus, we have found a feasible solution $(\bar{y}, \bar{s})$ for (P) such that

$$\inf P \leq \sum_{i=1}^{N} C^i(\bar{y}^i) + \sum_{i=1}^{N} D^i \bar{s}^i \leq \sup(D) + \sum_{i \in \bar{I}} \rho^i,$$

which, since $\bar{I} \leq 2n + 1$, yields the result. $$\blacksquare$$
Applications:

Power Systems: $x$ corresponds to turning generators on or off.

Uncertainty surrounds future demand. Many possible branches.

Results: Solved for Michigan system. Gaps to within 0.5% with $64 = N$ branches.
Conclusions:

• Integer Problems can cause Duality Gaps

• Stochastic Integer Programs can also have Duality Gaps

• As Sample Sizes increase, Gaps Decrease to Zero

• Convergence appears Rapid in Power System Applications.