Optimization in Financial Engineering in the Post-Boom Market

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Introduction

• History of financial engineering
  – Rapid expansion of derivative market (total now greater than global equity)
  – Rise in successful quantitative investors (e.g., hedge funds)
  – Applications in asset management and risk management
  – Boom market

• Current situation
  – Overall consolidation in the industry
  – Maintained asset management and risk management interest – steady use of optimization
Presentation Outline

• Selected applications of optimization
  • Option pricing
  • Portfolio/asset-liability models
  • Tracking and trading
  • Securitization
  • Risk management/Real options
• Future potential

Option Models

• “Derivative” securities
  – Example: Call: Buy a share at a given price at a specific time (European)
    • If by a specific time - American
  – Put: Sell; Straddle: Buy or sell
• Why?
  – Reduce risk (hedge)
  – Speculate
  – Arbitrage
• Original analysis - L. Bachelier (1900 - Brownian motion)
Results on European Options

- Black-Scholes-Merton formula
- Put-call parity for exercise price \( K \) and expiration \( T \)
  \[
  C_t - P_t = S_t - e^{-r(T-t)}K
  \]

American options:
- Can exercise before \( T \)
- No parity
- Calls not exercised early if no dividend
- Puts have value of early exercise

American Option Complications

- American options
  - Decision at all \( t \) - exercise or not?
- Find best time to exercise (optimize!)
American Options

- Difficult to value because:
  - Option can be exercised at any time
  - Value depends on entire sample path not just state (current price)

- Model (stopping problem):
  \[
  \max_{0 \leq t \leq T} e^{-rt} V_t(S_{0t})
  \]

- Approaches:
  - Linear programming, linear complementarity, dynamic programming

Formulating as Linear Program

- At each stage, can either exercise or not
  \[
  V_t(S) \geq K-S \text{ and } e^{-r\delta} (pV_{t+\delta}(uS)+(1-p) V_{t+\delta}(dS))
  \]
  If minimize over all \( V_t(S) \) subject to these bounds, then find the optimal value.

- Linear program formulation (binomial model)
  \[
  \min \sum_t \sum_{kt} V_{t,kt}
  \text{ s. t. } \begin{align*}
    V_{t,kt} &\geq K-S_{t,kt}, t=0,\delta,2\delta,\ldots,T; V_{T,kt} \geq 0 \\
    V_{t,kt} &\geq e^{-r\delta} (pV_{t+\delta}(u(kt))+(1-p) V_{t+\delta}(d(kt))) \\
    t=0,\delta,2\delta,\ldots,T-1; kt=1,\ldots,t+1; S_{t+\delta}(U(kt))=uS(kt); S_{t+\delta}(D(kt))=dS(kt); S_{0,1}=S(0).
  \end{align*}
  \]
  Result: can find the value in a single linear program
Extensions of LP Formulation

- General model:
  - Find a value function $v$ to
    \[
    \min <C,V> \text{ s.t. } V_t(S_t) \geq (K-S_t)^+, \\
    - \mathcal{L}V + \left( \frac{\partial V}{\partial t} \right) \geq 0, \\
    V_T(S_T) = (K-S_T)^+
    \]
    where $C>0$ and $\mathcal{L}$ denotes the Black-Scholes operator for price changes on a European option.
- Can consider in linear complementarity framework
- Solve with various discretizations
  - Finite differences
  - Finite element methods

General Option Pricing

Applications: Implied Trees

- Basic Idea:
  - Assume a discrete representation of the price dynamics (often binomial) but not with associated probabilities
  - Observe prices of all assets associated with this tree of sample paths (and imply probabilities)
  - Find price for new claim (or check on consistency of option in market)
- Methodology:
  - Minimize deviations in prices or maximize/minimize price subject to fitting different set of prices (linear programming)
Finding Implied Trees

- Given call prices \( \text{Call}(K_i, T_i) \) at exercise prices \( K_i \) and maturities \( T_i \) (assuming risk-neutral pricing)
- Find probabilities \( P_j \) on branches \( j \) to:

\[
\begin{align*}
\min & \quad \sum_i (u_i^+ + u_i^-) \\
\text{s.t.} & \quad \sum_j P_j (S_j - K_i) + u_i^+ - u_i^- = \text{FV}(\text{Call}(K_i, T_i)) \\
\sum_j P_j S_j & = \text{FV}(S) \\
\sum_j P_j & = 1, \quad P_j \geq 0.
\end{align*}
\]

OUTLINE

- Applications
  - Option pricing
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  - Securitization
  - Risk management/Real options
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Overview of Approaches

• General problem
  – How to allocate assets (and accept liabilities) over time?
  – Uses: financial institutions, pensions, endowments

• Methods
  – Static methods and extensions:
  – Dynamic extensions of static
  – Portfolio replication (duration matching)
  – DP policy based
  – Stochastic program based

Static Portfolio Model

Traditional model
  – Choose portfolio to minimize risk for a given return
  – Find the efficient frontier

  Quadratic program (Markowitz):

  find investments \( x=(x(1),\ldots,x(n)) \) to
  
  \[
  \min x^T Q x \\
  \text{s.t. } r^T x = \text{target}, \ e^T x=1, \ x>0.
  \]
Static Model Results

For a given set of assets, find
- fixed percentages to invest in each asset
- maintain same percentage over time
- implies trading but gains over “buy-and-hold”

Needs
- rebalance as returns vary
- cash to meet obligations

Problems
- transaction costs
- cannot lock in gains
- tax effects

Static Asset and Liability Matching: Duration +

- Idea: Find a set of assets to match liabilities (often WRT interest rate changes)
  - Duration (first derivative) and convexity (second derivative) matching
- Formulation:
  Given duration d, convexity v and maturity m of target security or liability pool, find investment levels x_i in assets of cost c_i to:
  \[
  \min \sum_i c_i x_i \\
  \text{s.t. } \sum_i d_i x_i = d; \sum_i v_i x_i = v; \\
  \sum_i m_i x_i = m; x_i \geq 0, i = 1 \ldots n
  \]
- Extensions:
  - Put in scenarios for the durations.. extend their application
- Problems:
  - Maintaining position over time
  - Asymmetry in reactions to changing (non-parallel yield curve shifts)
  - Assumes assets and liabilities face same risk
Extension to Liability Matching

• Idea (Black et al.)
  – Best thing is to match each liability with asset
  – Implies bonds for matching pension liabilities
• Formulation:
  Suppose liabilities are \( l_t \) at time and asset \( i \) has cash flow \( f_{it} \) at
time, then the problem is:
  \[
  \min \sum c_i x_i \\
  \text{s.t. } \sum f_{it} x_i = l_t \text{ all } t; x_i \geq 0, i = 1 \ldots n
  \]
• Advantages:
  – Liabilities matched over time
  – Can respond to changing yield curve
• Disadvantages
  – Still assumes same risk exposure
  – Does not allow for mix changes over time

Further Extensions to Liability Matching

• Include scenarios \( s \) for possible future liabilities and asset returns
• Formulation:
  \[
  \min \sum c_i x_i \\
  \text{s.t. } \sum f_{its} x_i = l_{ts} \text{ all } t \text{ and } s; x_i \geq 0, i = 1 \ldots n
  \]
• If not possible to match exactly then include some error
  that is minimized.
• Allows more possibilities in the future, but still not dealing
  with changing mixes over time.
• Also, does not consider possible gains relative to liabilities
  which can be realized by rebalancing and locking in
Extended Policies – Dynamic Programming Approaches

• Policy in static approaches
  – Fixed mix or fixed set of assets
  – Trading not explicit
• DP allows broader set of policies
• Problems: Dimensionality, Explosion in time
• Remedies: Approximate (Neuro-) DP
• Idea: approximate a value-to-go function and possibly consider a limited set of policies

Dynamic Programming Approach

• State: $x_t$ corresponding to positions in each asset (and possibly price, economic, other factors)
• Value function: $V_t(x_t)$
• Actions: $u_t$
• Possible events $s_t$, probability $p_{st}$
• Find:
  $$V_t(x_t) = \max \left\{ -c_t u_t + \sum s_t p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t)) \right\}$$

Advantages: general, dynamic, can limit types of policies
Disadvantages: Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
General Methods

• Basic Framework: Stochastic Programming
  – Allows general policies

• Model Formulation:

\[
\max \sum_\sigma p(\sigma) (U(W(\sigma, T) )
\]

s.t. (for all \(\sigma\), \(\sigma\)):
\[
\Sigma_k x(k, 1, \sigma) = W(o) \text{ (initial)}
\]
\[
\Sigma_k r(k, t-1, \sigma) x(k, t-1, \sigma) - \Sigma x(k, t, \sigma) = 0, \text{ all } t > 1;
\]
\[
\Sigma_k r(k, T-1, \sigma) x(k, T-1, \sigma) - W(\sigma, T) = 0, \text{ (final)};
\]
\[
x(k, t, \sigma) \geq 0, \text{ all } k, t;
\]

Nonanticipativity:
\[
x(k, t, \sigma') - x(k, t, \sigma) = 0 \text{ if } \sigma', \sigma \in \Sigma, \text{ for all } t, i, \sigma', \sigma
\]
This says decision cannot depend on future.

Advantages: General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, computational capabilities, insight into policies

General Model Properties

• Assume possible outcomes over time
  • discretize generally

• In each period, choose mix of assets

• Can include transaction costs and taxes

• Can include liabilities over time

• Can include different measures of risk aversion
Example: Investment to Meet Goal

• Proportion in stock versus bonds depends on success of market (no fixed fraction)

![Bar chart showing stock and bond fractions after 5 and 10 years for different market scenarios.]

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Tracking a Security/Index

- **GOAL:** Create a portfolio of assets that follows another security or index with maximum deviation above the underlying asset.

Asset Tracking Decisions

- **Pool of Assets:**
  - TBills
  - GNMA$s, Other mortgage-backed securities
  - Equity issues
- **Underlying Security:**
  - Mortgage index
  - Equity index
  - Bond index
- **Decisions:**
  - How much to hold of each asset at each point in time?
Traditional Approach

- **MODEL**: variant of Markowitz model
- **SOLUTION**: Nonlinear optimization
- **PROBLEMS**:
  - Must rebalance each period
  - Must pay transaction costs
  - May pay taxes
  - Reward on beating target?
- **RESOLUTION**:
  - Make transaction costs explicit
  - Include in dynamic model

Trading and Pricing

- **Situation**:
  - A can borrow 7% fixed or LIBOR+3%
  - B can borrow 6.5% fixed or LIBOR+2%
  - Dealer offers a swap of fixed interest rate for floating (LIBOR)
- **Questions**
  - How to price? Who pays what?
  - How to trade? How to identify partners?

Counterparty A
(Net: LIBOR+2.8%)

Counterparty B
(Net: 6.30% fixed)
Dynamic Trading Formulation

- **PRICES**: \( p(i) \) for asset \( i \) with future cash flows \( c(i,t,s) \) under scenario \( s \); required cash flow of \( b(t,s) \);
- Pay \( x(i) \) now (and perhaps in future)
- **PRICING MODEL** (like liability matching):
  \[
  \min \sum p(i) x(i) \\
  \text{s.t. (for all } s) \sum c(i,t,s) x(i) = b(t,s) \text{ all } t,s.
  \]

**Extensions**
- Different maturity on the securities
- Maintain hedge over time
- Trade securities and match as closely as possible
- Again, can include transaction costs.

Real-time Trading

- **Arbitrage searching**:
  - Assume a set of prices \( p_{ijk} \) for asset \( i \) to asset \( j \) trade in market \( k \) (e.g., currency)
  - Start with initial holdings \( x(i) \) and maximize output \( z \) from asset 1 over trades \( y \)
  \[
  \max z(1) \\
  \text{s.t. } x(i) - \sum_{j,k} p_{ijk} y_{ijk} + \sum_{j,k} p_{jik} y_{jik} = z(i) \\
  y \geq 0, z \geq 0
  \]
  (Generalized network: want to find negative cycles)
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Securitization

• Suppose you hold a collection of assets (loans, royalties, real properties) with different credit worthiness, maturities, and chance for early return of principal
• Idea: divide cash flows into marketable slices with different ratings, maturities
• Maximize value of division of asset cash flows:
  \[
  \max \sum_i p(i) x(i)
  \]
  \[
  \text{s.t. (for all } s): \sum_i c(i,t,s) x(i) = b(t,s) \text{ all } t,s.
  \]
Real Options for Comprehensive Risk Management

- Use real option approach to risks of the firm
- Combine operational and financial decisions
- Set levels for risk (insurance from buy and sell sides)
- Use of stochastic models on several levels and distributed optimization

Future Possibilities and Needs

- Better discretization methods (FEM v. finite differences)
- On-line (continual) optimization for real-time applications
- Inclusion of incomplete markets – distributed optimization
- Consideration of taxes – nonconvex and discrete optimization
- Integration of stochastic model/simulation and optimization
Conclusions

- Optimization continues to bring value to financial engineering
- Existing implementations in multiple areas of financial industry
- Potential for research, theory, methodology, and implementation in real options, incomplete markets, and broader pricing issues