Stochastic Programming Models in Asset-Liability Management

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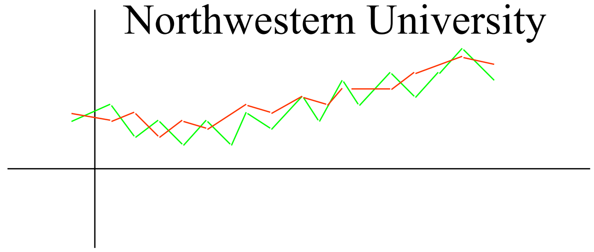


Background

- What is asset-liability management?
  - Deciding how to allocate assets and what liabilities to incur to obtain best performance (meet liabilities and grow net assets)
- Why interest?
  - Trillions of dollars in pension funds alone
  - Key function of insurance carriers
  - Endowments, corporations, individuals
- Why use stochastic programming?
  - Comprehensive
  - Specific

OUTLINE

- **Overview of approaches**
- **Static versus dynamic comparison**
- **Alternative approaches**
- **Analysis and critique**

Overview of Approaches

- Static and extensions
  - Mean-variance based
  - Rebalancing improvement
  - No lock-in/Loss on transaction fees
- Dynamic extensions of static
  - Fixed proportion plus trading window on static
- Portfolio replication (duration match)
- DP policy based
- Stochastic program based

Static Portfolio Model

Traditional model

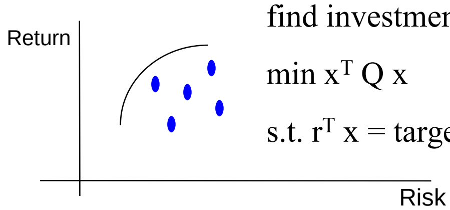
- **Choose portfolio to minimize risk for a given return**
- **Find the efficient frontier**

Quadratic program (Markowitz):

find investments x=(x(1),…,x(n)) to

min $x^T Q x$

s.t. $r^T x$ = target, $e^T x$=1, x>=0.



Static Model Results

For a given set of assets, find

- **fixed percentages to invest in each asset**
- **maintain same percentage over time**
- **implies trading but gains over "buy-and-hold"**

Needs

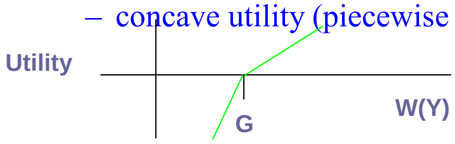
- **rebalance as returns vary**
- **cash to meet obligations**

Problems

- **transaction costs**
- **cannot lock in gains**
- **tax effects**

Example: Retirement Planning

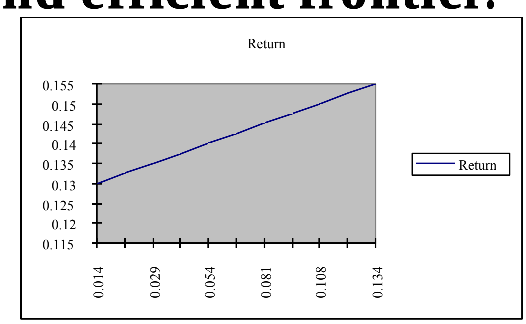
- GOAL: Accumulate $G Y years from now
- Assume:
  - $ W(0) - initial wealth
  - K - investments
  - concave utility (piecewise linear)



Note: Similar to meeting a target or benchmark

Static Markowitz Solution

1 **Find efficient frontier:**



Results with Static Model

- **Fixed proportion in stock and bonds in each period**
- **80% stock for 15% return**
- **40% stock for 14% return**
- **Results: no fixed proportion achieves target better than 50% of time**
- **Dynamic achieves target 87.5% of time**

Alternative Dynamic Model

- **Assume possible outcomes over time**
  - **discretize generally**
- **In each period, choose mix of assets**
- **Can include transaction costs and taxes**
- **Can include liabilities over time**
- **Can include different measures of risk aversion**

Formulation with No Transactions Fees

- SCENARIOS: $\sigma \in \Sigma$
  - Probability, $p(\sigma)$
  - Groups, $S^t_1, \ldots, S^t_{St}$ at t
- MULTISTAGE STOCHASTIC NLP FORM:

max $\Sigma_\sigma\ p(\sigma)\ (\ U(W(\sigma, T)\ )$

s.t. (for all σ): $\Sigma_k\ x(k,1,\sigma) = W(o)$ (initial)

$\Sigma_k\ r(k,t-1,\sigma)\ x(k,t-1,\sigma) - \Sigma_k\ x(k,t,\sigma) = 0$, all $t > 1$;

$\Sigma_k\ r(k,T-1,\sigma)\ x(k,T-1,\sigma) - W(\sigma, T) = 0$, (final);

$x(k,t,\sigma) >= 0$, all k,t;

Nonanticipativity:

$x(k,t,\sigma') - x(k,t,\sigma) = 0$ if $\sigma', \sigma \in S^t_i$ for all t, i, σ', σ

This says decision cannot depend on future.

DATA and SOLUTIONS

- ASSUME:
  - Y=15 years
  - G=$80,000
  - T=3 (5 year intervals)
  - k=2 (stock/bonds)
- Returns (5 year):
  - Scenario A: r(stock) = 1.25 r(bonds)= 1.14
  - Scenario B: r(stock) = 1.06 r(bonds)= 1.12
- Solution:

PERIOD	SCENARIO	STOCK	BONDS
1	1-8	41.5	13.5
2	1-4	65.1	2.17
2	5-8	36.7	22.4
3	1-2	83.8	0
3	3-4	0	71.4
3	5-6	0	71.4
3	7-8	64.0	0

Analysis of Dynamic Model

- With discrete outcomes, p.l. utility:
  - Optimal solution has number of investments at most equal to number of branches in each period
  - Constrain the number of positive investments with the number of outcomes per period
- Impact of transaction fees and taxes
  - Additional constraints
  - Creates potential for more active investments in each period
  - Additional constraints can be imposed with linearization (representation other variance information)
  - Number of constraints can be used to limit number of investments

Other Model Gains

- **Can include transaction costs**
  - **Fixed proportion requires transaction costs each period just to re-balance**
  - **can accumulate**
- **Can include tax considerations**
  - **Model size grows (lots of each asset)**
- **Maintain consistent utility**

Uses of Mean-Variance?

- Examples: Mulvey/Lattice Financial
  - Towers Perrin
  - California pension
  - GM?
- Advantage?
  - Improvement over buy and hold can be large
  - Rebalancing is key to maintaining diversification
- Disadvantages
  - No lock-in (relative to benchmark)
  - Transaction costs may be large (can use no-trade region but no fixed criteria in multiple dimensions)
  - Hard to generalize for taxes
  - Nonconvex optimization (approximated)

Duration Matching and Generalizations

- Duration matching
  - Idea to match first derivative (and with convexity second derivative)
- Formulation:

  Given duration d, convexity v and maturity m of target security or liability pool, find investment levels x_i in assets of cost c_i to:

min $\Sigma_i c_i x_i$

s.t. $\Sigma_i d_i x_i = d$; $\Sigma_i v_i x_i = v$;

$\Sigma_i m_i x_i = m$; $x_i >= 0$, $i = 1 \ldots n$

- Extensions: (Dembo/Algorithmics)
  - Put in scenarios for the durations.. extend their application
- Problems:
  - Maintaining position over time
  - Asymmetry in reactions to changing rates (non-parallel yield curve shifts)
  - Assumes assets and liabilities face same risk

Extension to Liability Matching

- Idea (Black et al.)
  - Best thing is to match each liability with asset
  - Implies bonds for matching pension liabilities
- Formulation:

  Suppose liabilities are l_t at time and asset i has cash flow f_{it} at time, then the problem is:

  $$\min \Sigma_i\, c_i\, x_i$$
  $$\text{s.t. } \Sigma_i\, f_{it}\, x_i = l_t \text{ all t; } x_i >= 0,\ i = 1\ldots n$$
- Advantages:
  - Liabilities matched over time
  - Can respond to changing yield curve
- Disadvantages
  - Still assumes same risk exposure
  - Does not allow for mix changes over time

Further Extensions to Liability Matching

- Include scenarios s for possible future liabilities and asset returns
- Formulation:

  $$\min \Sigma_i\, c_i\, x_i$$
  $$\text{s.t. } \Sigma_i\, f_{its}\, x_i = l_{ts} \text{ all t and s; } x_i >= 0,\ i = 1\ldots n$$
- If not possible to match exactly then include some error that is minimized.
- Allows more possibilities in the future, but still not dealing with changing mixes over time.
- Also, does not consider possible gains relative to liabilities which can be realized by volatility pumping and locking in

Extended Policies – Dynamic Programming Approaches

- Policy in duration and liability matching:
  - Fixed mix or fixed set of assets
  - No trading or dynamics
- DP allows broader set of policies
- Problems: Dimensionality, Explosion in time
- Remedies: Approximate (Neuro-) DP
- Idea: approximate a value-to-go function and possibly consider a limited set of policies

Dynamic Programming Approach

- State: x_t corresponding to positions in each asset (and possibly price, economic, other factors)
- Value function: $V_t(x_t)$
- Actions: u_t
- Possible events s_t, probability p_{st}
- Find:

$$V_t(x_t) = \max -c_t u_t + \boldsymbol{\Sigma}_{\mathbf{st}} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t))$$

Advantages: general, dynamic, can limit types of policies

Disadvantages: Dimensionality, approximation of V at some point needed, limited policy set may be needed, accuracy hard to judge

Other Restricted Policy Approaches

- Kusy-Ziemba ALM model for Vancouver Credit Union
- Idea: assume an expected liability mix with variation around it; minimize penalty to meet the variation
- Formulation:

$$\min \Sigma_i c_i \; x_i + \Sigma_{st} p_{st}(q_{st}^+ y_{st}^+ + q_{st}^- y_{st}^-)$$

$$\text{s.t. } \Sigma_i \; f_{its} \; x_i + y_{st}^+ - y_{st}^- = l_{ts} \text{ all t and s; } x_i \; y >= 0, i = 1\ldots n$$

Problems: Similar to liability matching.

General Methods

- Basic Framework: Stochastic Programming
- Model Formulation:

$$\max \quad \Sigma_\sigma \; p(\sigma) \, (U(W(\sigma , T))$$

$$\text{s.t. (for all } \sigma\text{): } \Sigma_k \; x(k,1, \sigma) = W(o) \text{ (initial)}$$

$$\Sigma_k \; r(k,t\text{-}1, \sigma) \, x(k,t\text{-}1, \sigma) - \Sigma_k \; x(k,t, \sigma) = 0 , \text{ all } t >1;$$

$$\Sigma_k \; r(k,T\text{-}1, \sigma) \, x(k,T\text{-}1, \sigma) - W(\sigma , T) = 0, \text{ (final);}$$

$$x(k,t, \sigma) >= 0, \text{ all } k,t;$$

Nonanticipativity:

$$x(k,t, \sigma') - x(k,t, \sigma) = 0 \text{ if } \sigma', \sigma \in S^t_i \text{ for all } t, i, \sigma', \sigma$$

This says decision cannot depend on future.

Advantages:

General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, computational capabilities, insight into policies

Examples of General Models

- Frank Russell-Ziemba-Yasuda Marine
  - Large model for insurance
  - Many side constraints on policies
  - Branching with ~10 initial branches then fewer in future periods
  - No arbitrage on branches?
  - Few total branches
  - Unclear on solution value (compared to Mean-Variance)
  - Current state unclear

More Examples (2)

- Zenios: Cyprus/Wharton
  - Multiple models – insurance, mortgages
  - General form with some sophistication in solution approaches (parallel etc.)
  - Improvement relative to fixed strategies
  - Comparisons on "efficient frontier"
  - Unclear on price dynamics
  - Hard to draw overall policy conclusions

More Examples (3)

- Dempster – Cambridge
  - Similar to Zenios model
  - Some effort in scenario generation
  - Unclear on no-arbitrage
  - Uses expected value of perfect information for tree "trimming"
  - Participation of London banks
- Mitra – Brunel
  - Similar in model to Dempster
  - Probably has multiple "shortcuts"
  - Unclear on price dynamics

More Examples (4)

- Frauendorfer – St. Gallen, CH
  - Other models just present approximation of sample paths
  - Uses bounding samples
  - Examples in income securities with interest rate dynamics
  - Unclear on no-arbitrage
- Klaassen – Erasmus, Netherlands
  - Approximations with no arbitrage
  - Still some lack of clarity on form of scenario generation

Objective Functions

- Previous examples
  - Forms of piecewise linear utility functions
  - Not clear whether consistent with financial evaluations
- Alternatives
  - Probability of beating benchmark..not coherent
- Coherent measures of risk (Heath et al.)
  - Lead to p.l. utility function forms
  - Expected downside risk or conditional value-at-risk (Uryasiev and Rockafellar)

Analysis and Critique

- Models
  - Not clear whether models consider all that is known, may have arbitrage possibilities
  - Generation of scenarios not clear, no universal methodology
  - Can include information about tails
  - Tax considerations have generally been avoided
  - P.L. utility versus more sophisticated.

Solution Method Critique

- Existing methods limited in size of problems
- No consistent estimators that are also efficient
- Decomposition methodology and combinations with DP approximation may have promise

Conclusions

- Static portfolio models have problems with:
  - benchmark targets
  - transaction costs and taxes
- Dynamic stochastic programming models can address difficulties
  - variety of objectives
  - can use structure to meet additional requirements
- Computation of large problems using decomposition and special structure