Introduction to Stochastic Programming

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Outline

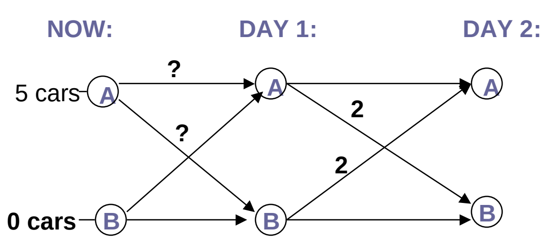
- Overview
- Examples
  - Vehicle Allocation
  - Financial planning
  - Manufacturing
- Methods
- View ahead

Overview

- Stochastic optimization
  - Traditional
    - Small problems
    - Impractical
  - Current
    - Integrate with large-scale optimization (stochastic programming)
    - Practical examples
    - Expanding rapidly

Vehicle Allocation

- Decision:
    - How to position empty freight cars?



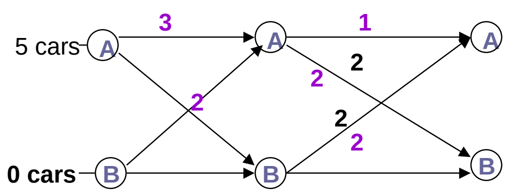
DEMAND: **DAY 1: B to A:Mean Value=2**
DAY 1: A to B:Mean Value=2

Vehicle Allocation: Mean Value Solution

Parameters: COST: 0.5 per empty car from A to B
REVENUE: 1.5 per full car from A to B

- **Maximize: Revenue-Cost**
  - » **MOVE TWO EMPTY CARS FROM A to B**

NOW: **DAY 1:** **DAY 2:**



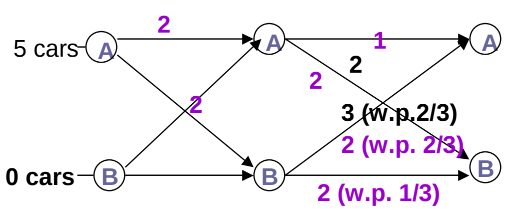
RESULT: **Net 2: A to B; Net 2: B to A**
TOTAL(*MV*) = 4

Expectation of Mean Value

Suppose: Demand is Random (Expectation from A to B=2)
- **0 from A to B with prob. 1/3**
- **3 from A to B with prob. 2/3**

- **Find: Expected (Revenue-Cost)**
  - » **MOVE Two EMPTY CARS FROM A to B**

NOW: **DAY 1:** **DAY 2:**



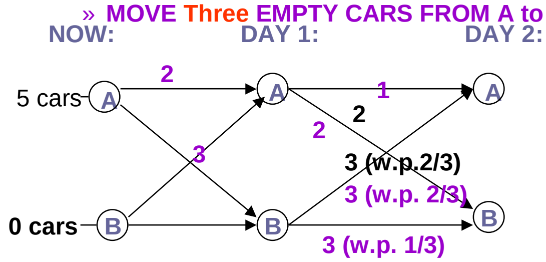
Expected Value:

Net 2: A to B;
Net 2: B to A (w.p. 2/3)
-1: B to A (w.p. 1/3)
TOTAL (*EMV*): 3

Stochastic Program Solution

Suppose: Demand is Random (as before)
GOAL: A solution to obtain highest expected value

- **Maximize: Expected (Revenue-Cost)**
  - » MOVE Three EMPTY CARS FROM A to B



Expected Value:

Net 2: A to B;
Net 3: B to A (w.p. 2/3)
-1.5 : B to A (w.p. 1/3)
TOTAL (*RP*): 3.5
RP=Recourse Problem

INFORMATION and MODEL VALUE

- **INFORMATION VALUE:**
  - **FIND Expected Value with Perfect Information or Wait-and-See (WS) solution:**
    - **Know demand: if 3, send 3 from A to B; If 0, send 0 from A to B:**
    - **Earn: 2 (AtoB) + (2/3) (3) + (1/3)0= 4 = WS**
  - **Expected Value of Perfect Information (EVPI):**
    - **EVPI = WS - RP = 4 - 3.5 = 0.5**
    - **Value of knowing future demand precisely**
- **MODEL VALUE:**
  - **FIND EMV, RP**
  - **Value of the Stochastic Solution (VSS):**
    - **VSS = RP - EMV=3.5 - 3 = 0.5**
    - **Value of using the correct optimization model**

INFORMATION/MODEL OBSERVATIONS

- EVPI and VSS:
  - ALWAYS >= 0 (WS >= RP>= EMV)
  - OFTEN DIFFERENT (WS=RP but RP > EMV and vice versa)
  - FIT CIRCUMSTANCES:
    - COST TO GATHER INFORMATION
    - COST TO BUILD MODEL AND SOLVE PROBLEM
- MEAN VALUE PROBLEMS:
  - MV IS OPTIMISTIC (MV=4 BUT EMV=3, RP=3.5)
    - ALWAYS TRUE IF CONVEX AND RANDOM
    - CONSTRAINT PARAMETERS
  - VSS LARGER FOR SKEWED DISTRIBUTIONS/COSTS

STOCHASTIC PROGRAM

- ASSUME: Random demand on AB and BA
- GOAL: maximize expected profits
  - (risk neutral)
- DECISIONS: x_{ij} - empty from i to j
  - $y_{ij}(s)$ - full from i to j in scenario s (RECOURSE)
  - (prob. p(s))
- FORMULATION:

Max -0.5xAB + $\Sigma_{s=s1,s2}$ p(s) (1.5 yAB(s) + 1.5 yBA(s))

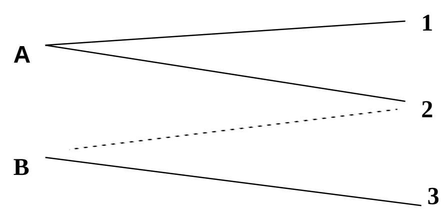
s.t.	xAB +xAA		= 5	(Initial)
	-xAB	+ yBA(s)	<= 0	(Limit BA)
	-xAA	+ yAB(s)	<= 0	(Limit AB)
		yBA(s)	<= DBA(s)	(Demand BA)
		+ yAB(s)	<= DAB(s)	(Demand AB)

xAA, XAB, yAA(s), yAB (s)>=0

EXTENSIONS: Multiple stages;Constraint/objective complexity (Powell et al.)

Manufacturing Capacity

- **Where to Install Capacity for Different Models among Different Plants?**



•Where to add flexibility? (multiple models)

Recourse Payoff Evaluation

- **Key: Evaluate Expected Optimal with Installed Capacity**

  Must choose best mix of models assigned to plants

  Maximize Σi Profit (i) Production(i)

  subject to: MaxSales(i) >= Σ j Production(i at j)

  Σ i Production(i at j) <= Capacity (i)

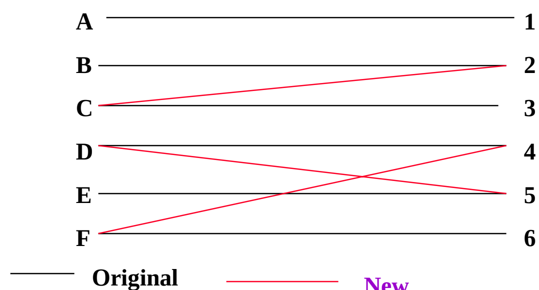
  Production(i at j) <= Capacity (i at j)

  Production(i at j) >= 0

- **Transportation Problem**
- **Need MaxSales(i) - random - unknown distribution**
  - **Capacity(i at j) - Decision in First Stage**

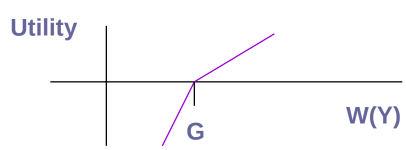
Solution Results

- **Model Data: from Graves/Jordan**
- **Vary: Model Lifetimes**
  - **Longer => More flexibility**
- **Start: 1 Year**



Financial Planning

- **GOAL:** **Accumulate $G for tuition Y years from now**
- **Assume:**
  - **$ W(0) - initial wealth**
  - **K - investments**
  - **concave utility (piecewise linear)**



RANDOMNESS: **returns r(k,t) - for k in period t**

where Y → T decision periods

FORMULATION

- **SCENARIOS:** $\sigma \in \Sigma$
  - **Probability, $p(\sigma)$**
  - **Groups, $S^t_1, \ldots, S^t_{St}$ at t**
- **MULTISTAGE STOCHASTIC NLP FORM:**

max $\Sigma_\sigma\ p(\sigma)\ (\ U(W(\sigma, T)\ )$

s.t. (for all σ): $\Sigma_k\ x(k,1,\sigma) = W(o)$ **(initial)**

$\Sigma_k\ r(k,t\text{-}1,\sigma)\ x(k,t\text{-}1,\sigma) - \Sigma_k\ x(k,t,\sigma) = 0$, **all t >1;**

$\Sigma_k\ r(k,T\text{-}1,\sigma)\ x(k,T\text{-}1,\sigma) - W(\sigma, T) = 0$, **(final);**

$x(k,t,\sigma) >= 0$, **all k,t;**

Nonanticipativity:

$x(k,t,\sigma') - x(k,t,\sigma) = 0$ **if** $\sigma', \sigma \in S^t_i$ **for all t, i,** σ', σ

This says decision cannot depend on future.

DATA and SOLUTIONS

- **ASSUME:**
  - **Y=15 years**
  - **G=$80,000**
  - **T=3 (5 year intervals)**
  - **k=2 (stock/bonds)**
- **Returns (5 year):**
  - **Scenario A: r(stock) = 1.25 r(bonds)= 1.14**
  - **Scenario B: r(stock) = 1.06 r(bonds)= 1.12**
- **Solution:**

PERIOD	SCENARIO	STOCK	BONDS
1	1-8	41.5	13.5
2	1-4	65.1	2.17
2	5-8	36.7	22.4
3	1-2	83.8	0
3	3-4	0	71.4
3	5-6	0	71.4
3	7-8	64.0	0

GENERAL MULTISTAGE MODEL

- FORMULATION:

MIN $\quad E\,[\,\Sigma_{t=1}^{T} f_t(x_t,x_{t+1})\,]$

s.t. $\quad x_t \in X_t$

$\quad x_t$ nonanticipative

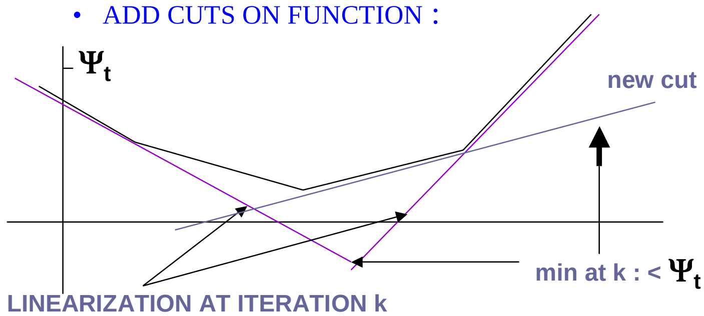
$\quad P[\,h_t\,(x_t,x_{t+1}) <= 0\,] >= a$ (chance constraint)

EXAMPLES:

Vehicle Allocation: Linear functions, continuous or integer variables
Capacity: Linear plus integer variables
Financial Planning: Nonlinear objective, continuous variables

DECOMPOSITION METHODS

- BENDERS IDEA
  - FORM AN OUTER LINEARIZATION OF Ψ_t - VALUE FUNCTION AT STAGE t
  - ADD CUTS ON FUNCTION :



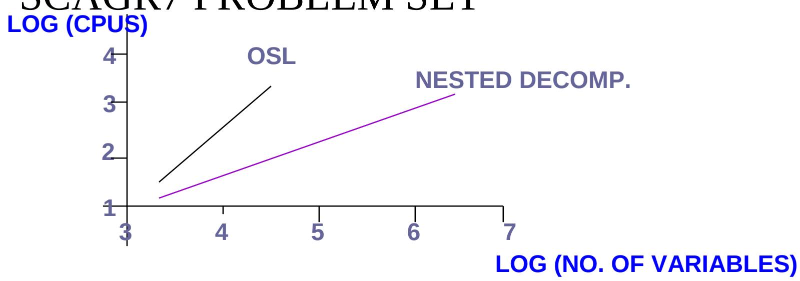
ITERATE BETWEEN STAGES UNTIL ALL MIN = Ψ_t

DECOMPOSITION IMPLEMENTATION

- NESTED DECOMPOSITION
  - LINEARIZATION OF VALUE FUNCTION AT EACH STAGE
  - DECISIONS ON WHICH STAGE TO SOLVE, WHICH PROBLEMS AT EACH STAGE
- LINEAR PROGRAMMING SOLUTIONS
  - USED OSL/CPLEX FOR LINEAR SUBPROBLEMS
  - USE MINOS FOR NONLINEAR PROBLEMS
- PARALLEL IMPLEMENTATION

RESULTS

- SCAGR7 PROBLEM SET



PARALLEL: 60-80% EFFICIENCY IN SPEEDUP

OTHER PROBLEMS: SIMILAR RESULTS
- ONLY < ORDER OF MAGNITUDE SPEEDUP WITH STORM
  - TWO-STAGES - LITTLE COMMONALITY IN SUBPROBLEMS
  - STILL ABLE TO SOLVE ORDER OF MAGNITUDE LARGER PROBLEMS

View Ahead

- New Trends
  - Methods for integer variables
    - Power system implementations
    - Vehicle routing
  - Integrating simulation
    - Sampling with optimization
    - On-line optimization
    - Low-discrepancy methods

More Trends

- Modeling languages
  - Ability to build stochastic programs directly
  - Integrating across systems
- Using application structure
  - Separation of problem (dimension reduction)
  - Network properties
  - Generalized versions of convexity

Summary

- Increasing application base
- Value for solving the stochastic problem
- Efficient implementations
- Opportunities for new results