Comparison of Static and Dynamic Asset Allocation Models

John R. Birge
University of Michigan

Outline

- Basic Models
  - Static Markowitz mean-variance
  - Dynamic stochastic programming
- Difficulties in static model
- Example results
- Other tests
**Static Model**

- Markowitz model
  - Choose portfolio to minimize risk for a given return
  - Find the **efficient frontier**

  ![Graph](image)

**Markowitz model**

- For a given set of assets, find
  - fixed percentages to invest in each asset
  - maintain same percentage over time

- **Needs**
  - rebalance as returns vary
  - cash to meet obligations
Dynamic Model

- Assume possible outcomes over time
  - discretize generally
- In each period, choose mix of assets
- Can include transaction costs
- Can include liabilities over time
- Can include different measures of risk aversion

FORMULATION

- **SCENARIOS**: \( \sigma \in \Sigma \)
  - Probability, \( p(\sigma) \)
  - \( \Gamma_{t_1}, \ldots, \Gamma_{t_r}, S_{s_k} \) at \( t \)

**MULTISTAGE STOCHASTIC NLP FORM**:

\[
\begin{align*}
\text{max} & \quad \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right) \\
\text{s.t.} (\text{for all } \sigma) & \quad \sum_k x(k, 1, \sigma) = W(\sigma) \quad \text{(initial)} \\
& \quad \sum_k r(k, t-1, \sigma) x(k, t-1, \sigma) - \sum_k x(k, t, \sigma) = 0, \quad \text{all } t > 1; \\
& \quad \sum_k r(k, T-1, \sigma) x(k, T-1, \sigma) - W(\sigma, T) = 0, \quad \text{(final)}; \\
& \quad x(k, t, \sigma) \geq 0, \quad \text{all } k, t; \\
\end{align*}
\]

**Nonanticipativity**:

\[
\begin{align*}
x(k, t, \sigma') - x(k, t, \sigma) = 0 \quad \text{if } \sigma', \sigma \in S_i, \text{for all } t, i, \sigma', \sigma
\end{align*}
\]

This says decision cannot depend on future.
GENERAL MULTISTAGE MODEL

**FORMULATION:**

\[
\begin{align*}
\text{MIN} & \quad E \left[ \sum_{t=1}^T f_t(x_t,x_{t+1}) \right] \\
\text{s.t.} & \quad x_t \in X_t \\
& \quad x_t \text{ nonanticipative} \\
& \quad P[ h_t(x_t,x_{t+1}) \leq 0 ] \geq a \text{ (chance constraint)}
\end{align*}
\]

**EXAMPLES:**

- **Vehicle Allocation:** Linear functions, continuous or integer variables
- **Capacity:** Linear plus integer variables
- **Financial Planning:** Nonlinear objective, continuous variables

**Problems in Static Approach**

- **Utility form**
  - Not consistent over multiple periods
  - If near end, may be conservative
  - Different behavior at beginning
- **Transaction costs**
  - Missing actual needs over time - target utility
Financial Planning

- **GOAL:** Accumulate $G$ for tuition $Y$ years from now
- **Assume:**
  - $W(0)$ - initial wealth
  - $K$ - investments
  - concave utility (piecewise linear)

![Utility diagram]

**RANDOMNESS:** returns $r(k,t)$ - for $k$ in period $t$

where $Y$ = decision period

DATA and SOLUTIONS

- **ASSUME:**
  - $Y=15$ years
  - $G=$80,000
  - $T=3$ (5 year intervals)
  - $k=2$ (stock/bonds)

- Returns (5 year):
  - Scenario A: $r$(stock) = 1.25   $r$(bonds)= 1.14
  - Scenario B: $r$(stock) = 1.06   $r$(bonds)= 1.12

- Solution:

<table>
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<th>PERIOD</th>
<th>SCENARIO</th>
<th>STOCK</th>
<th>BONDS</th>
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<td>64.0</td>
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</tr>
</tbody>
</table>

J.R. Birge, Industrial and Operations Engineering
University of Michigan
Static Markowitz Solution

- Find efficient frontier:

  ![Efficient Frontier Diagram]

Results with Static Model

- Fixed proportion in stock and bonds in each period
- 80% stock for 15% return
- 40% stock for 14% return
- Results: no fixed proportion achieves target better than 50% of time
- Dynamic achieves target 87.5% of time
Other Model Gains

- Include transaction costs
  - Fixed proportion has 0.1% per period just to re-balance
  - can accumulate
- Maintain consistent utility

Current Study

- Portfolios of major indexes
- Constructed efficient frontier
- Developed decision tree form for stochastic program
- Gains in basic model for stochastic program of 3-5% over 10 periods
Summary

- Static models have real problems for dynamic problems
- Biggest gains may be in ability to change positions over time
- Large study on indices to continue