Asymptotic Duality in Stochastic Integer Programming

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Outline

- Duality
- Integer Problems in Duality
- Stochastic Programs
- Result
- Application

Duality

Primal Problem:

P^*=

$$\begin{array}{lll} \min_{x \in X} & f(x) & \\ s.t. & g(x) & \leq \; 0 \end{array}$$

$$Typically, \; f : \Re^n \mapsto \Re, g : \Re^n \mapsto \Re^m$$

Dual Problem:

D^*=

$$\max_{\lambda \geq 0} \; \min_{x \in X} [f(x) + \lambda^T g(x)]$$

STRONG DUALITY RESULT: Under good conditions, $P^* = D^*$.

Dual Uses

Easier to Solve: Often, $g(x) \leq 0$ causes complications. In objective, less trouble.

Bounds available (weak duality):

$$P^* \geq D^*.$$

Suppose x^*,$\lambda^* \geq 0$ optimal:

$$\begin{aligned} P^* = f(x^*) \; & \geq f(x^*) + \lambda^{*T} g(x^*) \\ & \geq \min_{x \in X} f(x) + \lambda^{*T} g(x) \\ & = D^* \end{aligned}$$

DUALITY GAP: $P^* > D^*$. Possible for $X = Z_n^+$.

Duality Gap

Example:

Primal Problem:

P^*=

$$\begin{array}{lll} \min_{x_1,x_2 \in Z^+} & 1.5x_1 + x_2 & \\ s.t. & 5.5 - 6x_1 - 5x_2 & \leq \ 0 \end{array}$$

Solution: $x^* = (1, 0)$. P^*=1.5.

Dual Problem:

D^*=

$$\begin{array}{lll} \max_{\lambda \geq 0} & \min_{x_1,x_2 \in Z^+} & 1.5x_1 + x_2 + \lambda(5.5 - 6x_1 - 5x_2) \\ = \max_{\lambda \geq 0} & \min_{x_1,x_2 \in Z^+} & (1.5 - 6\lambda)x_1 + (1 - 5\lambda)x_2 + 5.5\lambda \end{array}$$

Solution: $\lambda^* = 0.2$. D^*=1.15.

Closing the Duality Gap in Integer Programs

Many variables:

Suppose: x has many possibilities all with different weights:

Primal Problem: $P^*=$

$$\begin{array}{ll} \min_{x_i \in Z_n^+} & \Sigma_i \, c_i x_i \\ s.t. & 5.5 - \Sigma_i \, w_i x_i \ \leq \ 0 \end{array}$$

Result: With random c_i and w_i or x_i binary, can close the gap as n increases. (Turnpike Result.)

Our problem:

$x : \Omega \mapsto Z_+^n$ where Ω is a sample space.

Now, x is a random variable onto integers (stochastic program).

What can we say now?

Stochastic Integer Program

Two-Stage Primal Problem:

P^*=

$$\begin{array}{lll} \min_{x \in X} & f(x) & \\ s.t. & x(\omega) - E[x(\omega)] & = & 0, a.s., \end{array}$$

where X represents the integer mapping, $f(x) = E[g(x(\omega))]$ is an expectation with a probability measure defined on Ω. The constraint forces nonanticipativity.

Dual Problem:

D^*=

$$\max_{\lambda(\omega) \geq 0} \ \min_{x \in X}[f(x) + E[\lambda \cdot [x(\omega) - E[x(\omega)]]]$$

NOTE: The dual problem separates into separate problems for each ω.

STRONG DUALITY RESULT:

In the limit, $|\Omega| \to \infty$, $P^* = D^*$, i.e., no duality gap as sample size increases.

Key Insights

Make x a binary mapping:

Note: $x(\omega) - E[x(\omega)] = 0$ for all ω is the same as $x(\omega') - E[x(\omega)] = 0$ for one $\omega' \in \Omega$.

If one $x(\omega)$ is 1, then all must be.

Implication: Just one constraint but $|\Omega|$ can increase.

Let $N = |\Omega| < \infty$. The problems are then:

$$
\begin{aligned}
P^* = \min P = \Sigma_{i=1}^{N} p^i F^i(x^i) & \\
\text{s. t. } x^i \in X^i, \qquad & i = 1, \ldots, N, \\
\Sigma_{i=1}^{N} H^i x^i \qquad & \leq b.
\end{aligned}
$$

$$
D^* = \max D(\lambda) = \begin{array}{lll} \min \Sigma_{i=1}^{N} [p^i F^i(x^i) & +\lambda^T H^i x^i] - \lambda^T b \\ \text{s. t. } x^i \in X^i, & i = 1, \ldots, N, \\ & \lambda \geq 0, \end{array}
$$

Key Insights (cont.)

Theorem. If P has a solution, for every i, the set $\{(x^i, F^i(x^i)) | x^i \in X^i\}$ is compact, and, for every $\hat{x} \in conv(X^i)$, there exists $x \in X^i$ such that $H^i x \leq H^i \hat{x}$, then

$$\inf P - \sup D \leq (q+1)\rho,$$

where $\rho = \max_{i=1,\ldots,N} \sup(p^i F^i(x^i) | x^i \in X^i) - \inf(p^i F^i(x^i) | x^i \in X^i)$ and q is the number of constraints.

Proof: Follows Bertsekas (1982).

Main Result

Theorem. Suppose the conditions of previous theorem, that $X^i = Y^i \times S^i$ where Y^i is convex and S^i is intersection of convex and integer,

$$\inf P - \sup D \leq (2n+1)\rho,$$

where $\rho = \max_{i=1,\ldots,N} \sup(p^i F^i(x^i) | x^i \in X^i) - \inf(p^i F^i(x^i) | x^i \in X^i)$.

PROOF OUTLINE:

Following Bertsekas, consider:

H^i split into L^i and G^i so that $H^i x^i \leq b$ is equivalent to $\Sigma_{i=1}^N L^i y^i \leq l$ and $\Sigma_{i=1}^N G^i s^i \leq g$ and F^i split so that $F^i(x^i) = C^i(y^i) + D^i(s^i)$.

Then, let

$W^i = \{w^i | w^i = [L^i y^i, C^i(y^i)], y^i \in Y^i\}$ and

$Z^i = \{z^i | z^i = [G^i s^i, D^i(s^i)], s^i \in S^i\}$.

Consider $W = \Sigma_i^N W^i$ and $Z = \Sigma_i^N Z^i$, then we have:

$$\inf P = \min\{u + v | \exists((w,u),(z,v)) \in W \times Z,$$

$$\text{such that } w \leq g, z \leq l\}.$$

PROOF OUTLINE (cont.):

From duality theory, we have that

$$\sup D = \min\{u+v | \exists((w,u),(z,v)) \in conv(W \times Z)$$
$$\text{such that } w \leq g, z \leq l\},$$

where *conv* denotes the convex hull.

Note $conv(W \times Z) = W \times conv(Z)$, since Y^i is convex.

Now use the Shapley-Folkman theorem to write every $z \in conv(Z)$ using a subset $I(z) \in \{1,\ldots,N\}$ with at most $2n+1$ indices such that

$$z \in [\sum_{i \notin I(z)} Z^i + \sum_{i \in I(z)} conv(Z^i)].$$

Now, suppose $((\bar{w},\bar{u}),(\bar{z},\bar{v})) \in W \times conv(Z)$ with $\bar{u}+\bar{v} = \sup D$ and $\bar{w} \leq g, \bar{z} \leq l$. Then we have $\bar{y}^i \in Y^i$ such that $\Sigma_{i=1}^N L^i \bar{y}^i \leq g$ and $\Sigma_{i=1}^N C^i(\bar{y}^i) = \bar{u}$.

PROOF OUTLINE (cont.):

From Shapley-Folkman, we also have some $\bar{I} \subset \{1, \ldots, N\}$ with $|\bar{I}| \leq 2n+1$ with $(\bar{l^i}, \bar{v}^i) \in conv(Z^i)$ and $\bar{s}^i \in S^i$, $i \notin \bar{I}$, such that

$$\sum_{i \notin \bar{I}} G^i(s^i) + \sum_{i \in \bar{I}} l^i = \bar{z} \leq l,$$

and

$$\sum_{i=1}^{N} C^i(\bar{y}^i) + \sum_{i \notin \bar{I}} D^i \bar{s}^i + \sum_{i \in \bar{I}} \bar{v}^i = sup(D).$$

Now, we can obtain for every $i \in \bar{I}$, some $\bar{s}^i$ such that $G^i \bar{s}^i \leq l^i$ and $f_i(\bar{s}^i) \leq \bar{v}^i + \rho^i + \epsilon$ for any $\epsilon > 0$. Thus, we have found a feasible solution $(\bar{y}, \bar{s})$ for (P) such that

$$\inf P \leq \sum_{i=1}^{N} C^i(\bar{y}^i) + \sum_{i=1}^{N} D^i \bar{s}^i \leq \sup(D) + \sum_{i \in \bar{I}} \rho^i,$$

which, since $\bar{I} \leq 2n+1$, yields the result. ∎

Applications:

Power Systems: x corresponds to turning generators on or off.

Uncertainty surrounds future demand. Many possible branches.

Results: Solved for Michigan system. Gaps to within 0.5% with $64 = N$ branches.

Conclusions:

- Integer Problems can cause Duality Gaps
- Stochastic Integer Programs can also have Duality Gaps
- As Sample Sizes increase, Gaps Decrease to Zero
- Convergence appears Rapid in Power System Applications.