Building Consistent Asset-Liability Management Models

John R. Birge
Northwestern University

Background

• Interest in asset-liability management
  – Deciding how to allocate assets and what liabilities to incur to obtain best performance (meet liabilities and grow net assets)
  – Trillions of dollars in pension funds alone
  – Key function of insurance carriers
  – Endowments, corporations, individuals

• Why use stochastic programming?
  – Comprehensive
  – Specific

• Problems in models and methods
OUTLINE

• Overview of approaches
• Critiques
• General decomposition approach
• Extensions

Overview of Approaches

• Static and extensions
  – Consistency of objective
  – Transaction costs
• DP Policies
  – Restricted policies (optimal – feasible?) Portfolio replication (duration match)
• General methods (Stochastic programs)
  – Price path consistency
  – Objective consistency
  – Method consistency
Dynamic Programming Approach

- State: \( x_t \) corresponding to positions in each asset (and possibly price, economic, other factors)
- Value function: \( V_t (x_t) \)
- Actions: \( u_t \)
- Possible events \( s_t \), probability \( p_{st} \)
- Find:
  \[
  V_t (x_t) = \max -c_t u_t + \sum_{s_t} p_{st} V_{t+1} (x_{t+1}(x_t, u_t, s_t))
  \]

**Advantages**: general, dynamic, can limit types of policies

**Disadvantages**: Dimensionality, approximation of \( V \) at some point needed, limited policy set may be needed, accuracy hard to judge

**Consistency questions**: Policies optimal? Policies feasible? Consistent future value?

Other Restricted Policy Approaches

- Kusy-Ziemba ALM model for Vancouver Credit Union
- Idea: assume an expected liability mix with variation around it; minimize penalty to meet the variation
- Formulation:
  \[
  \min \sum c_i x_i + \sum p_{st} (q_{st}^+ y_{st}^+ + q_{st}^- y_{st}^-)
  \]
  \[
  \text{s.t. } \sum f_{st} x_i + y_{st}^+ - y_{st}^- = l_t \text{ all } t \text{ and } s; x_i, y \geq 0, i = 1 \ldots n
  \]

**Problems**: Similar to liability matching.

**Consistency questions**: Possible to purchase insurance at cost of penalties? Best possible policy?
General Methods

• Basic Framework: Stochastic Programming

• Model Formulation:

\[
\max \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right)
\]

s.t. (for all \( \sigma \)):
\[
\Sigma_{k} x(k,1,\sigma) = W(o) \text{ (initial)}
\]
\[
\Sigma_{k} r(k,t-1,\sigma) x(k,t-1,\sigma) - \Sigma_{k} x(k,t,\sigma) = 0, \text{ all } t > 1;
\]
\[
\Sigma_{k} r(k,T-1,\sigma) x(k,T-1,\sigma) - W(\sigma, T) = 0, \text{ (final)};
\]
\[
x(k,t,\sigma) \geq 0, \text{ all } k,t;
\]

Nonanticipativity:
\[
x(k,t,\sigma') - x(k,t,\sigma) = 0 \text{ if } \sigma', \sigma \in S, \text{ for all } t, i, \sigma', \sigma
\]

This says decision cannot depend on future.

Advantages:
General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, insight

Consistency questions: Price dynamics appropriate?
objective appropriate? Solution method consistent?

Model Consistency

• Price dynamics may have inherent arbitrage
  – Example: model includes option in formulation that is not the present value of future values in model (in risk-neutral prob.)
  – Does not include all market securities available

• Policy inconsistency
  – May not have inherent arbitrage but inclusion of market instrument may create arbitrage opportunity
  – Skews results to follow policy constraints

• Lack of extreme cases
  – Limited set of policies may avoid extreme cases that drive solutions
Objective Consistency

• Examples
  – Mean and variance
  – Probability of beating benchmark
  – Not coherent (basic consistency)
• Coherent measures of risk (Heath et al.)
  – Lead to piecewise linear utility function forms
  – Expected downside risk or conditional value-at-risk (Uryasiev and Rockafellar)

Model and Method Difficulties

• Model Difficulties
  – Arbitrage in tree
  – Loss of extreme cases
  – Inconsistent utilities
• Method Difficulties
  – Deterministic incapable on large problems
  – Stochastic methods have bias difficulties
    • Particularly for decomposition methods
    • Discrete time approximations
  – Stopping rules and time hard to judge
Resolving Inconsistencies

- Objective: Coherent measures
- Model resolutions
  - Construction of no-arbitrage trees (Klaassen)
  - Extreme cases (Generalized moment problems and fitting with existing price observations)
- Method resolutions
  - Use structure for consistent bound estimates
  - Decompose for efficient solution

Model Consistency

- Construct consistent scenarios with observed prices
- Find prices and scenarios to fit observed data and include extreme events (e.g., max probability of large decline)
- Format of general moment problem:

\[
\max \int_{\Xi} g(\xi) \, P(d\xi)
\]

over \( P \in \mathcal{P} \) a set of probability measures on \((\Xi, \mathcal{B}^\Xi)\) s.t.

\[
\int_{\Xi} v_i(\xi) \, P(d\xi) \leq \alpha_i, \quad i=1,\ldots,s,
\]

\[
\int_{\Xi} v_i(\xi) \, P(d\xi) = \beta_i, \quad i=s+1,\ldots,M
\]

where \( M \) is finite and the \( v_i \) are bounded, continuous functions.
Extremal Probabilities

- Problem: find $p_j$ to
  
  $\text{Max } \sum_{j | S_j \geq 55} p_j$
  
  s.t. $\sum_j p_j = 1$
  $\sum_j p_j (S_j - K_i) = \text{FV}(C(K_i, T))$
  $\sum_j p_j S_j = \text{FV}(S_i)$, $p_j \geq 0$

  For example, suppose $S_j = 30, 35, 40, 45, 50, 55, 60$ and
  Call values: $C(35)=10.3, C(40)=5.5, C(45)=2, C(50)=0.5$

  Can solve via generalized linear programming:
  
  Result:
  $\text{Prob}(S_T \geq 55)=0.10$

  Can extend to find sets of probabilities and ranges for consistent choices, finite realizations.

Method Consistency: Abridged Nested Decomposition

- Incorporates sampling into the general framework of Nested Decomposition
- Samples both the subproblems to solve and the solutions to continue from in the forward pass of nested decomposition
- Eliminates inconsistency by use of deterministic lower bound and re-sampled upper bound (consistent check of optimality on each iteration)
Abridged Nested Decomposition

**Forward Pass**
1. Solve root node subproblem
2. Sample Stage 2 subproblems and solve selected subset
3. Sample Stage 2 subproblem solutions and branch in Stage 3 only from selected subset (i.e., nodes 1 and 2)
4. For each selected Stage t-1 subproblem solution, sample Stage t subproblems and solve selected subset
5. Sample Stage t subproblem solutions and branch in Stage t+1 only from selected subset

---

Abridged Nested Decomposition

**Backward Pass**
1. Starting in first branching node of Stage $t = N-1$, solve all Stage $t+1$ descendant nodes and construct new optimality cut for all stage $t$ subproblems. Repeat for all sampled nodes in Stage $t$, then repeat for $t = t - 1$

**Consistent Convergence Test**
1. Randomly select $H$ $N$-Stage scenarios. For each sampled scenario, solve subproblems from root to leaf to obtain total objective value for scenario
2. Calculate statistical estimate of the first stage objective value $\bar{z}$
   - algorithm terminates if current first stage objective value $c_1 x_1 + \theta_1$ is within a specified confidence interval of $\bar{z}$ else, a new forward pass begins
**Additional Features for Portfolio Problems**

- **Serial independence**
  - Increments are generally serial
  - Formulation is complex to address problem directly
  - Slows computation speed
- **Using structure to relax serial independence**
  - Can still use structure but assume some correlation of returns over time
  - Based on state space determining future price trajectory

---

**Sample Computational Results**

![CPU Time (seconds)](chart)
Conclusions

• Problems with arbitrage, coverage of paths, objective consistency and method consistency
• With some effort, models and methods can become consistent (so effort goes to efficiency)
• Moment problem solutions can give consistent sample paths
• Abridged nested decomposition provide consistent optimality checks
• Needs:
  – Extensions for serial correlation
  – Testing for early termination
  – Effective methods for taxable portfolios and nonconvexities