Stochastic Programming Models in Asset-Liability Management

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Background

• What is asset-liability management?
  – Deciding how to allocate assets and what liabilities to incur to obtain best performance (meet liabilities and grow net assets)

• Why interest?
  – Trillions of dollars in pension funds alone
  – Key function of insurance carriers
  – Endowments, corporations, individuals

• Why use stochastic programming?
  – Comprehensive
  – Specific
OUTLINE

• Overview of approaches
• Static versus dynamic comparison
• Alternative approaches
• Analysis and critique

Overview of Approaches

• Static and extensions
  – Mean-variance based
  – Rebalancing improvement
  – No lock-in/Loss on transaction fees
• Dynamic extensions of static
  – Fixed proportion plus trading window on static
• Portfolio replication (duration match)
• DP policy based
• Stochastic program based
Static Portfolio Model

Traditional model
- Choose portfolio to minimize risk for a given return
- Find the efficient frontier

Quadratic program (Markowitz):
find investments \( x=(x(1),\ldots,x(n)) \) to
\[
\begin{align*}
\text{min } & x^T Q x \\
\text{s.t. } & r^T x = \text{target}, e^T x=1, x\geq 0.
\end{align*}
\]

Static Model Results

For a given set of assets, find
- fixed percentages to invest in each asset
- maintain same percentage over time
- implies trading but gains over “buy-and-hold”

Needs
- rebalance as returns vary
- cash to meet obligations

Problems
- transaction costs
- cannot lock in gains
- tax effects
Example: Retirement Planning

- **GOAL**: Accumulate $G Y$ years from now

- **Assume**:
  - $W(0)$ - initial wealth
  - $K$ - investments
  - concave utility (piecewise linear)

**Note**: Similar to meeting a target or benchmark

Static Markowitz Solution

1. **Find efficient frontier**:

![Efficient Frontier Graph](image)
Results with Static Model

- Fixed proportion in stock and bonds in each period
- 80% stock for 15% return
- 40% stock for 14% return
- Results: no fixed proportion achieves target better than 50% of time
- Dynamic achieves target 87.5% of time

Alternative Dynamic Model

- Assume possible outcomes over time
  - discretize generally
- In each period, choose mix of assets
- Can include transaction costs and taxes
- Can include liabilities over time
- Can include different measures of risk aversion
Formulation with No Transactions Fees

• SCENARIOS: $\sigma \in \Sigma$
  – Probability, $p(\sigma)$
  – Groups, $S_t^1, ..., S_t^t$ at $t$

• MULTISTAGE STOCHASTIC NLP FORM:

$$\max \quad \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right)$$

s.t. (for all $\sigma$):
$$\sum_k x(k, 1, \sigma) = W(o) \quad \text{(initial)}$$
$$\sum_k r(k, t-1, \sigma) x(k, t-1, \sigma) - \sum_k x(k, t, \sigma) = 0, \quad \text{all } t > 1;$$
$$\sum_k r(k, T-1, \sigma) x(k, T-1, \sigma) - W(\sigma, T) = 0, \quad \text{(final)};$$
$$x(k, t, \sigma) \geq 0, \quad \text{all } k, t;$$

Nonanticipativity:
$$x(k, t, \sigma') - x(k, t, \sigma) = 0 \quad \text{if } \sigma', \sigma \in S_t^i \text{ for all } t, i,$$

This says decision cannot depend on future.

DATA and SOLUTIONS

• ASSUME:
  – $Y=15$ years
  – $G=$80,000
  – $T=3$ (5 year intervals)
  – $k=2$ (stock/bonds)

• Returns (5 year):
  – Scenario A: $r($stock$) = 1.25$  $r($bonds$)= 1.14$
  – Scenario B: $r($stock$) = 1.06$  $r($bonds$)= 1.12$

• Solution:

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>SCENARIO</th>
<th>STOCK</th>
<th>BONDS</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1-8</td>
<td>41.5</td>
<td>13.5</td>
</tr>
<tr>
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<td>1-4</td>
<td>65.1</td>
<td>2.17</td>
</tr>
<tr>
<td>2</td>
<td>5-8</td>
<td>36.7</td>
<td>22.4</td>
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<td>3</td>
<td>3-4</td>
<td>0</td>
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<td>5-6</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>3</td>
<td>7-8</td>
<td>64.0</td>
<td>0</td>
</tr>
</tbody>
</table>
Analysis of Dynamic Model

- With discrete outcomes, p.l. utility:
  - Optimal solution has number of investments at most equal to number of branches in each period
  - Constrain the number of positive investments with the number of outcomes per period
- Impact of transaction fees and taxes
  - Additional constraints
  - Creates potential for more active investments in each period
  - Additional constraints can be imposed with linearization (representation other variance information)
  - Number of constraints can be used to limit number of investments

Other Model Gains

- Can include transaction costs
  - Fixed proportion requires transaction costs each period just to re-balance
  - can accumulate
- Can include tax considerations
  - Model size grows (lots of each asset)
- Maintain consistent utility
Uses of Mean-Variance?

- Examples: Mulvey/Lattice Financial
  - Towers Perrin
  - California pension
  - GM?

- Advantage?
  - Improvement over buy and hold can be large
  - Rebalancing is key to maintaining diversification

- Disadvantages
  - No lock-in (relative to benchmark)
  - Transaction costs may be large (can use no-trade region but no fixed criteria in multiple dimensions)
  - Hard to generalize for taxes
  - Nonconvex optimization (approximated)

Duration Matching and Generalizations

- Duration matching
  - Idea to match first derivative (and with convexity second derivative)

- Formulation:
  Given duration \( d \), convexity \( v \) and maturity \( m \) of target security or liability pool, find investment levels \( x_i \) in assets of cost \( c_i \) to:

\[
\begin{align*}
\text{min} & \, \sum_i c_i x_i \\
\text{s.t.} & \, \sum_i d_i x_i = d; \, \sum_i v_i x_i = v; \, \sum_i m_i x_i = m; \, x_i \geq 0, \, i = 1...n
\end{align*}
\]

- Extensions: (Dembo/Algorithmics)
  - Put in scenarios for the durations.. extend their application

- Problems:
  - Maintaining position over time
  - Asymmetry in reactions to changing rates (non-parallel yield curve shifts)
  - Assumes assets and liabilities face same risk
Extension to Liability Matching

• Idea (Black et al.)
  – Best thing is to match each liability with asset
  – Implies bonds for matching pension liabilities

• Formulation:
  Suppose liabilities are $l_t$ at time and asset $i$ has cash flow $f_{it}$ at time, then the problem is:
  \[
  \min \sum_i c_i x_i \\
  \text{s.t. } \sum_i f_{it} x_i = l_t \text{ all } t ; x_i \geq 0, i = 1 \ldots n
  \]

• Advantages:
  – Liabilities matched over time
  – Can respond to changing yield curve

• Disadvantages
  – Still assumes same risk exposure
  – Does not allow for mix changes over time

Further Extensions to Liability Matching

• Include scenarios $s$ for possible future liabilities and asset returns

• Formulation:
  \[
  \min \sum_i c_i x_i \\
  \text{s.t. } \sum_i f_{its} x_i = l_{ts} \text{ all } t \text{ and } s ; x_i \geq 0, i = 1 \ldots n
  \]

• If not possible to match exactly then include some error that is minimized.

• Allows more possibilities in the future, but still not dealing with changing mixes over time.

• Also, does not consider possible gains relative to liabilities which can be realized by volatility pumping and locking in
Extended Policies – Dynamic Programming Approaches

- Policy in duration and liability matching:
  - Fixed mix or fixed set of assets
  - No trading or dynamics
- DP allows broader set of policies
- Problems: Dimensionality, Explosion in time
- Remedies: Approximate (Neuro-) DP
- Idea: approximate a value-to-go function and possibly consider a limited set of policies

Dynamic Programming Approach

- State: $x_t$ corresponding to positions in each asset (and possibly price, economic, other factors)
- Value function: $V_t(x_t)$
- Actions: $u_t$
- Possible events $s_t$, probability $p_{st}$
- Find:
  $$V_t(x_t) = \max \left\{ -c_t u_t + \sum_{s_t} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t)) \right\}$$

Advantages: general, dynamic, can limit types of policies

Disadvantages: Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
Other Restricted Policy Approaches

- Kusy-Ziemba ALM model for Vancouver Credit Union
- Idea: assume an expected liability mix with variation around it; minimize penalty to meet the variation
- Formulation:

\[
\min \sum_i c_i x_i + \sum_{st} p_{st}(q_{st}^+ y_{st}^+ + q_{st}^- y_{st}^-)
\]

\[
s.t. \sum_i f_{it} x_i + y_{st}^+ - y_{st}^- = l_{it} \text{ all } t \text{ and } s; \quad x_i, y \geq 0, i = 1 \ldots n
\]

Problems: Similar to liability matching.

General Methods

- Basic Framework: Stochastic Programming
- Model Formulation:

\[
\max \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right)
\]

\[
s.t. \text{(for all } \sigma): \sum_k x(k,1, \sigma) = W(\sigma) \text{ (initial)}
\]

\[
\sum_k r(k,t-1, \sigma) x(k,t-1, \sigma) - \sum_k x(k,t, \sigma) = 0, \text{ all } t > 1;
\]

\[
\sum_k r(k,T-1, \sigma) x(k,T-1, \sigma) - W(\sigma, T) = 0, \text{ (final)};
\]

\[
x(k,t, \sigma) \geq 0, \text{ all } k,t;
\]

Nonanticipativity:

\[
x(k,t, \sigma') - x(k,t, \sigma) = 0 \text{ if } \sigma', \sigma \in S; \text{ for all } t, i, \sigma', \sigma
\]

Advantages:

- General model, can handle transaction costs, include tax lots, etc.

Disadvantages:

- Size of model, computational capabilities, insight into policies
Examples of General Models

- Frank Russell-Ziemba-Yasuda Marine
  - Large model for insurance
  - Many side constraints on policies
  - Branching with ~10 initial branches then fewer in future periods
  - No arbitrage on branches?
  - Few total branches
  - Unclear on solution value (compared to Mean-Variance)
  - Current state unclear

More Examples (2)

- Zenios: Cyprus/Wharton
  - Multiple models – insurance, mortgages
  - General form with some sophistication in solution approaches (parallel etc.)
  - Improvement relative to fixed strategies
  - Comparisons on “efficient frontier”
  - Unclear on price dynamics
  - Hard to draw overall policy conclusions
More Examples (3)

- Dempster – Cambridge
  - Similar to Zenios model
  - Some effort in scenario generation
  - Unclear on no-arbitrage
  - Uses expected value of perfect information for tree “trimming”
  - Participation of London banks
- Mitra – Brunel
  - Similar in model to Dempster
  - Probably has multiple “shortcuts”
  - Unclear on price dynamics

More Examples (4)

- Frauendorfer – St. Gallen, CH
  - Other models just present approximation of sample paths
  - Uses bounding samples
  - Examples in income securities with interest rate dynamics
  - Unclear on no-arbitrage
- Klaassen – Erasmus, Netherlands
  - Approximations with no arbitrage
  - Still some lack of clarity on form of scenario generation
Objective Functions

• Previous examples
  – Forms of piecewise linear utility functions
  – Not clear whether consistent with financial evaluations
• Alternatives
  – Probability of beating benchmark..not coherent
• Coherent measures of risk (Heath et al.)
  – Lead to p.l. utility function forms
  – Expected downside risk or conditional value-at-risk
    (Uryasiev and Rockafellar)

Analysis and Critique

• Models
  – Not clear whether models consider all that is known, may have arbitrage possibilities
  – Generation of scenarios not clear, no universal methodology
  – Can include information about tails
  – Tax considerations have generally been avoided
  – P.L. utility versus more sophisticated.
Solution Method Critique

• Existing methods limited in size of problems
• No consistent estimators that are also efficient
• Decomposition methodology and combinations with DP approximation may have promise

Conclusions

• Static portfolio models have problems with:
  – benchmark targets
  – transaction costs and taxes
• Dynamic stochastic programming models can address difficulties
  – variety of objectives
  – can use structure to meet additional requirements
• Computation of large problems using decomposition and special structure