Decomposition Methods for Nonlinear Stochastic Integer Programs

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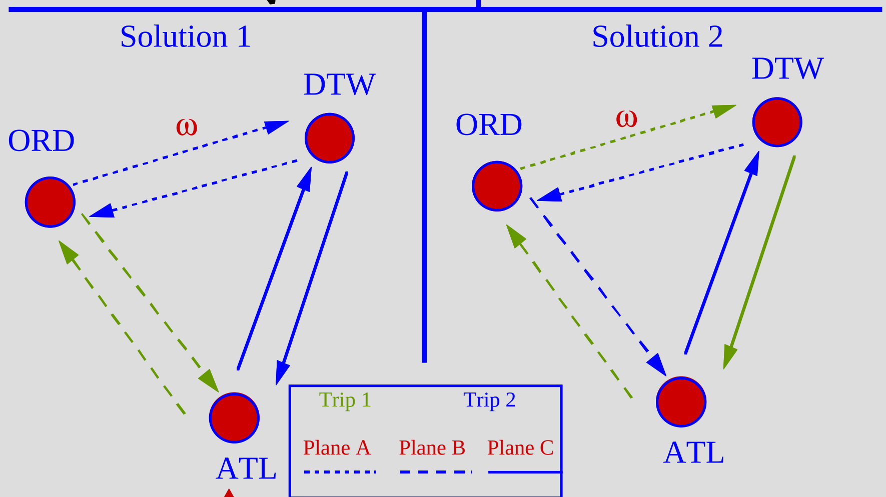
Presentation Overview

- Motivating Problem
- Example
- Decomposition Methods
- Future Work

Motivating Problem

- Airline Crew Scheduling
  - Extensively studied
  - Currently able to find good solutions to crew scheduling problems when given all information
  - What happens when delays are encountered?

Example



Problem Formulation

For each disruption ω, we have the following:

$$\begin{aligned} \text{minimize} \quad & c^T x + \mathcal{Q}(x) \\ \text{subject to} \quad & Ax = b \\ & 0 \le x \le 1 \\ & x \text{ integer} \end{aligned}$$

where $\mathcal{Q}(x) = \int_{\omega} Q(x,\omega) P(d\omega)$

is the expected value of future actions due to disruptions ω in the original schedule

Decomposition Methods

- Frank Wolfe Method of Feasible Directions
- Quadratic approximation of polynomial recourse function
- Benders Decomposition with cutting planes on linear program recourse form

Frank-Wolfe Method

- Begin with some feasible set of pairings, x_k, and find feasible point, y, that minimizes the gradient of the objective function evaluated at x_k
- next interate x_{k+1} is a convex combination of x_k and y which minimizes the objective function
- Problem: difficult to interpret fractional solutions

Quadratic Approximation

- Can construct delay recourse function $\mathcal{Q}(x)$ analytically as polynomial for given disruption ω
  - based on branching of disruption effects
- for each ω, minimize cost of delay subject to flight coverage constraints

Quadratic Approximation (2)

- Problems
  - quadratic approximations do not work well because delay function magnitude quickly grows
  - nonconvexity of polynomial form

Bender's Decomposition Method

- Can contruct an LP to find a recourse cost
  - Given the LP, want to construct cutting planes and derive a subgradient from repeated solutions to linear program under different delay scenarios
- Problem: x enters into the constraints directly
  - see ** on following slide
  - in general $\mathcal{Q}(x)$ is not convex

Recourse LP Formulation

$$Q(x,\omega) = \min \sum_{j \text{ segments}} penalty \cdot delay(j)$$

subject to:

$$time\_arr(j) - time\_dep(j) \geq time(j,\omega)$$

$$time\_arr(j) - delay(j) \leq sched\_arr(j)$$

$$time\_dep(j) - delay(j) \geq sched\_dep(j)$$

$$time\_dep(j) - time\_arr(plane\_pred(j)) \geq plane\_grd(j,\omega)$$

$$time\_dep(j) - \sum_{j \in pairing\ k} time\_arr(crew\_pred(j,k)\text{-}crew\_grd(j,k))x_k \geq 0^{**}$$

$$delay(j) \geq 0$$

ω : delay scenario   j : flight   k : round trip itinerary

General form of LP

$$Q(x,\omega) = \min q^T y$$

$$s.t. \quad (W + Gx)y = h(\omega)$$

$$y \geq 0$$

- This form is not convex in general in x
- Does the structure of the crew problem yield convexity?

Dealing with Nonlinearity

- Want to know the convexity of the delay function
  - Can use the recourse LP formulation for given solutions x to construct pseudogradient.
    - use finite different methods
    - inconclusive, some instances of nonconvexity
  - Example of nonconvexity for polynomial form

Convexity of LP

- Suppose an optimal solution y_1 to the recourse LP includes *time_dep(j,1)* given an input x_1 and optimal y_2 includes *time_dep(j,2)* given an input x_2
- Consider a convex combination of y_1 and y_2, $\lambda y_1 + (1-\lambda) y_2$
  - feasible for all constraints without x
  - for constraint ** note:

Convexity of LP (cont)

$$time\_dep(j) - \sum_{j \in pairing\ k} time\_arr(crew\_pred(j,k) - crew\_grd(j,k))x_k \geq 0 \quad **$$

- Rewrite as $time\_dep(j) \geq \sum_{j \in pairing\ k} G(j,k)x_k$
- Then

$$\begin{aligned} time\_dep(j,1) + (1-\lambda)time\_dep(j,2) &\geq \lambda G(j,k_1)x_{k_1} + (1-\lambda)G(j,k_2)x_{k_2} \\ &= \lambda \sum_{j \in pairing\ k_1} G(j,k_1)x_{k_1} + (1-\lambda) \sum_{j \in pairing\ k_2} G(j,k_2)x_{k_2} \\ &= \sum G(j,k)(\lambda x_{k_1} + (1-\lambda)x_{k_2}) \end{aligned}$$

- Hence, $\lambda y_1 + (1-\lambda) y_2$ is feasible for input $\lambda x_1 + (1-\lambda) x_2$, but not necessarily optimal.

Convexity of LP (cont)

- So,

$Q(\lambda x_1 + (1-\lambda) x_2) <= \lambda Q(x_1) + (1-\lambda) Q(x_2)$

gives convexity.

- Result: Convexity holds with the LP formulations but not the polynomial form. Both values agree on integer x points, so the LP formulatoin is still valid and can be used for optimization.

Example of Convexity

Towards an Algorithm

- Given an input x, can construct a valid subgradient for Q(x)
- Add a constraint (cut) to the master crew scheduling problem in x
  - this creates a crew scheduling problem with additional constraints and additional variable(s) corresponding to Q(x)

Algorithm (cont)

$$\begin{aligned} \min \quad & c^T x + \theta \\ s.t. \quad & Ax = b \\ & Ex + \theta \geq e \qquad \text{(these are additional constraints)} \\ & x \text{ binary} \end{aligned}$$

- Constraints are added until $\theta^k = Q(x^k)$ as in standard Benders' or L-shaped methods

Future Work

- Refine algorithm - subgradient construction
- Testing with actual data
- Generalization to other network design problems with similar structure

Recourse LP Formulation

$$Q(x,\omega) = \min \sum_{j \text{ segments}} penalty \cdot delay(j)$$

subject to:

$$\begin{aligned}
time_arr(j) - time_dep(j) &\geq time(j,\omega) \\
time_arr(j) - delay(j) &\leq sched_arr(j) \\
time_dep(j) - delay(j) &\geq sched_dep(j) \\
time_dep(j) - time_arr(plane_pred(j)) &\geq plane_grd(j,\omega)
\end{aligned}$$

$$time\_dep(j) - \sum_{j \in pairing\ \mathrm{k}} time\_arr(crew\_pred(j,k) - crew\_grd(j,k))x_k \geq 0 \quad **$$

$$delay(j) \geq 0$$

ω : delay scenario j : flight k : round trip itinerary