Stochastic Programming Models in Design

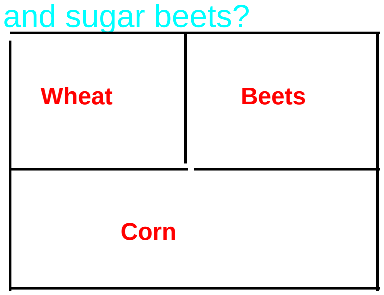
John R. Birge
University of Michigan

OUTLINE

- Models
  - General - Farming
  - Structural design
  - Design portfolio
  - General
- Approximations
- Solutions
- Revisions

European Farming

- Decision:
  - How to plant 500 acres with wheat, corn and sugar beets?



•**Issues: Livestock needs, quotas, costs, yields, prices**

Farm Parameters

- Livestock requirements
  - 200 Tons of wheat
  - 240 Tons of corn
- Prices
  - Wheat: $170 /ton to sell/ $238/ton to buy
  - Corn: $150 /ton to sell/ $210/ton to buy
  - Beets: $36/ton up to 6000 ton (quota); $10/ton if over
- Planting costs
  - Wheat: $150/acre; Corn: $230/acre
  - Beets: $260/acre
- Yields (means)
  - Wheat: 2.5 tons/acre; Corn: 3 tons/acre
  - Beets: 20 tons/acre

Deterministic Farmer's Problem

- Formulation

Min 150 x1 + 230 x2 + 260 x3 + 238 a1 - 170 v1 + 210 a2 - 150 v2 -36 v3 - 10 v4

s.t.	x1	+x2	+ x3			≤	500	(acres)
	2.5 x1			+ a1	-v1	≥	200	(wheat)
		3 x2		+a2	-v2	≥	240	(corn)
			20 x3	-v3	-v4	≥	0	(beets)
				v3		≤	6000	(quota)
			x1,x2,x3,a1,a2,v1,v2,v3,v4			≥	0	

SOLUTION:	WHEAT	CORN	BEETS
ACRES (xi)=	120	80	300
YIELD =	300	240	6000

PROFIT = $118,600 per season

Tight constraints

Scenario Solutions

- Random Factor: Weather
  - Yield variations: +/- 20% of the mean
- Scenario Approach
  - A - Optimistic - Assume +20%

SOLUTION:	WHEAT	CORN	BEETS
ACRES (xi)=	183	67	250
YIELD =	550	240	6000

PROFiT= $167,667 per season

  - B - Pessimistic - Assume -20%

–

SOLUTION:	WHEAT	CORN	BEETS
ACRES (xi)=	100	25	375
YIELD =	200	60	6000

PROFiT= $59,950 per season

STOCHASTIC PROGRAM

- ASSUME: Plant without knowing future
  - Suppose each scenario equally likely (prob.= 1/3 each)
  - Place in single mathematical program
- GOAL: maximize expected profits
  - (risk neutral)
- FORMULATION:

Min 150 x1 + 230 x2 + 260 x3 + 1/3 $\Sigma_{i=1,3}$ (238 a1i - 170 v1i + 210 a2i - 150 v2i -36 v3i - 10 v4i)

s.t. x1 +x2 + x3 $\le$ 500 **(acres)**

(1+.2(2-i))2.5 x1 + a1i -v1i $\ge$ 200 **(wheat)**

(1+.2(2-i))3 x2 +a2i -v2 i $\ge$ 240 **(corn)**

(1+.2(2-i))20 x3 -v3i -v4i $\ge$ 0 **(beets)**

v3 i $\le$ 6000 **(quota)**

x1,x2,x3,a1i,a2i,v1i,v2i,v3i,v4i $\ge$ 0

Axle Design Example

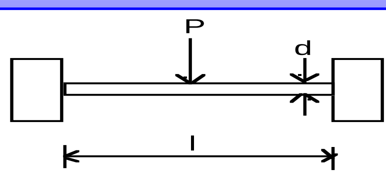


Figure 1. An axle of length l, diameter d, with a central load P.

- Random: $\mathbf{d}(\bar{d})$
- Density

$$f_{\bar{d}}(d) = \begin{cases} \frac{100}{\bar{d}^2}(d - 0.9\bar{d}) & \text{if } 0.9\bar{d} \le d < \bar{d}; \\ \frac{100}{\bar{d}^2}(1.1\bar{d} - d) & \text{if } \bar{d} \le d \le 1.1\bar{d}; \\ 0 & \text{otherwise.} \end{cases} \quad (2.4.1)$$

- Decision: $\bar{d} \le d^{\max}$ and $l \le l^{\max}$
- Selling price:

$$s(1 - e^{-0.1\bar{l}}), \quad (2.4.2)$$

- Manufacturing cost:

$$c(\frac{l\pi\bar{d}^2}{4}). \quad (2.4.3)$$

Axle cont.

- Stress constraint:

$$\frac{l}{d^3} \leq 39.27. \tag{2.4.4}$$

- Deflection constraint: obtain:

$$\frac{l^3}{d^4} \leq 63169. \tag{2.4.5}$$

- Nonlinear recourse function:

$$Q(l, \bar{d}, d) := \min_y \{wy^2 \text{ s. t. } \frac{l}{d^3} - y \leq 39.27,\ \frac{l^3}{d^4} - 300y \leq 63169\}, \tag{2.4.6}$$

- Expected recourse function:

$$\mathcal{Q}(l, \bar{d}) = \int_d Q(l, \bar{d}, d) f_{\bar{d}}(d) \mathrm{d}d, \tag{2.4.7}$$

Full Problem

$$\max \text{ (total revenue per item } - \text{ manufacturing cost per item}$$
$$- \text{ expected future cost per item).} \tag{2.4.9}$$

Formulation and Value

$$\max z(l, \bar{d}) = s(1 - e^{-0.1l}) - c(\frac{l\pi \bar{d}^2}{4}) - \mathcal{Q}(l, \bar{d})$$
$$\text{s. t. } 0 \leq l \leq l^{\max},\ 0 \leq \bar{d} \leq d^{\max}. \tag{2.4.10}$$

Stochastic Solution

- Data:

$$l^{\max} = 35, d^{\max} = 1.25, s = 10$$
$$c = .025, w = 1$$

- Solution:

$$l^* = 33.6, \bar{d}^* = 1.038, z^* = z(l^*, \bar{d}^*) = 8.94$$

Deterministic Solution ■

$$l^{Det} = 35.0719, \bar{d}^{Det} = 0.963, z^{Det}(l^{Det}, \bar{d}^{Det}) = 9.07, z(l^{Det}, \bar{d}^{Det}) = 5.88$$ ■

Value of the Stochastic Solution ■

$$z^* - z(l^{Det}, \bar{d}^{Det}) = 3.06$$

Example for Yacht Design

- Yacht velocity prediction program - A. Philpott
  - Determines velocity based on input parameters
  - Can be optimized for various conditions
  - Includes design parameters
- Stochastic variables
  - Wind velocity
  - Angle to wind
  - Hydrodynamic resistance

Deterministic Problem - Example

- Decision variables:
  - Length: Long, medium, short
- Conditions:
  - Wind: Strong or light
- Outcomes:

Wind	Length	Prob. of Win
Strong	L	0.8 -Optimal /Strong
Strong	M	0.6
Strong	S	0.2
Light	L	0.2
Light	M	0.6
Light	S	0.8 -Optimal /Light

Deterministic Value

- Suppose equal likelihood on conditions
- Prob. of win:
  - If Long, (1/2)(0.8) + (1/2)(0.2)=0.5
  - If Short, (1/2)(0.8) + (1/2)(0.2)=0.5
  - If Medium, (1/2)(0.6) + (1/2)(0.6)=0.6
- Note: Deterministic is not optimal, also no deterministic opt. value is opt. overall
- Value of Stochastic Solution=0.6-0.5=0.1

Portfolio Problem

- Suppose two boats possible
- Suppose choice given previous conditions,
  - If strong now, then P(Strong at race)=0.8
  - If light now, then P(Light at race)=0.8.
- Prob. of win=
  - If Strong now, choose Long, P(Win/Lo andSt)=(0.8)(0.8)+ (0.2)(0.2)=0.68
  - If Light now, choose Short, P(Win/Sh and Li)=0.68
  - If Light or Strong now, choose Medium, P(Win/Medium)=0.6

Portfolio Observations

- Portfolio allows:
  - Increased prob. of win
  - Use of learning about uncertainty
  - Partial hedging
- Note: Change in solution structure
- Difficulties in probability evaluation and integration with yacht design problem

GENERAL MULTISTAGE MODEL

- FORMULATION:

MIN $E[\sum_{t=1}^{T} f_t(x_t, x_{t+1})]$

s.t. $x_t \in X_t$

x_t nonanticipative

$P[h_t(x_t, x_{t+1}) \leq 0] \geq a$ (chance constraint)

EXAMPLES:

FARM: Linear functions, continuous variables

AXLE: Nonlinear plus continuous variables

YACHT: Nonlinear objective, integer variables

DYNAMIC PROGRAMMING VIEW

- STAGES: t=1,...,T
- STATES: x_t -> $B_t x_t$(or other transformation)
- VALUE FUNCTION:
  - $\Psi_t(x_t) = E[\psi_t(x_t,\xi_t)]$ where
  - ξ_t is the random element and
  - $\psi_t(x_t,\xi_t) = \min f_t(x_t,x_{t+1},\xi_t) + \Psi_{t+1}(x_{t+1})$
  - s.t. $x_{t+1} \in X_{t+1t}(\xi_t)$ x_t given
- SOLVE : iterate from T to 1
- PROBLEM: How to find $E[\psi_t(x_t,\xi_t)]$?
  - ξ_t may have high dimension

ALTERNATIVES FOR FINDING Ψ_t

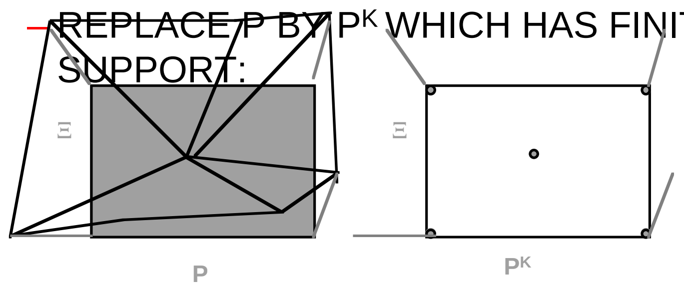
- DIRECT NUMERICAL INTEGRATION
  - Possible only if very small or special structure
  - Not applicable to general, large problems
- SIMULATION
  - Limited convergence rate (1/ √n error for n samples)
  - Difficult estimates of confidence intervals on solutions
- BOUNDING APPROXIMATIONS
  - Find $\Psi_t^{l,k}$ and $\Psi_t^{u,k}$ such that:
    - $\Psi_t^{l,k} \leq \Psi_t \leq \Psi_t^{u,k}$
    - $\lim_k \Psi_t^{l,k} = \Psi_t = \lim_k \Psi_t^{u,k}$
    - where limit is "epigraphical"

BOUNDING APPROXIMATIONS

- GOALS
  - MAINTAIN SOLVABLE SYSTEM
  - ENSURE SOLUTION VALUE WITHIN BOUNDS
  - CONVERGENCE OF BOUNDS
- BASIC IDEA
  - USE CONVEXITY/DUALITY
  - CONSTRUCT FEASIBLE:
    - DUAL SOLUTIONS
      - LOWER BOUNDS
    - PRIMAL SOLUTIONS
      - UPPER ROUNDS
- CONVERGENCE
  - NO DUALITY GAP
  - IMPROVING REFINEMENTS

DISCRETIZATIONS

- SIMPLIFY THE DISTRIBUTION
  - REPLACE P BY P^K WHICH HAS FINITE SUPPORT:



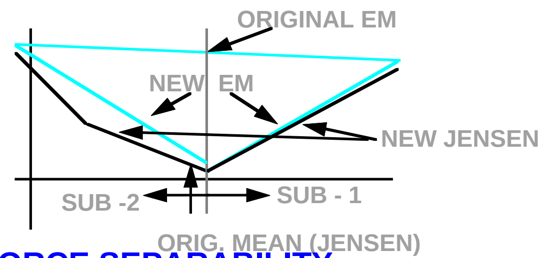
MIAIN PROCEDURES:

LOWER: JENSEN (MEAN)

UPPER: EDMUNDSON-MADANSKY (EXTREME POINTS)

BOUND IMPROVEMENTS

- PARTITIONING
  - SPLIT Ξ (SUPPORT OF RANDOM VECTOR) INTO SUBREGIONS
  - MAKE FUNCTION Ψ AS LINEAR AS POSSIBLE ON EACH SUBREGION



ENFORCE SEPARABILITY:

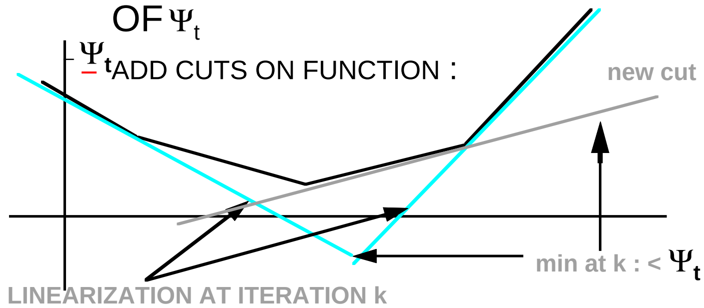
- FIND SEPARABLE RESPONSES TO ALL RANDOM PARAMETER CHANGES

SOLVING AS LARGE-SCALE MATHEMATICAL PROGRAMS

- ORIGIN:
  - DISCRETIZATION LEADS TO MATHEMATICAL PROGRAM BUT LARGE-SCALE
  - USE STANDARD METHODS BUT EXPLOIT STRUCTURE
- DIRECT METHODS
  - TAKE ADVANTAGE OF SPARSITY STRUCTURE
    - SOME EFFICIENCIES
  - USE SIMILAR SUBPROBLEM STRUCTURE
    - GREATER EFFICIENCY - DECOMPOSITION
- SIZE
  - UNLIMITED (INFINITE NUMBERS OF VARIABLES)
  - STILL SOLVABLE (CAUTION ON CLAIMS)

DECOMPOSITION METHODS

- BENDERS IDEA
  - FORM AN OUTER LINEARIZATION OF Ψ_t
  - ADD CUTS ON FUNCTION :



USE AT EACH STAGE TO APPROXIMATE VALUE FUNCTION

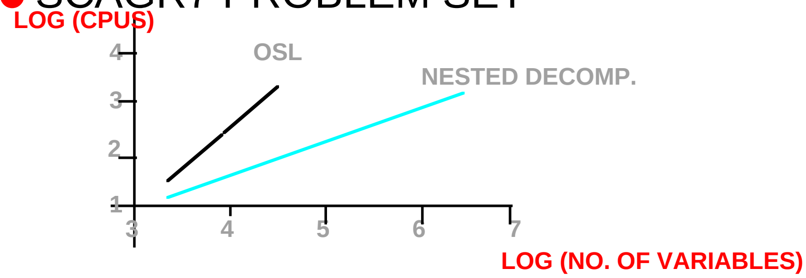
- ITERATE BETWEEN STAGES UNTIL ALL MIN = Ψ_t

DECOMPOSITION IMPLEMENTATION

- NESTED DECOMPOSITION
  - LINEARIZATION OF VALUE FUNCTION AT EACH STAGE
  - DECISIONS ON WHICH STAGE TO SOLVE, WHICH PROBLEMS AT EACH STAGE
- LINEAR PROGRAMMING SOLUTIONS
  - USE OSL FOR LINEAR SUBPROBLEMS
  - USE MINOS FOR NONLINEAR PROBLEMS
- PARALLEL IMPLEMENTATION
  - USE NETWORK OF RS6000S
  - PVM PROTOCOL

RESULTS

- SCAGR7 PROBLEM SET



PARALLEL: 60-80% EFFICIENCY IN SPEEDUP

OTHER PROBLEMS: SIMILAR RESULTS

- **ONLY < ORDER OF MAGNITUDE SPEEDUP WITH STORM**
  - **TWO-STAGES - LITTLE COMMONALITY IN SUBPROBLEMS**
  - **STILL ABLE TO SOLVE ORDER OF MAGNITUDE LARGER PROBLEMS**

SOME OPEN ISSUES

- MODELS
  - IM PACT ON METHODS
  - RELATION TO OTHER AREAS
- APPROXIMATIONS
  - USE WITH SAMPLING METHODS
  - COMPUTATION CONSTRAINED BOUNDS
  - SOLUTION BOUNDS
- SOLUTION METHODS
  - EXPLOIT SPECIFIC STRUCTURE
  - MASSIVELY PARALLEL ARCHITECTURES
  - LINKS TO APPROXIMATIONS

CRITICISMS

- UNKNOWN COSTS OR DISTRIBUTIONS
  - FIND ALL AVAILABLE INFORMATION
  - CAN CONSTRUCT BOUNDS OVER ALL DISTRIBUTIONS
    - FITTING THE INFORMATION
  - STILL HAVE KNOWN ERRORS BUT ALTERNATIVE SOLUTIONS
- COMPUTATIONAL DIFFICULTY
  - FIT MODEL TO SOLUTION ABILITY
  - SIZE OF PROBLEMS INCREASING RAPIDLY (OVER 20 MILLION VARIABLES)

CONCLUSIONS

- STOCHASTIC PROGRAMS CAN BE:
  - LINEAR, NONLINEAR, INTEGER PROGRAMS
  - CONTINUOUS OR DISCRETE R.V.'S
  - OF SIGNIFICANT VALUE (VSS) OVER DETERMINISTIC MODELS
  - INTEGRATION INTO DESIGN PROBLEMS
  - PORTFOLIOS to HEDGE
- RANDOMNESS =>
  - VALUE OF MODELING
  - DIFFICULTY IN EVALUATING OBJECTIVES
  - MOTIVATION FOR APPROXIMATION
- SOLUTIONS
  - DECOMPOSITION FOR LINEAR PROBLEMS
  - SPEEDUPS OF ORDERS OF MAGNITUDE