Optimal Consumption – A Stochastic Programming Approach

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Problem Statement

- Determine asset allocation and consumption policy to maximize the expected discounted utility of spending
  - Two asset classes
    - Risky asset, with lognormal return distribution
    - Riskfree asset, with given return r_f
  - Infinite horizon
  - Power utility function

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

  - Consumption rate constrained to be non-decreasing

Existing Research

- Dybvig '95*
  - Continuous-time approach
  - Solution Analysis
    - Consumption rate remains constant until wealth reaches a new maximum
    - The risky asset allocation α is proportional to $w-c/r_f$, which is the excess of wealth over the perpetuity value of current consumption
    - α decreases as wealth decreases, approaching 0 as wealth approaches c/r_f (which is in absence of risky investment sufficient to maintain consumption indefinitely).
- Dybvig '01
  - Considered similar problem in which consumption rate can decrease but is penalized (soft constrained problem)

* "Duesenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living" *Review of Economic Studies* 62, 1995, 287-313.

Objectives

- Replicate Dybvig results using discrete time approach
  - For both hard and soft consumption rate constraint cases
- Consider additional problem features
  - Transaction Costs
  - Multiple risky assets
    - Allocation constraints

Approach

- Application of typical stochastic programming approach complicated by infinite horizon

$$Q(\bar{x}) = \max \sum_i p_{\xi_i}\left(c_{\xi_i} x_{\xi_i} + e^{-\delta t} Q\left(x_{\xi_i}\right)\right)$$
$$s.t. \quad A x_{\xi_i} = b_{\xi_i} - T_{\xi_i} \bar{x}$$

- Initialization.
  - Define a valid constraint on *Q(x)*

$$Q(\bar{x}) \le -E^0 x_{\xi_i} + e^0$$

Requires problem knowledge. For optimal consumption problem, assume extremely high rate of consumption forever

Approach (cont.)

- Iteration k

$$\text{Find} \quad \gamma^k = \min_x \left(U^k(x) - V^k(x) \right), \text{ where}$$

$$V^k(x) = \min_{0 \le j \le k} \left\{ -E^j x + e^j \right\}, \text{ and}$$

$$U^k(x) = \max \quad \sum_i p_{\xi_i} \left(c_{\xi_i} x_{\xi_i} + e^{-\delta t} \Theta_{\xi_i} \right)$$

$$\text{s.t.} \quad A x_{\xi_i} = b_{\xi_i} - T_{\xi_i} \bar{x}$$

$$E^j x_{\xi_i} + \Theta_{\xi_i} \le e^j, \quad j = 0, \ldots, k-1$$

$$\text{If } \gamma^k > -\varepsilon, \text{ terminate.}$$

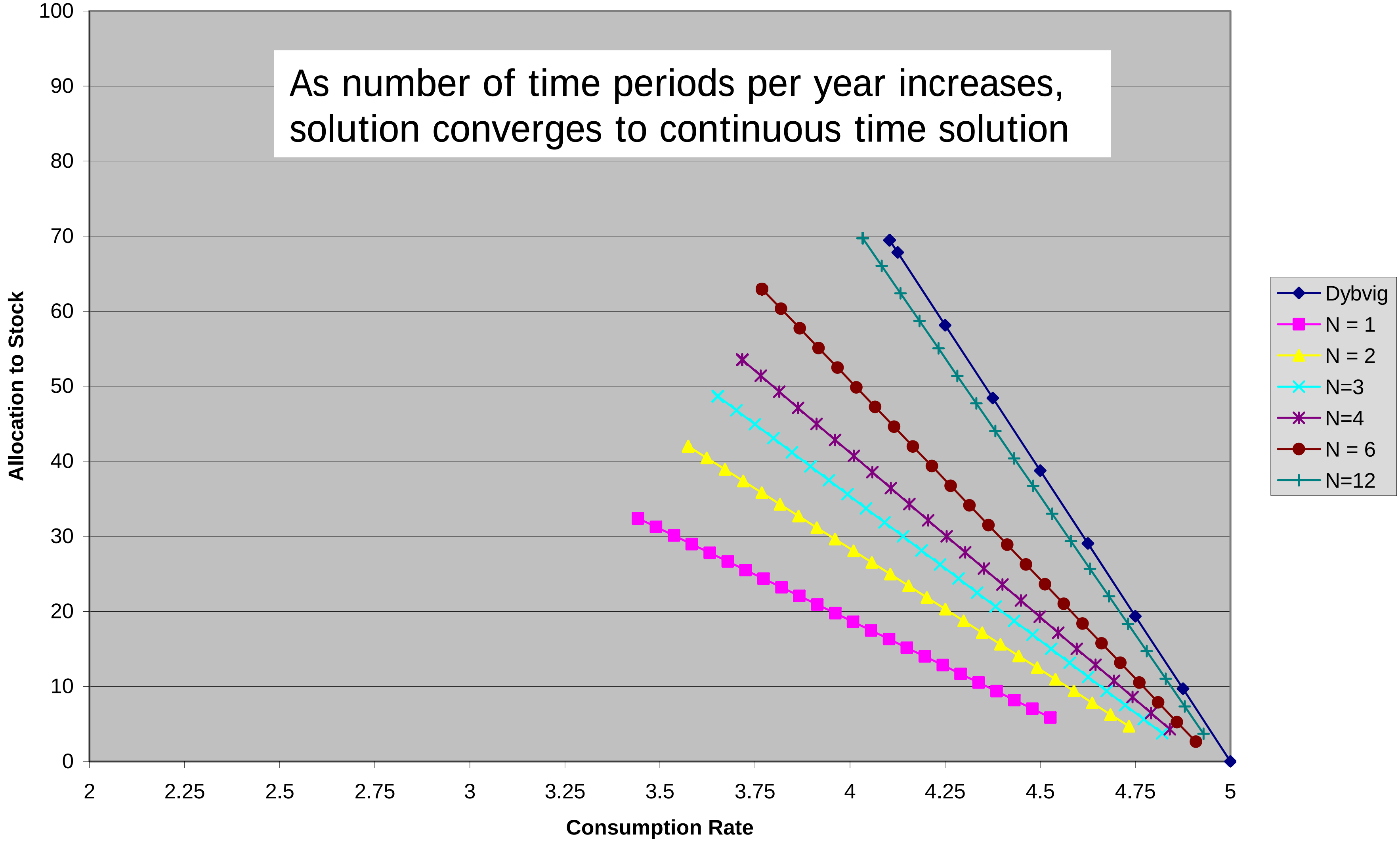
Else, define a new cut

$$E^k = \sum_i - p_{\xi_i} \mu_{\xi_i} T_{\xi_i}$$

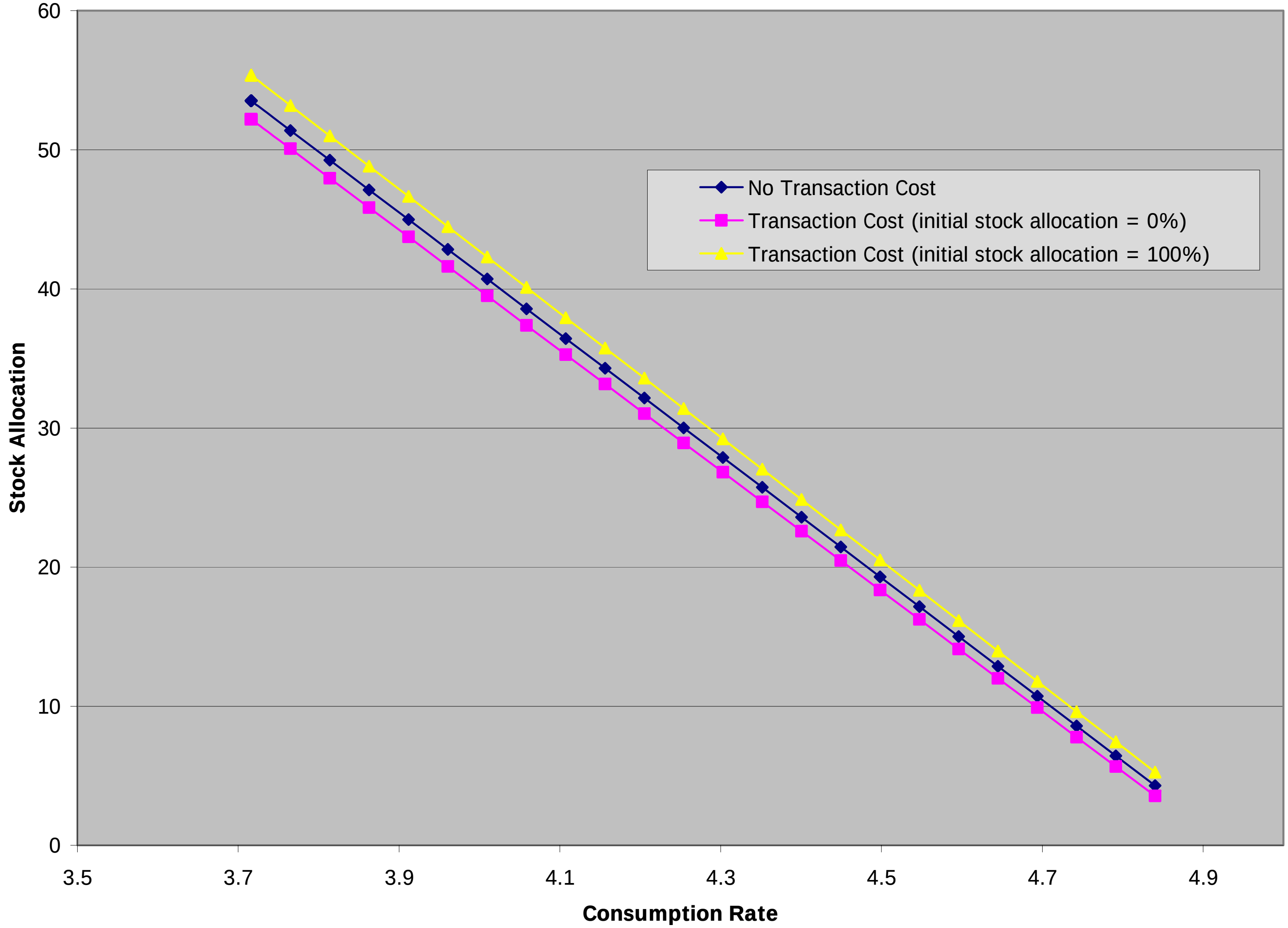
$$e^k = \sum_i p_{\xi_i} \left(\mu_{\xi_i} b_{\xi_i} + \rho_{\xi_i} e \right)$$

Expensive search over x, possible for the optimal consumption problem because of small number of variables

Results – Non-decreasing Consumption



Results – Non-Decreasing Consumption with Transaction Costs



To be continued ...

- Soft constraint on decreasing consumption
- Multiple assets
  - Allocation bounds