Stochastic Optimization in Asset-Liability Management

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OUTLINE

• Overview of approaches
• Mean-variance extensions (Mulvey)
• Duration matching and scenarios (Dembo)
• Bond portfolio versus other
• DP methods
• Single policy plus penalties (Kusy-Ziemba)
• Multiple PL objective (Russell-Yasuda-Ziemba)
• Alternative objectives (coherent risk – Heath, et al.)
• General model approaches (Zenios, Dempster, Mitra, Frauendorfer)

• Analysis and critique
Overview of Approaches

• Extensions of static:
  – Mean-variance based
  – Rebalancing improvement
  – No lock-in/Loss on transaction fees

• Dynamic extensions of static
  – Fixed proportion plus trading window on static

• Portfolio replication (duration match)
• DP policy based
• Stochastic program based
Static Portfolio Model

Traditional model

– Choose portfolio to minimize risk for a given return
– Find the efficient frontier

Quadratic program (Markowitz):

find investments $x=(x(1),\ldots,x(n))$ to

$$\min x^T Q x$$

s.t. $r^T x = \text{target}$, $e^T x=1$, $x\geq 0$. 
Static Portfolio Model

1. Markowitz model
   - Choose portfolio to minimize risk for a given return
   - Find the efficient frontier
Static Model Results

For a given set of assets, find

- fixed percentages to invest in each asset
- maintain same percentage over time
- implies trading but gains over “buy-and-hold”

Needs

- rebalance as returns vary
- cash to meet obligations

Problems

- transaction costs
- cannot lock in gains
- tax effects
Example: Retirement Planning

- **GOAL**: Accumulate $G_Y$ years from now

- **Assume**:
  - $W(0)$ - initial wealth
  - $K$ - investments
  - concave utility (piecewise linear)

Note: Similar to meeting a target or benchmark
1 \textbf{Find efficient frontier:}

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    title={Return},
    ylabel=Return,
    xlabel=Return,
    xmin=0.014, xmax=0.155,
    ymin=0.115, ymax=0.155,
    xtick={0.014, 0.029, 0.054, 0.081, 0.108, 0.134},
    ytick={0.115, 0.12, 0.125, 0.13, 0.135, 0.14, 0.145, 0.15},
]
\addplot[mark=none, blue, line width=1.5pt]
coordinates {
(0.014, 0.115) (0.029, 0.12) (0.054, 0.125) (0.081, 0.13) (0.108, 0.135) (0.134, 0.15)
};
\end{axis}
\end{tikzpicture}
\end{center}
Results with Static Model

1. Fixed proportion in stock and bonds in each period
2. 80% stock for 15% return
3. 40% stock for 14% return
4. Results: no fixed proportion achieves target better than 50% of time
5. Dynamic achieves target 87.5% of time
Alternative Dynamic Model

1. Assume possible outcomes over time
   - discretize generally
2. In each period, choose mix of assets
3. Can include transaction costs and taxes
4. Can include liabilities over time
5. Can include different measures of risk aversion
Formulation with No Transactions Fees

• **SCENARIOS:** ?? ??
  - Probability, \( p(?) \)
  - Groups, \( S^t_1, ..., S^t_{St} \) at \( t \)

• **MULTISTAGE STOCHASTIC NLP FORM:**

\[
\text{max } \sum_k p(?) W(?, T) \\
\text{s.t. (for all ?): } \quad \begin{align*}
    & x(k,1,?) = W(o) \quad \text{(initial)} \\
    & \quad \begin{cases} 
        & r(k,t-1,?) x(k,t-1,?) - x(k,t,?) = 0, \quad \text{all } t > 1; \\
        & r(k,T-1,?) x(k,T-1,?) - W(?, T) = 0, \quad \text{(final)}; \\
    & x(k,t,?) \geq 0, \quad \text{all } k,t;
    \end{cases}
\end{align*}
\]

**Nonanticipativity:**

\[
x(k,t,?) - x(k,t,?) = 0 \quad \text{if } ?, ?, ? S^t_i \text{ for all } t, i, ?, ?
\]

This says decision cannot depend on future.
DATA and SOLUTIONS

**ASSUME:**
- $Y=15$ years
- $G=80,000$
- $T=3$ (5 year intervals)
- $k=2$ (stock/bonds)

**Returns (5 year):**
- Scenario A: $r(\text{stock}) = 1.25$  $r(\text{bonds})= 1.14$
- Scenario B: $r(\text{stock}) = 1.06$  $r(\text{bonds})= 1.12$

**Solution:**

<table>
<thead>
<tr>
<th>PERIOD</th>
<th>SCENARIO</th>
<th>STOCK</th>
<th>BONDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1-8</td>
<td>41.5</td>
<td>13.5</td>
</tr>
<tr>
<td>2</td>
<td>1-4</td>
<td>65.1</td>
<td>2.17</td>
</tr>
<tr>
<td>2</td>
<td>5-8</td>
<td>36.7</td>
<td>22.4</td>
</tr>
<tr>
<td>3</td>
<td>1-2</td>
<td>83.8</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3-4</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>3</td>
<td>5-6</td>
<td>0</td>
<td>71.4</td>
</tr>
<tr>
<td>3</td>
<td>7-8</td>
<td>64.0</td>
<td>0</td>
</tr>
</tbody>
</table>
Analysis of Dynamic Model

• With discrete outcomes, p.l. utility:
  – Optimal solution has number of investments at most equal to number of branches in each period
  – Constrain the number of positive investments with the number of outcomes per period

• Impact of transaction fees and taxes
  – Additional constraints
  – Creates potential for more active investments in each period
  – Additional constraints can be imposed with linearization (representation other variance information)
  – Number of constraints can be used to limit number of investments
Other Model Gains

1. Can include transaction costs
   - Fixed proportion requires transaction costs each period just to re-balance
   - can accumulate

2. Can include tax considerations
   - Model size grows (lots of each asset)

3. Maintain consistent utility
Uses of Mean-Variance?

• Examples: John Mulvey/Lattice Financial
  – Towers Perrin
  – California pension
  – GM?

• Advantage?
  – Improvement over buy and hold can be large
  – Rebalancing is key to maintaining diversification

• Disadvantages
  – No lock-in (relative to benchmark)
  – Transaction costs may be large (can use no-trade region but no fixed criteria in multiple dimensions
  – Hard to generalize for taxes
  – Nonconvex optimization (approximated)
Duration Matching and Generalizations

- Duration matching
  - Idea to match first derivative (and with convexity) second derivative
- Formulation:
  Given duration $d$, convexity $v$ and maturity $m$ of target security or liability pool, find investment levels $x_i$ in assets of cost $c_i$ to:

  $$\min S_i c_i x_i$$

  s.t. $S_i d_i x_i = d; S_i v_i x_i = v;$
  $S_i m_i x_i = m; x_i \geq 0, i = 1 \ldots n$

- Extensions: (Dembo – Algorithmics)
  - Put in scenarios for the durations.. extend their application
- Problems:
  - Maintaining position over time
  - Asymmetry in reactions to changing (non-parallel yield curve shifts)
  - Assumes assets and liabilities face same risk
Extension to Liability Matching

• Idea (Black et al.)
  – Best thing is to match each liability with asset
  – Implies bonds for matching pension liabilities
• Formulation:
  Suppose liabilities are $l_t$ at time and asset $i$ has cash flow $f_{it}$ at time, then the problem is:
  \[
  \begin{align*}
  \min & \quad S_i \ c_i \ x_i \\
  \text{s.t.} \quad S_i \ f_{it} \ x_i &= l_t \quad \text{all } t; \quad x_i \geq 0, \ i = 1\ldots n
  \end{align*}
  \]
• Advantages:
  – Liabilities matched over time
  – Can respond to changing yield curve
• Disadvantages
  – Still assumes same risk exposure
  – Does not allow for mix changes over time
Further Extensions to Liability Matching

• Include scenarios s for possible future liabilities and asset returns

• Formulation:

\[
\begin{align*}
\min & \quad S_i \ c_i \ x_i \\
\text{s.t.} & \quad S_i \ f_{its} \ x_i = l_{ts} \text{ all } t \text{ and } s; \ x_i \geq 0, \ i = 1\ldots n
\end{align*}
\]

• If not possible to match exactly then include some error that is minimized.

• Allows more possibilities in the future, but still not dealing with changing mixes over time.

• Also, does not consider possible gains relative to liabilities which can be realized by volatility pumping and locking in
Extended Policies – Dynamic Programming Approaches

- Policy in duration and liability matching:
  - Fixed mix or fixed set of assets
  - No trading or dynamics
- DP allows broader set of policies
- Problems: Dimensionality, Explosion in time
- Remedies: Approximate (Neuro-) DP
- Idea: approximate a value-to-go function and possibly consider a limited set of policies
Dynamic Programming Approach

- State: $x_t$ corresponding to positions in each asset (and possibly price, economic, other factors)
- Value function: $V_t(x_t)$
- Actions: $u_t$
- Possible events $s_t$, probability $p_{st}$
- Find:

$$V_t(x_t) = \max -c_t u_t + S_{st} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t))$$

**Advantages:** general, dynamic, can limit types of policies

**Disadvantages:** Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
Other Restricted Policy Approaches

• Kusy-Ziemba ALM model for Vancouver Credit Union

• Idea: assume an expected liability mix with variation around it; minimize penalty to meet the variation

• Formulation:

$$\min \sum_{i} c_i x_i + \sum_{st} p_{st} (q_{st}^+ y_{st}^+ + q_{st}^- y_{st}^-)$$

s.t. $$\sum_{i} f_{its} x_i + y_{st}^+ - y_{st}^- = l_{ts} \text{ all } t \text{ and } s; \ x_i y >= 0, i = 1...n$$

Problems: Similar to liability matching.
General Methods

• Basic Framework: Stochastic Programming

• Model Formulation:

\[
\begin{align*}
\text{max} \quad & \quad \sum_{i} p_i U(W(\omega, T)) \\
\text{s.t. (for all } \omega & \text{): } \sum_{k} x(k,1, \omega) = W(0) \text{ (initial)} \\
& \quad \sum_{k} r(k,t-1, \omega) x(k,t-1, \omega) - \sum_{k} x(k,t, \omega) = 0, \text{ all } t > 1; \\
& \quad \sum_{k} r(k,T-1, \omega) x(k,T-1, \omega) - W(\omega, T) = 0, \text{ (final)}; \\
& \quad x(k,t, \omega) \geq 0, \text{ all } k,t;
\end{align*}
\]

**Nonanticipativity:**

\[
x(k,t, \omega') - x(k,t, \omega) = 0 \text{ if } \omega, \omega' \in S_i \text{ for all } t, i, \omega', \omega
\]

This says decision cannot depend on future.

**Advantages:**

General model, can handle transaction costs, include tax lots, etc.

**Disadvantages:** Size of model, computational capabilities, insight into policies
Examples of General Models

• Frank Russell-Ziemba-Yasuda Marine
  – Large model for insurance
  – Many side constraints on policies
  – Branching with ~10 initial branches then fewer in future periods
  – No arbitrage on branches?
  – Few total branches
  – Unclear on solution value (compared to Mean-Variance)
  – Current state unclear
More Examples (2)

- Zenios: Cyprus/Wharton
  - Multiple models – insurance, mortgages
  - General form with some sophistication in solution approaches (parallel etc.)
  - Improvement relative to fixed strategies
  - Comparisons on “efficient frontier”
  - Unclear on price dynamics
  - Hard to draw overall policy conclusions
More Examples (3)

• Dempster – Cambridge
  – Similar to Zenios model
  – Some effort in scenario generation
  – Unclear on no-arbitrage
  – Uses expected value of perfect information for tree “trimming”
  – Participation of London banks

• Mitra – Brunel
  – Similar in model to Dempster
  – Probably has multiple “shortcuts”
  – Unclear on price dynamics
More Examples (4)

- Frauendorfer – St. Gallen, CH (UBS?)
  - Other models just present approximation of sample paths
  - Uses bounding samples
  - Examples in income securities with interest rate dynamics
  - Unclear on no-arbitrage

- Klaassen – Erasmus, Netherlands
  - Approximations with no arbitrage
  - Still some lack of clarity on form of scenario generation
Objective Functions

• Previous examples
  – Forms of piecewise linear utility functions
  – Not clear whether consistent with financial evaluations

• Alternatives
  – Probability of beating benchmark..not coherent

• Coherent measures of risk (Heath et al.)
  – Lead to p.l. utility function forms
  – Expected downside risk or conditional value-at-risk
    (Uryasiev and Rockafellar)
Analysis and Critique

• Models
  – Not clear whether models consider all that is known, may have arbitrage possibilities
  – Generation of scenarios not clear, no universal methodology
  – Can include information about tails
  – Tax considerations have generally been avoided
  – P.L. utility versus more sophisticated.
Solution Method Critique

• Existing methods limited in size of problems
• No consistent estimators that are also efficient
• Decomposition methodology and combinations with DP approximation may have promise
Conclusions

• Static portfolio models have problems with:
  – benchmark targets
  – transaction costs and taxes

• Dynamic stochastic programming models can address difficulties
  – variety of objectives
  – can use structure to meet additional requirements

• Computation of large problems using decomposition and special structure