Optimization Models in Financial Mathematics

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Introduction

• Trends in financial mathematics and engineering
  – Rapid expansion of derivative market (total now greater than global equity)
  – Rise in successful quantitative investors (e.g., hedge funds)
  – Applications in asset management and risk management
  – Boom market (and meltdown)
  – Current focus on managing risks
• What to answer? (3 questions)
Big Three Questions

How much should you pay?
What should you buy?
How much can you lose?

Main application areas:
• Pricing: How much should you pay?
• Portfolio Optimization: What should you buy?
• Risk Management: How much can you lose?

Themes

• Optimization is a key part of the fundamental questions
• Duality and optimal solution properties provide a foundation for finding the answers
Presentation Outline

• How much to pay?
  – Fundamental Theorem of Asset Pricing
  – Pricing the American option

• What to buy?
  – Dynamic portfolio optimization

• How much to lose?
  – Value-at-Risk and the Moment Problem

Axiom of the Market

• Market Axiom: There is no free lunch
  – The market does not allow arbitrage
  – No one can trade assets, never lose money, and sometimes make a profit

• How to write this mathematically?
• Assume prices $S_t(1),...,S_t(n)$ for $n$ assets at times $t$
• Own (owe) $x_t = x_t(1),...,x_t(n)$ shares of each
• Trades at $t$ change our position from $x_{t-1}$ to $x_t$ and must satisfy conservation of funds:
  \[ \sum_{i=1}^{n} S_i(i) x_{t-1}(i) = \sum_{i=1}^{n} S_i(i) x_t(i) \text{ or } S_t x_{t-1} = S_t x_t \]
Linear Program for “No Free Lunch”

- Prices and share decisions are random variables, some distribution on events $P$
- No losses means $S_T x_T \geq 0$ almost surely;
- No positive profits without losses means:
  $$0 \geq \max_{P} E_P [S_T x_T]$$
  s.t. $S_0 x_0 = 0$, $S_T x_T \geq 0$ (a.s.)
  $$S_t x_{t-1} = S_t x_p \quad t=1, \ldots, T$$

Linear Program Dual

- Primal problem:
  $$0 \geq \max c^T x$$
  s.t. $A x = 0$, $B x \geq 0$
- Dual problem:
  $$\exists \pi, \rho \quad \text{s.t.} \quad \pi^T A - \rho^T B = c^T, \rho \geq 0$$

What does that mean for no-arbitrage problem?
Discrete Scenario Tree

- Suppose $i=1 \ldots N_t$ outcomes at time $t$
- Probability of each outcome $i$ at $t$ is $p(i,t)$
- Each outcome $i$ at $t$ has ancestor $a(i,t)$ at $t-1$ and descendants $D(i,t)$ at $t+1$
- Prices $S_t(i)$ and actions $x_t(i)$ depend on outcomes $i$

Constraints for No Arbitrage

- $A x = 0$ corresponds to:
  \[ S_0 x_0 = 0 \]
  \[-S_t(i) x_{t-1}(a(i,t))+S_t(i) x_t(i) = 0, \quad \forall i, t \]
- $B x \geq 0$ corresponds to $S_T(i) x_T(i) \geq 0, \quad \forall i$
- Starting structure of $A$:
  \[
  \begin{bmatrix}
  S_0 & 0 & 0 & 0 \\
  -S_1 & S_1 & 0 & 0 \\
  -S_2 & 0 & S_2 & 0 \\
  0 & -S_1 & 0 & S_1 \\
  \end{bmatrix}
  \]
Dual for No Arbitrage

• $\pi^T A + \rho^T B = c^T$ becomes
  $$\pi_0 S_0 - \sum_{i=1}^{N_1} \pi_i S_i = 0$$
  $$\pi_t(i) S_t(i) - \sum_{j \in D(i,t)} \pi_{t+1}(j) S_{t+1}(j) = 0, \ t=1..T-1$$
  $$\pi_T(i) S_T(i) - \rho_T(i) S_T = p(i,T) S_T(i)$$

• Suppose Asset 1 is a mattress (riskfree investment – Treasury bonds), price of Asset 1 is $S_1(1,i) = 1$ for all $t,i$

This means: $\pi_t(i) = \sum_{j \in D(i,t)} \pi_{t+1}(j), \ t=0..T-1$

Let $q_{t+1}(j) = \sum_{j \in D(i,t)} \pi_{t+1}(j)/\pi_t(i)$ then

$S_t(i) = \sum_{j \in D(i,t)} q_{t+1}(j) S_{t+1}(j) = E_Q[S_{t+1}], \ t=1..T-1$

where $Q$ is called a risk-neutral equivalent measure to $P$

Fundamental Theorem of Asset Pricing

• The absence of arbitrage is equivalent to the presence of a risk-neutral equivalent measure such that the expected return on all assets is the same with respect to this measure

• $Q$ is also called a martingale measure

• Can price any asset (in fact, derivative) using $Q$
Results on European Options

- Black-Scholes-Merton formula can be found using $Q$
- Find values of Calls and Puts for buying and selling at $K$ at $T$:
  \[ \text{Call} = e^{-rT}E_Q[(S_T-K)^+] \]

American options:
- Can exercise before $T$
- No parity
- Calls not exercised early if no dividend
- Puts have value of early exercise

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American Option Complications

- American options
  - Decision at all $t$ - exercise or not?
- Find best time to exercise Put (optimize!)
American Options

- Difficult to value because:
  - Option can be exercised at any time
  - Value depends on entire sample path not just state (current price)
- Model (stopping problem):
  \[ \sup_{0 \leq t \leq T} e^{-rt} V_t(S_0) \]
- Approaches:
  - Linear programming, linear complementarity, dynamic programming, duality

Formulating as Linear Program

- At each stage, can either exercise or not
- Suppose price goes from \( S \) to \( uS \) or \( dS \) each \( \delta \) time step (binomial model):
  \[ V_t(S) \geq K-S \text{ and } e^{-r\delta} (pV_{t+\delta}(uS)+(1-p) V_{t+\delta}(dS)) \]
  If minimize over all \( V_t(S) \) subject to these bounds, then find the optimal value.
- Linear program formulation (binomial model)
  \[
  \min \sum_t \sum_{kt} V_{t,kt} \\
  s. t. \ V_{t,kt} \geq K-S_{t,kt}, \ t=0, \delta, 2\delta, ..., T; \ V_{T,KT} \geq 0 \\
  V_{t,kt} \geq e^{-r\delta} (pV_{t+\delta,U(kt)}+(1-p) V_{t+\delta,D(kt)}) \\
  t=0, \delta, 2\delta, ..., T-1; \ kt=1, ..., t+1; \ S_{t+\delta,U(kt)}=uS(kt); \ S_{t+\delta,D(kt)}=dS(kt); \ S_{0,1}=S(0). 
  \]
  Result: can find the value in a single linear program
Extensions of LP Formulation

• General model:
  – Find a value function \( v \) to
    \[
    \min C, V \text{ s.t. } V_t(S_t) \geq (K-S_t)^+, \\
    - \mathcal{L}V + (\partial V / \partial t) \geq 0, \\
    V_T(S_T) = (K-S_T)^+
    \]
    where \( C > 0 \) and \( \mathcal{L} \) denotes the Black-Scholes operator for price changes on a European option.

• Can consider in linear complementarity framework

• Solve with various discretizations
  – Finite differences
  – Finite element methods

The American Option Dual

• Approach from Haugh/Kogan and Rogers

• Primal problem:
  \[
  V_0 = \sup_{\tau \in [0,T]} E[e^{-r\tau} h(\tau) S_{\tau}],
  \]
  where \( h(S_{\tau}) = (K-S_{\tau})^+ \)

• Dual problem where \( E[\pi] = 0 \) (martingale):
  \[
  \sup_{\tau \in [0,T]} E[e^{-r\tau} h(\tau) S_{\tau}] = \sup E[e^{-r\tau} h(\tau) S_{\tau} - \pi] \\
  \leq E[\sup_e e^{-r\tau} h(\tau) S_{\tau} - \pi] \\
  \]
  So, \( V_0 \leq \inf_{\pi, \text{martingale}} E[\sup_e e^{-r\tau} h(\tau) S_{\tau} - \pi] \)
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Finding Optimal Growth

• Let $W_t$ be wealth at $t$, $W_t = R_t \cdot W_{t-1}$ for $R_t$ a return process
• $W_t = R_t \cdot R_{t-1} \cdots R_1 \cdot W_0$
• Take log’s
  \[
  \log W_t = \log W_0 + \sum_{s=1}^{t} R_s
  \]
  or \[
  \log(W_t/W_0)^{1/t} = \frac{1}{t} \sum_{s=1}^{t} \log R_s
  \]
• By Law of Large Numbers:
  \[
  (1/t) \sum_{s=1}^{t} \log R_s \rightarrow E[\log R_t] = m \text{ (if i.i.d.)}
  \]
  So, \[
  \log(W_t/W_0)^{1/t} \rightarrow m.
  \]
• Implication: Maximize $m$ to maximize growth
  \[
  (W_t/W_0) \text{ (Equivalent to Max } E[U(W_t)] \text{ for } U = \log W)
  \]
Kelly System

- Suppose double or nothing betting system (bet $x$ and win $2x$ with prob. $p$ or lose $x$ with prob. $1-p$)
- Let $\alpha$ be fraction to bet of wealth $W$
  \[
  \text{Max } E[U(X)] = p \log(\alpha+1) + (1-p)\log(1-\alpha)
  \]
  or $\alpha=2p-1$.
- (Luenberger…Blackjack.. $p=.5075$.. time to double = 6440 hands)

Continuous Portfolio Dynamics

- Suppose $n$ investments, weights $x_i$ on $i$
- Continuous time results:
  \[
  dW/W = \sum_i x_i \left( dS_i/S_i \right) = \sum_i x_i \mu_i \, dt + x_i \, dZ_i
  \]
- Result:
  \[
  E[\log(W_t/W_0)] = \nu t = \sum_i x_i \mu_i \, t - (1/2) \sum_{ij} x_i \sigma_{ij} x_j \, t
  \]
- To maximize growth rate:
  \[
  \max \sum_i x_i \mu_i - (1/2) \sum_{ij} x_i \sigma_{ij} x_j \\
  \text{s. t. } \sum_i x_i = 1.
  \]
Optimal Solution Results

- Log-optimal if riskfree asset with return $r_f$
  \[ x_i \text{ s.t. } \sum \sigma_{ij} x_j = \mu_i - r_f \]
  \[ \text{for } i = 1, \ldots, n. \]
- Should you do this?
  \[ \mu = 0.1, \sigma^2 = 0.04 \text{ for stocks, } r_f = 0.04 \]
  \[ x_{\text{stocks}} = \frac{(0.1 - 0.04)}{0.04} = 1.5 \text{ (with borrowing 0.5 at } r_f) \]

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Finding Distributions Based on Market Prices

- How much can you lose?
- Idea:
  - Assume the market correctly interprets probabilities into prices
  - These will be like the moments in a moment problem
  - Can find what probabilities are implied by market prices
  - Find the probability of a given loss (or the maximum probability of that loss or greater) or Value-at-Risk (VaR, the maximum loss with a given probability)
- First: implied binomial trees (Rubinstein)

Implied Binomial Trees

- Assume that you have a set of option prices:
  Example: Share price=45, r=5%, T-t=56/365
  Observe: Call(45,T)=2, Call(40,T)=5.5
  Assume binomial with ending prices: 52, 45, 39
  What are the probabilities with these branches?

\[
\begin{align*}
P_0 &+ P_1 + P_2 = 1 \\
P_1 &P_1(5) + P_2(12) = e^{r(T-t)} \cdot 5.5 \\
P_2 &P_2(7) = e^{r(T-t)} \cdot 2
\end{align*}
\]
Implied Trees Example

- So, $P_2 = .28$, $P_1 = .43$, $P_0 = .29$
- Could also go back to find probabilities on branches
- General idea: create a tree
- Consistent with market price?
  \[(.29(39) + .43(45) + .28(52))e^{-0.05(56/365)} = 44.9\]
- In general, might have more options than branches
- Fit the observed prices as closely as possible

Fitting Implied Trees

- Suppose additional prices:
  - e.g. $Call(35,T) = 10.3$, $Call(50,T) = 0.5$
- Find $P_0$, $P_1$, $P_2$, … to:
  \[
  \begin{align*}
  \min & \sum_j (u_i^+ + u_i^-) \\
  \text{s.t.} & \sum_j P_j (S_j - K_i)^+ + u_i^+ - u_i^- = FV(Call(K_i,T)) \\
  & \sum_j P_j S_j = FV(S_0) \\
  & \sum_j P_j = 1, P_j \geq 0.
  \end{align*}
  \]
Problems with Implied Binomial Trees

- Assumes that the binomial is followed
- Generalization: Use extremal probabilities with generalized linear programming.
- Results: can find maximum and minimum (risk-neutral) probabilities implied by market
- Example: Suppose we have written a call option at 50 and want to know the maximum probability of paying more than 5 (i.e., price is above 55)
- Like VaR – chance of losing a certain amount (fraction) in a specified time

Solving the Moment Problem by Linear Programming

- Semi-infinite L.P. (all prices possible)
- Initialize: Start with a set of prices $S_i, g$ is function of prices to maximize in expectation, other constraints by $v$
- Step 1: Generalize to Moment Problem:
- Master problem: Find $p_i \geq 0, \ldots, p_r \geq 0, \sum p_i = 1$, to
  \[
  \max \sum_{i=1}^{r} g(S_i) p_i \\
  s.t \sum_{i=1}^{s} v_i(S_i) p_i \leq \beta_i, i=1,\ldots,s, \]
  \[
  \sum_{i=s+1}^{M} v_i(S_i) p_i = \beta_i, i=s+1,\ldots,M;
  \]
  Let $\{p_1,\ldots,p_r\}$ attain the max and $\{\sigma,\pi_1,\ldots,\pi_M\}$ be the associated dual multipliers.
Subproblem: Step 2

Subproblem solution: Find new price, $S^{r+1}$ that maximizes

$$\gamma(S, \sigma^j, \pi^j) = g(S) - \sigma^j - \sum_{i=1}^{M} \pi^j_i v_i(S)$$

If $\gamma(S^{r+1}, \sigma^j, \pi^j) > 0$, let $r = r+1$, $v = v+1$ and go to Step 1.

Otherwise stop; $\{p_j^1, \ldots, p_j^{r+1}\}$ are the optimal probabilities associated with $\{S_j^1, \ldots, S_j^{r+1}\}$.

Extremal Probabilities

- Problem: find $p_j$ to

$$\text{Max } \sum_{j \mid S_j \geq 55} p_j$$

s.t. $\sum p_j = 1$

$$\sum_j p_j (S_j - K_i)^+ = FV(C(K_i, T))$$

$$\sum_j p_j S_j = FV(S_j), p_j \geq 0$$

For our example, suppose we have $S_j = 30, 35, 40, 45, 50, 55, 60$ and $C(35)=10.3, C(40)=5.5, C(45)=2, C(50)=0.5$.

Solve to find:

$$P(60) = 0.082, P(50) = 0.21, P(45) = 0.40, P(40) = 0.26,$$

$P(30) = 0.05$ with multipliers .165 on $K_i = 50$ constraint.

Subproblem: Max $\{1 - .165(S-50), S \geq 55; 0 - .165(S-50)^+, S < 55\}$ $\Rightarrow$ new $S_j = 55$

Result: $P(55) = .10$, and multiplier .2 on $K=50$ constraint

Subproblem: Max $\{1 - .2(S-50), S \geq 55; 0 - .2(S-50)^+, S < 55\}$ $\Rightarrow$ optimal
VaR Calculations

- Generalized programming (moment problem) framework allows calculation of bounds on implied probabilities
- Note: these are the risk-neutral probabilities
- How to bound actual probabilities?

For values below risk-free rate of growth, with positive risk premium, actual probabilities will be lower so these are also bounds on those probabilities.

Conclusions

- Optimization brings value to financial analysis in answering the Big 3 Questions
- Existing implementations in multiple areas of financial industry
- Significant potential for research, theory, methodology, and implementation