

Optimal Bidding Strategies in Non-Sealed Bid Online Auctions of Common Products with Quantity Uncertainty *

Chonawee Supatgiat[†]
Research Group, Enron Corporation
Houston, TX 77002

John R. Birge
Department of Industrial Engineering and Management Sciences
Northwestern University, Evanston, IL 60208

Rachel Q. Zhang
Johnson Graduate School of Management
Cornell University, NY 14850

December 7, 2001

Abstract

We consider non-sealed bid online auctions of common products with quantity uncertainty. Both first-price (also known as pay-as-you-bid) and uniform-price auctions are considered. In these auctions, all bidders have the same valuation of the products but may have different demand quantities. The number of units being auctioned can be random with a known and common distribution. Each bidder decides on a bidding price to maximize her profit. We derive Nash equilibrium solutions, i.e., bidders' optimal bidding strategies, and the resulting market clearing prices.

*This work was supported in part by the National Science Foundation under grants DMI-9501740 and DMI-9523275 and the University of Michigan Rackham Predoctoral Fellowship.

[†]Current affiliation: RWE Trading Americas, Houston, TX 77010

Keywords: multi-unit auctions, pure common value auction, discrete bid level, bid increment, E-commerce

1 Introduction

Auctions have been widely used as channels to sell unique products, e.g., paintings, artwork, junk cars, and fish. By selling these products via public auctions, the sellers can be assured of competitive prices for their products. Because of the uniqueness of the products, the sellers actually need to show the bidders the products they are selling at the sites. Moreover, due to the characteristics of traditional auctions, buyers and sellers must be present at the sites to submit bids and make decisions. Such inconvenience is believed to be one of the reasons that many common products with standard quality, such as books, CDs, computers, electronic appliances, and airline tickets, were not usually sold through auctions.

Over the past few years, advanced information technology and e-commerce have enabled individuals and organizations to exchange goods and services through the internet. The internet provides business and customer transactions with flexibility, convenience, accuracy and real-time interactions among sellers and potential buyers at significantly lower transaction and overhead costs than traditional retail trade. Thus, more and more businesses and customers are transacting online. Some of the web sites, such as ebay.com, Amazon.com, Yahoo! Auction, onsale.com, uBid.com, and dealbench.com, provide auction service through the internet (referred to as online auctions). One of the advantages of online auctions over traditional auctions is that buyers and sellers can make transactions on any computer that has internet access. Since online buyers cannot actually see the product physically to help evaluating the product, online auctions are well suited for common products, of which the quality is standard and the valuation is common.

Note that online auctions are not only outlets for trading common products, they are also used for products that are traditionally traded on auction houses such as used cars and diamond rings at web sites like ebaymotors.com and sothebys.com. In this case, the sellers usually provide detailed descriptions of the products and their photographs.

There are two main types¹ of auctions, selling auctions and buying auctions. In selling auctions, *bidders* are producers or suppliers competing with each other to sell their products to a buyer or consumer. In buying auctions, buyers or consumers bid to buy products from a supplier or producer. Most online auctions are buying auctions. In a buying online auction, the seller posts the description of the product being sold and the quantity that is available. The seller may also set a minimum acceptable bid, minimum bid increment, and closing time. The buyers then submit bids. If there is only one unit, the highest-price bidder wins. If there are q units to be sold, the units go to the bidders with the q highest bids. There are several rules to break a tie. The most commonly used rule is to award the units to the bidders who submitted their bids earlier.

There are two commonly used auction mechanisms that determine the buyers' payments to the seller, first-price (i.e., pay-as-you-bid) and uniform price auctions. In a first-price (or pay-as-you-bid) auction, units are sold to the winners at their bid prices; while, in a uniform pricing auction, units are sold at the market clearing price – the lowest successful bid. (Vickrey [12] is known for introducing uniform price auctions where the market clearing price in a multi-unit auction is the highest *unsuccessful* bid.) Ebay.com and Amazon.com run both first-price and uniform-price auctions where the uniform-price auction is referred to as “Dutch Auction”. The seller can choose between the first-price or uniform-price formats.

Most online auctions are non-sealed where all bids are disclosed to public. This transparency of bid disclosure encourages people to participate in the auctions. Currently, most online auctions are conducted in a non-sealed fashion. There are some auction sites, e.g., ebay.com and dealbench.com, that runs sealed bid auctions where the bidders do not have knowledge of others' bids.

Due to easy access to the information via the internet, price gaps among different online stores are becoming very small. Some web sites (e.g., pricewatch.com, dealpilot.com, netbuyer.com, altavista.com) provide comparison shopping by displaying the prices offered by various online stores. Such service further reduces price discrepancies, thus promoting an “efficient market” or “one price for one product”. As a result, in auctions of common products, valuations of a particular item from buyers should be very close, if not the same;

¹Another auction type is Double Auction where both buyers and sellers bid in the auction

otherwise, buyers can always go to other stores for lower prices. The valuation represents the price that the buyers can get elsewhere outside the auction side. On the other hand, the valuation also represents the selling price that the auction winners can re-sell the items elsewhere outside the auction market.

We assume that each participant behaves in a way to maximize her own expected profit. The goal of this paper is to find optimal bidding strategies for bidders and the resulting Nash equilibrium (or simply equilibrium) market spot price in non-sealed bid online auctions. In this paper, the bidders have the same valuation for the common product being sold in the online auction and this valuation is commonly known to the bidders, i.e. a pure common value auction [2]. Both first-price and uniform-price auctions are considered.

Almost all non-sealed online auctions have a minimum increment rule to avoid infinite adjustments of the bid prices and allow the auction to close at a reasonable time. Ebay.com and Yahoo.com have 50c bid increment for the current winning bid between \$5-\$24.99 in a buying auction. This means that the new bid must be at least \$5.50 if the current winning bid is \$5. The minimum bid increment rule motivates us to assume a finite and discrete set of possible bid prices, which represents a departure from most previous work that assumes continuous bidding space. With a continuous bidding space, one allows bidders to undercut each other's bids and, therefore, destroys most potential Nash equilibria that can be obtained from a discrete set assumption. However, there has been limited research on bidding in a discrete bid space. Yamey [13] investigates the effects of some bid increment rules and Rothkopf and Harstad [8] derive the bid increment level that optimizes the trade-off between the auctioneer's revenue and the cost associated with the auction duration.

In most auctions, there are irrational bidders and uninformed bidders. Irrational bidders are those who try to win an auction even if they have to bid higher in a buying auction (lower in a selling auction) than the actual value of the product. Uninformed bidders are bidders who don't have the correct valuation of the common product being auctioned. To incorporate irrational and uninformed bidders into our models, we introduce a random variable, called *supply* in a buying auction and *demand* in a selling auction, which is defined as the remaining units after subtracting the units that go to uninformed and irrational bidders. Because the quantity of irrational and uninformed bidders is uncertain, supply or demand is uncertain.

We assume that the probability distribution of the supply or demand is common to all bidders. In a selling auction, the demand is in-elastic where all demand must be satisfied. If demand is elastic, the market becomes an exchange. Likewise, the supply in a buying auction is in-elastic.

Our previous work [9] provides a detailed analysis of equilibrium points in a non-sealed bid uniform-price market, where bidders have distinct costs (or valuations), under both stochastic and deterministic demand assumptions. Unlike that paper, here we focus on the case where all bidders have the same valuation of the product. In this case, the previous results do not apply. We present our models in Section 2. The characteristics of Nash equilibrium points in non-sealed bid first-price auctions and uniform-price auctions are derived in Sections 3 and 4, respectively. The paper concludes in Section 5.

2 Model Description

We first point out that there is no analytical difference between selling auctions and buying auctions. In selling auctions, each bidder has a cost c (or value v) of the product, which represents the price that the bidders can get elsewhere outside the auction site or the buying price that the auction winners can buy from other markets to cover their short positions. The profit of a seller is equal to the selling price π that the seller gets from the auction subtracted by the cost. In buying auctions, the cost is the negative valuation of the product, and the profit is the negative purchasing price she gets from the auction minus the negative valuation of the product, or $(-\pi) - (-v)$. Hence, without loss of generality, our terminology used throughout the rest of the paper is based on selling auctions.

Both stochastic and deterministic versions of the demand are considered. Demand is denoted by d in the deterministic case and by a random variable D in the stochastic case. We assume also that all bidders have the same belief regarding the demand distribution, which we denote by F_D . The realization of the demand in the stochastic case is also denoted by d .

There are N bidders; each submits a quantity of units for sale and an acceptable price for a unit of this product, called the *bid price*. Every bidder i has the same cost c for each unit

and bids for a fixed quantity x_i at a price p_i . We denote the bid vector $[(x_1, p_1), \dots, (x_N, p_N)]$ by \mathbf{b} . If a bidder wins an auction, her selling quantity is called a *dispatched quantity*. If a bidder cannot sell any units, she is called an *undispatched bidder*. Bidders who offer lower prices are dispatched first. The goal of bidder i is to choose x_i and p_i so that her expected profit is maximized.

In first-price auctions, each winner gets paid at her own bid price. In uniform-price auctions, every winner gets paid at the same price – the market clearing price (MCP) or *market spot price*, which is the highest bid price among the dispatched units. When demand D is revealed, the market clearing price π , which is a function of the demand realization d and bid vector $\mathbf{b} = [(x_1, p_1), \dots, (x_N, p_N)]$, can be computed mathematically as

$$\pi(\mathbf{b}, d) = \left\{ \min_j p_j : \sum_{i \in I(j)} x_i \geq d \right\}, \quad (1)$$

where $I(j) = \{i : p_i \leq p_j\}$.

A bidder whose bid price is equal to the market-clearing price is called a *marginal bidder*, while a bidder whose bid price is less than the market-clearing price is called an *under-bidder*. Clearly, any under-bidder is fully dispatched, i.e., the market consumes x_i . On the other hand, a bidder i whose price p_i is higher than the spot price π is not dispatched.

The most commonly used method in determining the dispatch quantities of the bidders bidding at the same price is the *FIFO tie-breaking* method, where the bid with earlier time-stamp gets a higher priority than later time-stamp bids. Bidders can adjust their bids as many times as they prefer, but the time-stamp on the bid is the time of its last modification. This method is used in Yahoo.com, EBay.com, and Amazon.com.

There is another tie-breaking method, called *fixed FIFO tie-breaking*, where bidders can adjust their bids without changing their time stamp. In this method, the bidder who bids first at any price when the market is opened will always have the highest priority among the other bidders bidding at the same price when the market closes. UBid.com currently uses this tie-breaking method. Onsale.com, one of the pioneers in online auction, was also using this method before it was bought out. One supporting argument for using this method might be that it encourages early participation. We, however, show later in the paper that this method should not be used in online auctions.

Since both tie-breaking rules are time sensitive, we consider the *random tie-breaking* method, where the same-price bidders are dispatched randomly. Hence, each of the tie bidders has equal chance of being dispatched, regardless of their bid submission time. It is clear that this rule encourages bidders who would otherwise not bother to participate if they are late already. In an independent study, the authors found that the random tie-breaking rule is preferred by the buyer in a sealed-bid selling auction. We will show later in this paper that the random tie-breaking rule yields reasonable market clearing prices in uniform price auctions. Let \mathcal{S} be the set of marginal bidders $\{i|p_i = \pi\}$. The notation $|\mathcal{S}|$ denotes the number of bidders in \mathcal{S} . We denote by $x(\mathcal{S})$ the set of x_i submitted by marginal bidders; i.e., $x(\mathcal{S}) = \{x_i|i \in \mathcal{S}\}$.

In order to compute the expected dispatch of bidder i under the random tie breaking, we first define the recursive function

$$\bar{q}_i(d', x(\mathcal{S}), n) = \begin{cases} 0 & \text{if } d' \leq 0, \\ \frac{1}{n} \min(x_i, d') + \frac{1}{n} \sum_{j=1, j \neq i}^n \bar{q}_i(d' - x_j, x(\mathcal{S} - \{j\}), n - 1) & \text{otherwise.} \end{cases}$$

where d' and n are real number arguments of the function.

Then, the expected dispatch quantity of bidder i for a given \mathbf{b} and d is

$$q_i(\mathbf{b}, d) = \begin{cases} 0 & \text{if } p_i > \pi(\mathbf{b}, d), \\ x_i & \text{if } p_i < \pi(\mathbf{b}, d), \\ \bar{q}_i(d - \sum_{j \in \{i|p_i < \pi(\mathbf{b}, d)\}} x_j, x(\mathcal{S}), |\mathcal{S}|) & \text{if } p_i = \pi(\mathbf{b}, d). \end{cases} \quad (2)$$

If $E_D[q_i(\mathbf{b}, D)] = x_i$, bidder i is *fully dispatched*. Otherwise, when $E_D[q_i(\mathbf{b}, D)] < x_i$, bidder i is *partially dispatched*. The expected dispatch quantity of bidder i under FIFO tie-breaking method, for a given \mathbf{b} and d , is

$$q_i(\mathbf{b}, d) = \min \left(x_i, d - \sum_{j \in \{l|l < i, p_l \leq p_i\}} x_j \right). \quad (3)$$

where the bids are labeled in an ascending order of their time stamps.

In a non-sealed bid market, each bidder knows the other bidders' bid prices. At each iteration, the bidders can adjust their bid prices from the previous iteration. In most online auctions, if the proposed closing time is reached but there are still bidders bidding, auctions usually remain open until no one wants to adjust her bid. Therefore, we assume that they

can adjust their bid prices as many times as they prefer and the market is closed when no bidder wants to adjust her bid. If everyone maximizes her own expected profit, the market will be closed at a Nash equilibrium point as in (6). In other words, the bidders continue to adjust their bids until a Nash equilibrium is reached. Since all the bids are revealed to the public, all bidders play pure strategy and the market is closed at a pure strategy Nash equilibrium point. We know that one equilibrium point in a non-sealed bid market is a point where every bidder bids at the cost c . However, the bidders prefer not to play this equilibrium point because no one gains any profit.

In a non-sealed bid market where bidders play pure strategies, bidder i 's expected payoff is

$$f_i(\mathbf{b}) = E_D[(\pi(\mathbf{b}, D) - c) q_i(\mathbf{b}, D)], \quad (4)$$

in uniform-price auctions, and

$$f_i(\mathbf{b}) = E_D[(p_i - c) q_i(\mathbf{b}, D)], \quad (5)$$

in first-price auctions.

We search for a Nash equilibrium, $\{p_i^*, i = 1, \dots, N\}$, such that

$$f_i([(x_1, p_1^*), \dots, (x_i, p_i), \dots, (x_N, p_N^*)]) \leq f_i([(x_1, p_1^*), \dots, (x_i, p_i^*), \dots, (x_N, p_N^*)]) \quad (6)$$

for all feasible bid prices, $p_i, i = 1, \dots, N$.

In most non-sealed bid markets, the bidding rules indicate that a new bid price must be less than the previous winning bid minus a small margin or bid increment. Therefore, we assume that the bid price must be in the discrete set $\{l\epsilon \mid l = 0, 1, \dots, O\}$, where O is a large integer and ϵ is a small number. The discrete bid price assumption is a change from previous analysis, which assumes a continuous set of possible bid prices. The discrete version, however, gives us insight into the structure of the equilibrium and reflects reality. We assume also that the integer O is chosen large enough so that $O\epsilon > c + 2\epsilon$. Furthermore, for simplicity, we assume that c takes value in the possible bid price set. Hence, bidders can bid at their own cost c .

Since there are two stochastic parts in the model, the definition of an *under* bidder is valid only when the demand and the prices are realized; however, to be used in the general

case, the definition of a *marginal* bidder is adjusted to be the one who bids at the market clearing price in at least one realization of demand and prices.

A market is called *stable* if its clearing price is bounded above, regardless of the demand realization or the behavior of its participants. Although some markets have an upper bound on the bid price, these bounds are artificial barriers that prevent the theoretical market clearing price from reaching infinity. Note that the stability condition does not require all market participants to behave rationally; i.e., it expects some of them to bid at certain prices even though such prices do not maximize the expected profit. This is often the case in markets with a large number of participants and speculators who would like to capitalize on unusual events – price spikes – that may take place with relatively small probabilities.

As indicated above, our analysis assumes that the market clearing price is bounded above by $O\epsilon$. In our analysis, a market is called *stable* if its clearing price is guaranteed to be less than $O\epsilon$. From our previous work [9], the necessary condition for market stability is

Proposition 1 *If a market is stable, then following condition is satisfied*

$$\bar{D} \leq \sum_{\forall i \neq j} x_i, \text{ for } j = 1, \dots, N, \quad (7)$$

where \bar{D} is the maximum possible realization of D .

If the market stability condition is not satisfied, one bidder is guaranteed to be dispatched at the highest demand realization and she may set her bid price as high as possible.

The market stability condition (7) can be enforced by adding new bidders into the market and dividing bidders with large supply. Throughout this study, we assume that the market stability condition is satisfied.

We now explain the characteristics of the Nash equilibrium solutions in non-sealed bid markets in the following sections.

3 Non-sealed bid first-price auctions

The non-sealed bid first-price auction is the most common auction type in online auctions. It is clear that no bidder would bid below c . It is also possible that a winning bidder bids at

c ; however, this situation is unlikely because the winner receives no profit from this auction. We exclude this zero-profit case from our further consideration of the first-price auction.

The FIFO and random tie-breaking methods yield different bidding strategies and hence, different equilibrium points. Under FIFO tie-breaking rule, we show in the following proposition that every bidder bids at $c + \epsilon$.

Proposition 2 *At an equilibrium point in a non-sealed bid first-price auction with FIFO tie-breaking rule, all winning bidders bid at $c + \epsilon$.*

Proof : It is clear that no bidder bids below c . The market stability condition requires that at least one bidder, say i , at the highest bids cannot win in any demand realization. Thus, if there is a winner j whose $p_j > c + \epsilon$ then bidder i would decrease p_i . ■

The strategy for the bidders is to bid at this level as fast as possible. By bidding at this level, you are guaranteed that no one would outbid you since they will not get any profit. This result, however, does not apply to the fixed FIFO tie-breaking rule.

One supporting argument for implementing the fixed FIFO tie-breaking is that it encourages early participation. It, however, discourages later participants and may cause the auction to close at a non-equilibrium point. Under the fixed FIFO tie-breaking rule, the auction becomes a game of speed – who bids first wins. There are many bidders who bid very high at the market open just to get this privilege. They can bid as high as they want since they can move down later without losing anything. These bidders then do not participate in the bidding process until right before the auction close. At that time, these bidders then modify their bids to be equal to the current winning price. Because they have earlier time-stamp, their bids get higher priority and subsequently win in the auction. Knowing this fact, the later bidders have no motivation to compete in the auction since they know that they will lose anyway. Therefore, the later bidders will just give up and not even place a bid, causing the auction to close at a non-competitive price, which can be very high. This situation is similar to the fortress hub in the airline industry, where no one wants to start a price war by lowering prices to and from the hub because they know that if one of them reduces the price, the others will do the same.

The online auctions under fixed FIFO tie breaking still survive today because there are

irrational or uninformed bidders participating in the auction. Once this is not the case, then the non-sealed bid first-price auction with fixed FIFO tie breaking will not be the right choice for the buyer in a selling auction.

Unlike in the FIFO tie breaking, in the random tie breaking, the winners may bid higher than $c + \epsilon$, as shown in the following example.

Example 1 Assume five bidders with the same cost c of 1. Each has a bid quantity of 1. The demand is 4 and the bid increment ϵ is 1. An equilibrium point is where every bidder bids at 6, or $c + 5\epsilon$. Each gets an expected profit of $5(4/5) = 4$. ■

With the random tie-breaking rule, the buyer pays 6 per unit. She would have paid only 2 if the FIFO tie breaking is used. Since deterministic demand is a special case of stochastic demand, we conclude that a non-sealed bid first price auction with random tie breaking yields no higher profit, and in many cases, lower profit, to the buyer than the auction with FIFO tie breaking.

The buyer in a non-sealed bid first-price auction would prefer the FIFO tie breaking rule because it guarantees the lowest clearing price; however, the FIFO tie-breaking rule is time sensitive. That is, the auction is also a game of speed where early bidders bidding at $c + \epsilon$ win. If the auctioneer's main concern is not to have a game-of-speed auction, she should consider a uniform-price auction with random tie breaking.

4 Non-sealed bid uniform-price auctions

The uniform-price auction is widely used in commodities markets, such as treasury bill auctions, spectrum auctions, and electricity markets. At the moment, however, the uniform-price online auction is not as popular as the first-price online auction. One reason is that non-rational or uninformed bidders may bid below their costs. First-price auctions would capture this opportunity better than uniform-price auctions.

However, we believe that, in the future, uniform-price online auctions will become more popular than the first price auction for two main reasons. First, non-rational and uninformed bidders will fade away due to rapid improvement of global communications. Second, the

fairness of uniform-price auctions will attract more auction participants, and hence, improve price competition.

In a special case where all bidders have the same bid quantity, the following lemma explains the behavior of bidders trying to bid low in the uniform-price auction, regardless of the type of the tie-breaking rule used.

Lemma 1 *In an equilibrium point in a uniform-price auction where all bidders have the same bid quantity, the lower bidders never get a lower profit than the higher bidders.*

This lemma follows from the fact that, for each demand realization, all bidders get paid at the same price and the lower bidders get dispatched before the higher ones. This lemma is used in the later sections.

4.1 Non-sealed bid uniform-price auction with FIFO tie breaking

The following lemma shows that we can restrict ourselves to the case where every bidder bids at least c .

Lemma 2 *In a non-sealed bid uniform-price auction with FIFO tie breaking, if there is an equilibrium point with a bidder bidding below c , there is another equilibrium point where everyone bids at least c and none of them gets a lower profit.*

Proof : If a bidder i bids below c at an equilibrium point, bidder i 's profit will not be lower if $p_i = c$. When $\pi \geq c$, bidder i 's profit is unchanged and when $\pi < c$, bidder i 's profit is higher. ■

The following proposition gives an upper bound on the equilibrium market clearing price.

Proposition 3 *An equilibrium market clearing price cannot be higher than $c + \epsilon$.*

Proof : At an equilibrium point \mathbf{b}^* , suppose there is a demand realization d such that $\pi(\mathbf{b}^*, d) > c + \epsilon$; thus, there is a bidder i whose $p_i > c + \epsilon$ and who can be dispatched at d ; however, from the market stability condition (7), there must be a bidder j who cannot be dispatched at any demand realization. Bidder j can increase her profit by bidding at $c + \epsilon$ and take the quantity from bidder i , yielding a contradiction. ■

Lemma 2 and Proposition 3 indicate that we can restrict ourselves to an equilibrium point where bidders bid at either c or $c + \epsilon$. One interesting characteristic of the FIFO tie-breaking rule is that the last bidder never makes any profit in the auction. This fact helps us prove the following proposition.

Proposition 4 *Label the bids in order of their dispatch priorities. At an equilibrium point, the following condition*

$$\sum_{i < l | p_i = c} x_i \geq \bar{D} - x_k \quad (8)$$

must be satisfied, where

$$k = \arg \min_{j | \sum_{i=1}^{j-1} x_i \geq \bar{D}} x_j,$$

$$l = \min_{j | \sum_{i=1}^{j-1} x_i \geq \bar{D}} j.$$

Proof : From the market stability condition (7), there must be at least one bidder who will never be dispatched. Bidder l is the first undispached bidder at the highest demand realization. Bidder k is the smallest quantity bidder among the bidders at $c + \epsilon$. If the condition is not met, bidder k can set $p_k = c$ and receive some positive profit at the highest demand realization. ■

The buyer gets a higher profit when the market clears at c than when it clears at $c + \epsilon$. The higher the bid quantity at c , the higher chance that the market closes at c , and so the higher profit the buyer gets. Thus, the bid quantity at c is important to the buyer. Proposition 4 gives the lower bound on the total bid quantity at c . It is interesting that this lower bound does not depend on the demand distribution. From Proposition 4, we have the following corollary.

Corollary 1 *In any equilibrium point, the market clearing price must be c when the demand realization is less than $\bar{D} - x_k$, where k is defined in Proposition 4.*

Corollary 1 indicates that, if the demand distribution is skew with a long right tail, i.e., when $P(D > \bar{D} - x_k)$ is very small, or x_k is small, the auction will close at c most of the time. The following proposition explains the bidding strategy of the bidders under deterministic demand. The strategy is to always bid at c if the remaining demand is higher than the bid quantity; otherwise, bid at $c + \epsilon$.

Proposition 5 *When demand d is known, an equilibrium point that gives the highest market clearing price $c + \epsilon$ is the point where the bidders bid at:*

$$p_j = \begin{cases} c & \text{if } \sum_{i < j} x_i + x_j < d, \\ c + \epsilon & \text{otherwise.} \end{cases}$$

for each $j = 1, \dots, N$ where the bidders are labeled according to their bid submission time.

Proof : When the bidders adopt this strategy, the market closes at $c + \epsilon$, which is the highest possible market clearing price according to Proposition 3. Given the bids of the earlier bidders, the later bidders never gain higher profit by deviating from this strategy. Given the bids of the later bidders, the earlier bidders never gain a higher profit by adjusting their bids. ■

However, if the bidders are allowed to reduce their bid quantities, then the optimal bid strategy is to bid at c for full capacity until the remaining demand is one unit. The marginal bidder bids at $c + \epsilon$.

Proposition 6 *When demand d is known and the bidders can lower their bid quantities, $q_j, j = 1, \dots, N$, an equilibrium point that gives the highest market clearing price $c + \epsilon$ is the point where bidder j bids at*

$$\begin{aligned} x_j &= \begin{cases} \min(q_j, d - \sum_{i < j} x_i - 1) & \text{if } \sum_{i < j} x_i + 1 < d, \\ q_j & \text{otherwise;} \end{cases} \\ p_j &= \begin{cases} c & \text{if } \sum_{i < j} x_i + 1 < d, \\ c + \epsilon & \text{otherwise.} \end{cases} \end{aligned}$$

where the bidders are labeled according to their bid submission time.

Proof : Similar steps as in the proof of Proposition 5. ■

When demand is stochastic, the optimal bid price of a bidder depends on the quantities at c of the later bidders and hence, the order of their bid submission time, as shown in Example 2.

Example 2 *Suppose the demand can be 25 or 40 with some probability distribution. There are four bidders with quantity $(x_1, x_2, x_3, x_4) = (25, 1, 3, 38)$. Suppose bidder 1 submits the bid first. Conditioning on the bid submission time of the remaining bidders, bidder 1 decides on p_1 that maximizes his profit.*

If $p_1 = c$, bidder 1's expected profit is $25\epsilon P(D = 40)$. But if $p_1 = c + \epsilon$, the expected profit depends on the bid submission time of the later bidders. If bidder 3 bids second, bidder 4 will never bid at c . Thus, bidder 1's expected profit is $21\epsilon P(D = 25) + 25\epsilon P(D = 40)$. However, if bidder 2 bids second and bidder 4 third, the expected profit of bidder 1 becomes $\epsilon P(D = 40)$. Nevertheless, if bidder 2 bids second and bidder 3 third, the expected profit of bidder 1 remains $21\epsilon P(D = 25) + 25\epsilon P(D = 40)$. However, if bidder 4 bids second, bidder 1 gets only $\epsilon P(D = 40)$ of profit.

When $P(D = 40)$ is low or when bidder 3 is likely to bid second, bidder 1 has a higher chance of getting a better profit by bidding at $c + \epsilon$. This example indicates that the optimal bidding strategy for the deterministic demand case does not apply to the stochastic demand case. It also shows that the order of bid submission time and the demand distribution affect profits and optimal bid prices.

Since the bid submission time is revealed before the market closes, the earlier bidders can adjust their bid prices if it can increase their profits. As in Example 2, suppose $P(D = 40)$ is low, bidder 1 bids at $c + \epsilon$ hoping that bidder 3 bids second. If bidder 3 bids second, bidder 1 will not change p_1 . But if bidder 2 bids second, bidder 1's profit may drop to $\epsilon P(D = 40)$ if bidder 3 does not bid third. Hence, bidder 1 might lower p_1 to c and gets $25\epsilon P(D = 40)$. If bidder 1 decides to lower p_1 , he/she has to compete with bidder 4. If bidder 4 places the bid before bidder 1, bidder 1 still gets $\epsilon P(D = 40)$.

Suppose the bidders are equally likely to bid in any order, the optimal bid price of bidder 1 can be determined using a decision tree approach as shown in Figure 1.

From the decision tree, if bidder 1 initially bids at $c + \epsilon$ and bidder 2 bids second, bidder 1 prefers to lower p_1 to c if $7\epsilon P(D = 25) + 17\epsilon P(D = 40) > 10.5\epsilon P(D = 25) + 13\epsilon P(D = 40)$. Similarly, the optimal initial bid price for bidder 1 can be calculated by simply rolling back the tree.

The following example shows that it is possible that two earlier bidders at $c + \epsilon$ want to adjust their bids.

Example 3 *Suppose demand can be 12, 25, or 40, with some probability distribution. There are five bidders with quantity $(x_1, x_2, x_3, x_4, x_5) = (12, 25, 1, 3, 38)$. Suppose bidder 1 submits the bid first and bidder 2 submits the bid second.*

Figure 1: Bidder 1's decision tree

Assume that $P(D = 12)$ is high enough and $P(D = 40)$ is low enough so that $p_1^* = p_2^* = c + \epsilon$. If bidder 4 bids third, then $p^* = c$ and bidder 5 will never bid at c . Both bidder 1 and bidder 2 will not lower their bid prices. However, if bidder 3 bids third, it is possible that bidder 5 will bid fourth at c and take away bidders 1 and 2's profits. Therefore, when bidders 1 and 2 see bidder 3 bids third, one of them might want to lower the bid price to c . If bidder 1 lowers p_1 , bidder 1's profit is $12\epsilon P(D = 25, 40)$ and bidder 2's profit is $9\epsilon P(D = 25) + 24\epsilon P(D = 40)$. If bidder 2 lowers p_2 , bidder 1's profit is $11\epsilon P(D = 40)$ and bidder 2's profit is $25\epsilon P(D = 40)$. If $P(D = 40)$ is low enough, both bidders 1 and 2 prefer to lower p_1 instead of p_2 .

In addition to demand uncertainty, the uncertainty of the later bids' submission time makes the earlier bidders' bidding decision complicated. An algorithm to find the optimal bidding strategy in the non-sealed bid uniform-price auction with FIFO tie breaking under stochastic demand is left for future research. In the case where bidders can adjust their bid quantities, the optimal bidding strategy is the same as in the deterministic demand case with $d = \bar{D}$.

Proposition 7 *When demand D is stochastic and the bidders can lower their bid quantities, $q_j, j = 1, \dots, N$, an equilibrium point that gives the highest market clearing price $c + \epsilon$ is the point where bidder j bids at*

$$x_j = \begin{cases} \min(q_j, \bar{D} - \sum_{i < j} x_i - 1) & \text{if } \sum_{i < j} x_i + 1 < \bar{D}, \\ q_j & \text{otherwise;} \end{cases}$$

$$p_j = \begin{cases} c & \text{if } \sum_{i < j} x_i + 1 < \bar{D}, \\ c + \epsilon & \text{otherwise,} \end{cases}$$

where the bidders are labeled according to their bid submission time.

Proof : From Proposition 4, the bid quantity at c is $\bar{D} - 1$, hence no one gains any profit when $D < \bar{D}$. As a result, the optimal bidding strategy is the same as in the deterministic demand case of Proposition 6, where $d = \bar{D}$. ■

As a special case where all bidders have the same bid quantity, we show that the later bidders never bid below the earlier bidders.

Proposition 8 *In an equilibrium point in a non-sealed bid uniform-price auction with FIFO tie breaking where all bidders have the same quantity, if the bidders are labeled according to their bid submission time, the bid prices are monotone with respect to the bid submission times, i.e., $p_1 \leq p_2 \leq \dots \leq p_N$, or there is another equilibrium point where $p_1 \leq p_2 \leq \dots \leq p_N$ and non bidder get a lower profit. Furthermore, the i^{th} bidder's optimal bidding strategy is to bid at c if the expected profit by bidding at c is higher than that of bidding at $c + \epsilon$. That is, the following condition*

$$xP(D > ix) > \int_{(i-1)x}^{ix} dF - (i-1)xP((i-1)x < D \leq ix) \quad (9)$$

is met. Otherwise, the i^{th} bidder bids at $c + \epsilon$.

Proof : The proof follows from Lemma 1 and Proposition 3, and then by applying similar steps as in the proof of Proposition 5. ■

It is possible, however, that there might be an equilibrium point where $p_i > p_{i+1}$. This can happen only in the case when both bidder i and $i + 1$ receive the same profit for every demand realization, e.g., where $\pi(\underline{D}) > p_i$ or $\pi(\bar{D}) < p_{i+1}$. Both bidders are indifferent between bidding at p_i or p_{i+1} .

Propositions 7 and 8 also explain that the quantity-adjustable bidders or the same quantity bidders never adjust their bid prices upon the realization of the order of bid submission time.

The non-sealed bid uniform-price auction with FIFO tie breaking might be the most preferable by the buyer. The equilibrium market clearing price is not higher than $c + \epsilon$. Moreover, the total bid quantity at c is not less than $\bar{D} - x_k$, where k is the smallest quantity bidder at $c + \epsilon$. In the case where the bidders can lower their bid quantities, the total bid quantity at c is $\bar{D} - 1$. That is, the market clearing price is $c + \epsilon$ only at the highest demand realization; in all other realizations, the market closes at c .

The bidding strategy for a bidder under deterministic demand is simple, i.e., bid at c if the remaining demand is more than the bid quantity, otherwise bid at $c + \epsilon$. In the case where bidders can lower their bid quantities, for both deterministic and stochastic demand, the optimal strategy is to bid at c at the maximum quantity as long as the total quantity at c is still below the highest demand realization; otherwise, bid at $c + \epsilon$. In the case where bidders have the same bid quantity, their decisions follow a threshold strategy where it is optimal to bid at c when the total quantity at c is below some threshold computed from (9).

Under a stochastic demand case where bidders have various quantities and cannot lower their quantities, the optimal bid prices depend on both demand distribution and the order of bid submission time. Hence, the bidding strategy is dynamic. We show some examples where bidders lower their initial bid prices after the next bidder is revealed.

4.2 Non-sealed bid uniform-price auction with random tie breaking

In a non-sealed bid uniform price auction with random tie breaking, we show that there are only two groups of bidders. The first contains bidders bidding at or below their cost. The second contains bidders bidding above their costs. These bidders, however, do not bid higher than $c + 2\epsilon$. We first introduce the following lemma.

Lemma 3 *At an equilibrium point, the condition $\pi(\mathbf{b}, \bar{D}) \geq c$ must hold. Furthermore, if $\pi(\mathbf{b}, \bar{D}) \neq c$, then all bidders bid at or below $\pi(\mathbf{b}, \bar{D})$.*

Proof : If $\pi(\mathbf{b}, \bar{D}) < c$, then the dispatched bidders can reduce their losses by increasing their bid prices. Thus, $\pi(\mathbf{b}, \bar{D}) \geq c$. If $\pi(\mathbf{b}, \bar{D}) > c$ and one of the bidders bids above the market clearing price at the highest demand, she can gain some positive profit by bidding at the market clearing price. ■

Lemma 3 indicates that, if $\pi(\mathbf{b}, \bar{D}) \neq c$, a bidder in an equilibrium point at the highest demand realization is either an under-bidder or a marginal bidder. No bidder bids more than the market clearing price at the highest demand realization.

Proposition 9 *There must be more than one marginal bidder in an equilibrium point at the highest demand realization when $\pi(\mathbf{b}, \bar{D}) \neq c$.*

Proof : At an equilibrium point at the highest demand realization with $\pi(\mathbf{b}, \bar{D}) \neq c$, the total bid quantity for the under-bidder group must be less than \bar{D} ; otherwise, there must be some bidders in the under-bidder group bid at the market clearing price, yielding a contradiction. Thus, by the market stability condition (7), there are at most $N - 2$ under-bidders. By Lemma 3, there are at least two marginal bidders. ■

The following lemma gives an upper bound on the equilibrium market clearing price. This bound helps characterize the equilibrium bid prices.

Lemma 4 *If the highest demand realization, \bar{D} , requires $N - 1$ suppliers and if there are two bidders bidding at $c + 2\epsilon$, then the equilibrium market clearing prices at all demand realizations are at most $c + 2\epsilon$; otherwise, the equilibrium market clearing prices must be less than or equal to $c + \epsilon$.*

Proof : Suppose the spot market price at the highest demand realization is $c + k\epsilon$, where $k \geq 2$. Without loss of generality, we label the bidders in ascending order of their bid prices; that is, $p_1 \leq p_2 \leq \dots \leq p_N$. By Proposition 9, the number of marginal bidders, m , bidding at $c + k\epsilon$, is greater than or equal to 2.

Let $r(d)$ be the demand left after being supplied by the under bidders, i.e., $r(d) = d - \sum_{i=1}^{N-m} x_i$. The marginal bidder with the lowest supply is labeled as bidder j . Since bidder j has the lowest bidding quantity, she can supply at most $\min(\frac{r(D)}{m}, x_j)$ and receive

an expected profit that does not exceed

$$k\epsilon \int_{\sum_{i=1}^{N-m} x_i}^{\bar{D}} \min\left(\frac{r(D)}{m}, x_j\right) dF_D.$$

If bidder j , however, reduces p_j to $c + (k - 1)\epsilon$, her expected profit will be at least:

$$(k - 1)\epsilon \int_{\sum_{i=1}^{N-m} x_i}^{x_j + \sum_{i=1}^{N-m} x_i} r(D) dF_D + k\epsilon \int_{x_j + \sum_{i=1}^{N-m} x_i}^{\bar{D}} x_j dF_D.$$

Since $k \geq 2$ and $m \geq 2$, we have that $m(k - 1) \geq k$ or $r(D)(k - 1)\epsilon \geq r(D)k\epsilon/m$. This inequality is strict if $m > 2$ or $k > 2$. In other words, bidder j can gain a higher profit by lowering p_j .

Now we move to the case where $m = 2$ and $k = 2$. If there exists a demand realization that can be supplied by bidders who bid at $c + \epsilon$ or less, i.e., $\underline{D} \leq \sum_{k|p_k \leq c+\epsilon} x_k$. Consider a sub-game of bidders at or below $c + \epsilon$. We show that this sub-game has a positive profit equilibrium point. As a result, there exists a demand realization where the market clearing price is $c + \epsilon$ and bidder j can reduce p_j to $c + \epsilon$ and gain a higher profit. That is, we have $\underline{D} > \sum_{k|p_k \leq c+\epsilon} x_k$ and all bidders bidding below $c + 2\epsilon$ are fully dispatched. Moreover, the highest demand realization \bar{D} must be less than $\sum_{k|p_k \leq c+\epsilon} x_k + x_j$; otherwise, bidder j can gain a higher expected profit by bidding at $c + \epsilon$.

In conclusion, an equilibrium market clearing price is at most $c + 2\epsilon$. The case where it is $c + 2\epsilon$ occurs only when there are two bidders at $c + 2\epsilon$ and the other bidders at or below $c + \epsilon$. The other bidders are fully dispatched in all demand realizations. The demand distribution is such that $\bar{D} - \sum_{k|p_k \leq c+\epsilon} x_k < x_i$, $\bar{D} - \sum_{k|p_k \leq c+\epsilon} x_k < x_j$, and $\underline{D} > \sum_{k|p_k \leq c+\epsilon} x_k$. When the market clearing price is $c + 2\epsilon$ in one demand realization, it must be $c + 2\epsilon$ in all realizations. ■

From Lemmas 3 and 4, no bidder bids above the highest possible MCP and the MCPs can only be c , $c + \epsilon$ or $c + 2\epsilon$. $\text{MCP} = c + 2\epsilon$ is possible at certain demand realizations if and only if there are exactly two bidders bidding at $c + 2\epsilon$.

If the market clearing price is c , then all bidders gain nothing. Hence, they prefer equilibrium points that result in the market clearing price being $c + \epsilon$ or $c + 2\epsilon$, which award positive expected payoffs; however, since only under-bidders are guaranteed to sell all of their supply, every bidder wants to be one of the under-bidders.

It is possible that there exist equilibrium points with no under-bidders. One example is when $\bar{D} \leq x_i, \forall i = 1, \dots, N$. In this case, it is obvious that $p_i^* = c + \epsilon$ for all i is an equilibrium point. The point $p_i^* = c$ for all i is also an equilibrium point without under-bidders. In fact, when $\bar{D} > x_i$ for some i , the point $p_i^* = c$ for all i is the only symmetric pure strategy equilibrium point.

If there are two groups of bidders when $D = \bar{D}$, then Lemma 4 dictates that all bidders have positive expected profits. Any bidder in the under-bidder group can bid at any price below c and remain in equilibrium, as long as the market clearing price corresponding to the lowest demand realization does not change. Consider an equilibrium point with some bidders bidding below c . These bidders can increase their prices to c without violating the equilibrium condition. Hence, we can restrict ourselves to the class of equilibrium points where all bidders bid at least c .

We summarize all possible equilibrium points and their resulting MCPs in the following proposition.

Proposition 10 *There exist only three pure strategy equilibrium points as described below.*

1. *The point $c \leq p_i^* < p_{N-1}^* = p_N^* = c + \epsilon$ or $c + 2\epsilon$, for all $i = 1, \dots, N - 2$, with MCPs equal to $c + 2\epsilon$. This point can occur only when $\bar{D} - \sum_{k=1}^{N-2} x_k < x_N, x_{N-1}$, and $\underline{D} > \sum_{k=1}^{N-2} x_k$;*
2. *the point $p_i^* = c < p_{m+1}^* = \dots = p_N^* = c + \epsilon$, for $i = 1, \dots, m$, for some m , with MCPs equal to either c or $c + \epsilon$; and*
3. *the point $p_i^* = c$, for all $i = 1, \dots, N$, with MCP equal to c .*

In all of the three cases, some bidders at c may lower their prices to any number without changing the equilibrium condition, as long as the MCP remains unchanged. Proposition 10 indicates that, in the case of positive expected profits, there are two groups of bidders bidding at c and at $c + \epsilon$ or $c + 2\epsilon$.

When demand is stochastic, the following example shows that a bidder with a higher bid quantity may bid lower than a lower-quantity bidder.

Example 4 Assume three bidders with the same cost c . Bidder 1's quantity is 3, bidder 2's quantity is 2, and bidder 3's quantity is 2. The demand can take the value of 2 with

probability 0.625 and 4 with probability 0.375. The only positive profit equilibrium point for this problem is when bidder 1 bids at c and the others bid at $c + \epsilon$. ■

When demand is known to all bidders, the characteristics of bidders in Point 2 of Proposition 10 are explained in Propositions 11 and 12.

Proposition 11 *When demand is known, a positive profit equilibrium point exists and is such that*

$$\begin{aligned} p_i &< c, \quad \forall i = 1, \dots, m, \\ p_{m+1} &= \dots = p_N = c + \epsilon, \\ \sum_{i=1}^m x_i &< d, \\ d - \sum_{i=1}^m x_i &\leq x_j, \quad \forall j = m + 1, \dots, N. \end{aligned}$$

Proof: The proof of equilibrium is trivial. To show existence, we sort the bidders according to their bidding quantities so that $x_1 \leq x_2 \leq \dots \leq x_N$; m is then the integer such that $\sum_{i=1}^m x_i < d$ and $\sum_{i=1}^{m+1} x_i \geq d$. ■

A bidder with the lowest bid quantity does not always have to be the under bidder. Similarly, a bidder with the highest bid quantity does not have to be the marginal bidder. Any combination of bidders that satisfies the conditions in Proposition 11 is an equilibrium point.

Proposition 12 *At an equilibrium point in a known demand environment, the market clearing price is at $c + \epsilon$ or at $c + 2\epsilon$. The total bid quantity at c must be at least $d - x_k$, where k is the lowest quantity bidder bidding above c , i.e., a marginal bidder.*

Proof: From Propositions 10 and 11, the bidders bid to reach a positive profit equilibrium point and hence the market clearing price is either at $c + \epsilon$ or at $c + 2\epsilon$. Suppose there is a bidder i with bid quantity x_i less than $d - \sum_{j|p_j=c} x_j$ bidding above c , then the bidder can reduce p_i to c and gain a higher expected profit by guaranteed to be fully dispatched without changing the market clearing price. ■

Proposition 12 implies that, at the end, exactly one marginal bidder is dispatched. If the market has a progressive rule, e.g., the bid adjustment can be done if it does not increase the

current MCP, then the optimal bidding strategy is to bid at c until the remaining demand $d - \sum_{j|p_j=c} x_j$ is less than or equal to the bid quantity, otherwise, bid at $c + \epsilon$ (or at $c + 2\epsilon$ under a special condition explained in Proposition 10). If the market does not have this rule, the bidders might play a chicken game where everyone bids at c waiting for someone to take a role of marginal bidder.

The following sub-section discusses some market properties under a special case where bidders have the same bid quantity.

4.3 Identical Bidders in a non-sealed bid uniform price auction with random tie-breaking

We consider a case where the N bidders also have the same bid quantity x . We show an existence of a positive-profit pure-strategy equilibrium point with stochastic demand. We begin by focusing on the number of bidders bidding above c at point 2 in Proposition 10. In this case, we consider only two decisions for each bidder, i.e., bidding at c or at $c + \epsilon$. Consider a point where there are two groups of bidders. The bidders in the first group bid at c and the bidders in the second group bid at $c + \epsilon$. Suppose that there are m bidders bidding at c , then the expected profit of a bidder at c , denoted by $f_c(m)$, is $\int_{m^x}^{\bar{D}} \epsilon x dF$. The expected profit of a bidder at $c + \epsilon$, denoted by $f_{c+\epsilon}(m)$, is $\int_{m^x}^{\bar{D}} \epsilon(D - mx)/(N - m) dF$. Both $f_c(m)$ and $f_{c+\epsilon}(m)$ are decreasing functions of m .

We introduce a new function $g(m)$ where $g(m) = f_c(m) - f_{c+\epsilon}(m - 1)$. Given that there are m bidders bidding at c , a bidder at c would not want to increase her bid price if $f_c(m) \geq f_{c+\epsilon}(m - 1)$ or $g(m) \geq 0$. Likewise, a bidder at $c + \epsilon$ would not want to decrease her bid price if $f_c(m + 1) \leq f_{c+\epsilon}(m)$ or $g(m + 1) \leq 0$. Therefore, an equilibrium point with m^* bidders bidding at c must satisfy the following two conditions:

$$g(m^*) \geq 0, \quad \text{if } m^* > 0, \tag{10}$$

$$g(m^* + 1) \leq 0, \quad \text{if } m^* < N. \tag{11}$$

Any point with m^* bidders bidding at c satisfying both Condition (10) and Condition (11) is also an equilibrium point. We now prove the existence of a positive expected profit pure strategy equilibrium point.

Proposition 13 *There exists at least one pure strategy equilibrium point that gives each bidder a strictly positive expected profit.*

Proof : If $g(1) \leq 0$ then $m^* = 0$ satisfies Conditions (10) and (11) and the point where no bidder bids at c is a positive expected profit pure strategy equilibrium point. Assume $g(1) > 0$. Let s be the minimum number of bidders needed to supply \bar{D} . That is, $(s-1)x < \bar{D}$ and $sx \geq \bar{D}$. By the market stability condition (7), we have that $s < N$. Since $f_c(s) = 0$ and $f_{c+\epsilon}(s-1) > 0$, we have that $g(s) < 0$. There then exists $0 \leq m = s-1 < N$ such that $g(m+1) < 0$. We have that $g(1) > 0$ and $g(m') < 0$ for some $1 < m' \leq N$; hence, there exists $1 \leq m^* < N$ that satisfies Condition (10) and Condition (11) and the point where there are m^* bidders bidding at c is a positive expected profit pure strategy equilibrium point. ■

In some stochastic demand cases, the positive expected profit pure strategy equilibrium point may not be unique, as shown in the following example.

Example 5 *Consider a problem with six suppliers. Each of them has a unit production cost of 5 and bids for two units. Demand is stochastic with probability 0.9 for 5 units and probability of 0.05 each for 7 and 9 units. ϵ is set to 0.1.*

m	$f_c(m)$	$f_{c+\epsilon}(m)$	$g(m)$
0	-	0.0883	-
1	0.2	0.066	0.11167
2	0.2	0.0325	0.134
3	0.02	0.0067	-0.0125
4	0.01	0.0025	0.0033
5	0	0	-0.0025
6	0	-	0

Table 1: Profit functions of Example 5

The expected profits for each bidder under different numbers of bidders at c are shown in Table 1. The points where $m = 2$ and $m = 4$ are both pure strategy equilibrium points with strictly positive expected profits. The point with two bidders at c gives a better payoff to every bidder than the point with four bidders at c ; hence, it is the best equilibrium point for

all bidders. When there are four bidders at c , bidders at $c + \epsilon$ receives positive profits only when $d = 9$; they would never decrease their bid prices. On the other hand, one bidder at c would not increase her bid price alone; however, if *two* of the bidders at c agree to increase their bid prices, all bidders will obtain better payoffs.

In this example, suppose the market reaches equilibrium when there are four bidders bidding at c , and all four of them gain an expected payoff of 0.001. At this point, suppose one of the bidders at c , labeled as the “leader”, increases her bid price to $c + \epsilon$; she receives an expected payoff of 0.0067, which is lower than the expected payoff she previously would obtain; however, at this new position, every bidder can gain a higher expected payoff by changing her bid price. A bidder at c can increase her bid price to increase her expected profit by 0.0125. Likewise, a bidder at $c + \epsilon$ can lower her bid price to increase her expected profit by 0.0033. As a result, one bidder will change her bid price. We label this bidder as the “follower”.

Suppose a bidder at c is the follower, then every bidder, including the leader, receives a better expected payoff; however, if a bidder at $c + \epsilon$ is the follower, she is the only one who receives a better expected payoff while the others obtain lower expected payoffs. In addition to that, if the bidder at $c + \epsilon$ instead remains at $c + \epsilon$ and waits for a bidder at c to be the follower, the bidder at $c + \epsilon$ is able to improve her expected profit by $0.0325 - 0.01 = 0.0225$ for not being the follower himself/herself. As a result, bidders at $c + \epsilon$ prefers to wait for a bidder at c to increase the bid price. The market will move to the equilibrium point with two bidders at c . That is, bidders at $c + \epsilon$ would never want to be the follower. Thus, the leader finally receives a better expected payoff than what she originally receives at the equilibrium point with four bidders at c .

These arguments lead to the conclusion that the equilibrium points with four bidders at c are not “stable”. It is not stable in the sense that one bidder at c would want to increase her bid price to trigger another bidder at c to follow him/her and force the market towards an equilibrium point with a higher expected payoff to each bidder.

When there are four bidders at c , a new question of who should be the leader arises. If a bidder at c stays at c and the market eventually reaches the equilibrium point with two bidders at c , she would get 0.2, which is much higher than 0.0325, the expected profit that

the leader and the follower would receive. This becomes a game where all bidders at c hold on to the last moment, hoping other bidders will take the roles as the leader and the follower.

This is interesting in that a bidder can deviate from her current equilibrium bid price, given her beliefs in the rationality of the others, although it is not easy for a bidder at c to take the role of the leader or the follower. It is possible, however, that no bidder is brave enough to move and the market closes at the original equilibrium point.

In a progressive auction which has a rule that bidders can change their bids but not make them less attractive, e.g., cannot increase their bid prices, the bidding strategies are clearer for the problem in Example 5. In this case, all bidders compete to be an early bidder. The first two bidders place their bids at c . Knowing that the first two bidders cannot move up, the later bidders would not bid at c because 0.0325 is better than 0.01.

The case where demand is known to all bidders is a special case of a stochastic demand environment. All the results in the stochastic case hold. However, in contrast to the stochastic demand case where there can be multiple positive expected profit pure strategy equilibrium points, when demand is deterministic and less than or equal to $(N - 2)x$, the positive expected profit pure strategy equilibrium point is unique.

Proposition 14 *Let m be an integer such that $mx < d$ and $(m + 1)x \geq d$, i.e., $m + 1$ is a minimum number of bidders needed to supply deterministic demand d . There exist only three types of pure strategy equilibrium points as described below.*

1. *If $m = N - 2$, then $c \leq p_i^* \leq p_{N-1}^* = p_N^* = c + \epsilon$ or $c + 2\epsilon$, for all $i = 1, \dots, N - 2$, is an equilibrium point with the MCP equal to either $c + \epsilon$ or $c + 2\epsilon$;*
2. *If $m < N - 2$, then $p_i^* = c < p_{m+1}^* = \dots = p_N^* = c + \epsilon$, for $i = 1, \dots, m$, is an equilibrium point with the MCP equal to $c + \epsilon$; and*
3. *$p_i^* = c$, for all $i = 1, \dots, N$ is an equilibrium point with the MCP equal to c .*

Proof : If $\pi(\mathbf{b}, d) > c$, then the total bid quantity by the under-bidders should be at least $d - x$; otherwise, at least one marginal bidder can gain a higher profit by lowering her bid price. The proposition follows directly from Proposition 10 and the market stability condition (7). ■

Consider Point 1 where $d > (N - 2)x$ and $d \leq (N - 1)x$; there are two possible equilibrium

points. The first is when both marginal bidders bid at $c + \epsilon$ and the second is when both of them bid at $c + 2\epsilon$. At the point where two of them bid at $c + 2\epsilon$, one of them can lower the bid price to $c + \epsilon$ and still get the same payoff; however, the other bidder will receive nothing. Hence, in the market where the bidders are trying not only to maximize their own profits but also to hurt each other, the market will never close at the point where both marginal bidders bid at $c + 2\epsilon$. On the other hand, in the market where the bidders are trying only to maximize their own profits, the market may close at the point where both marginal bidders bid at $c + 2\epsilon$ because this point yields the maximum profit to them.

In any equilibrium point with deterministic demand, the number of bidders bidding at c is equal to the minimum number of bidders that can supply the demand subtracted by one; hence, only the earliest marginal bidder can be partially dispatched. That is, only one marginal bidder can be dispatched. The others cannot be dispatched at all. Proposition 14 also implies the following corollary.

Corollary 2 *When demand is known and is less than or equal to $(N - 2)x$, there exists only one pure strategy equilibrium point that gives all bidders strictly positive profits.*

Although there may exist multiple pure strategy equilibrium points with strictly positive expected profits to all bidders in the stochastic case in general, we can show that under certain conditions, a unique equilibrium point exists. Consider a stochastic demand environment where $D \in (mx, (m + 1)x]$ for some integer m . At each demand realization d , the positive profit equilibrium point for the deterministic demand problem associated with d can be determined by Proposition 14. Since $D \in (mx, (m + 1)x]$, Proposition 14 indicates that the equilibrium points for each deterministic problem are the same, yielding the following proposition.

Proposition 15 *If $D \in (mx, (m + 1)x]$ for some integer m , the stochastic demand problem reduces to a deterministic problem with demand $d = E[D]$.*

By Proposition 15 and Corollary 2, a unique equilibrium point with positive expected payoffs to all bidders exists not only in the deterministic case, but also in the stochastic case when $D \in (mx, (m + 1)x]$ for some integer $m < N - 2$. Thus, the market clearing equilibrium point can be anticipated.

In both stochastic and deterministic demand environments with identical bidders, bidders may play different pure strategies. Since under-bidders will obtain higher expected profits than marginal bidders, every bidder prefers to be one of the under-bidders; thus, the bidding rules of the market have a substantial effect on the bidders' bidding strategies. A market where bidders are allowed to adjust their bids infinitely without any further restriction on their bid prices may become a game where all bidders hang on to the last moment with bids at cost (or the lowest possible bid price) hoping others will take on the role as marginal bidders.

In a deterministic demand market under a progressive rule that constrains the adjustment of the bid prices, such as a new bid submitted must improve the MCP in the previous round, the *first* m bidders are in the under-bidder group and the remaining bidders are in the marginal bidder group, where $m + 1$ is the minimum number of bidders needed to supply the demand. The later bidders would not want to bid at c because they know that those first m bidders cannot move up and the MCP will become c . The key to this game is to be in the under-bidder group; hence, the bidders who bid earlier have a better market position.

In a stochastic demand market with a progressive rule, such as a bidder cannot increase the bid price, the auction becomes a game of speed. The earlier bidders bid at c . The later bidders, knowing that the earlier bidders cannot move up, have to bid above c .

The optimal bidding strategy in this type of auction can be thought as follows. "Just making sure you get the demand and leaving the price-setting task to the later bidders". Because the bidders know that everyone wants to make profits, early bidders would not care much about bidding at the right price since they know that the marginal bidder would bid at the best possible price anyway. Knowing this fact, the early bidders bid in a way to make sure that they can still get dispatched. Bidding at c guarantees that because the marginal bidder will not bid at or below c . The marginal bidder then bids at $c + \epsilon$, otherwise later bidders can replace him/her and take away the profit.

Without a progressive rule, an optimal strategy in a uniform-price auction under the random tie-breaking rule is not sensitive to the bid submission time. The auction closes at a reasonably low clearing prices ($\leq c + 2\epsilon$).

5 Conclusion

In this paper, online auctions are modeled as multi-round non-sealed bid auctions. Both stochastic and deterministic versions of the demand are considered. Assuming that bidders are rational, we derive the characteristics of the optimal bids in this market as well as the resulting market clearing price at an equilibrium point.

In a non-sealed bid first-price auction under fixed FIFO tie-breaking rule, where the tie-breaking priority is based on the bid submission time, the auction might close at a very high price since later bidders are discouraged from participating in the auction. For the non-sealed bid first-price auction with FIFO tie-breaking, it is a game of speed where every bidder bids, as fast as possible, at $c + \epsilon$ and the auction closes at $c + \epsilon$. The random tie-breaking rule should not be used in a non-sealed bid first-price auction because the equilibrium bid price can be much higher than $c + \epsilon$.

In a non-sealed bid uniform-price auction under FIFO tie-breaking rule, the equilibrium MCP cannot be higher than $c + \epsilon$. Uncertainty in demand benefits the buyer because, for any demand realization that is less than $\bar{D} - x_k$, where x_k is the smallest quantity among the undispached bids, the MCP is c . Hence, if the demand distribution is skew with a long right tail, the auction will close at c most of the time. Due to uncertainty of the later bidders' submission time, the bidders in this auction play dynamic bidding strategies, where they adjust their bid prices after the next bidder is revealed. When demand is known, each bidder always bids at c if the remaining demand is higher than her bid quantity, otherwise she bids at $c + \epsilon$. This static strategy is also optimal in a stochastic demand case if bidders can adjust their bid quantities. In this case, the bidders play a similar strategy as in the deterministic demand case where $d = \bar{D}$. Furthermore, when all bidders are identical, the optimal bidding strategy is a static threshold strategy.

In a non-sealed bid uniform-price auction with random tie breaking, the resulting MCP is either c , $c + \epsilon$, or $c + 2\epsilon$, depending on the demand and strategy. There are only two groups of bidders in an equilibrium point. The first contains bidders bidding at or below their cost. The second contains bidders bidding above their cost, but not exceed $c + 2\epsilon$. All bidders have positive expected dispatched quantities. A progressive auction under this setting is a

game of speed where the early bidders are in the first group. The later bidders have to be in the second group. However, if a progressive rule is not implemented, the auction may become a game of chicken where all bidders are in the first group and wait for someone to move to the second group.

When demand is known, a pure-strategy positive-profit equilibrium point exists. We also discover that a bidder with a higher quantity may bid lower than the one with a lower quantity.

In the special case where all bidders are identical, there exists at least one pure strategy equilibrium point that gives positive expected payoffs to all bidders. The positive expected payoff equilibrium point is unique if demand is known to all bidders or the demand distribution has a special structure. We also found that, when demand is stochastic, a bidder can deviate from her current equilibrium decision to trigger other bidders to move to another equilibrium point with a better payoff.

Table 2 provides a summary of the market prices and bidding strategies under different selling-auction formats. Buying auctions commonly held on the internet have the opposite effects. For example, the winning price in a non-sealed first-price format with FIFO tie breaking is $c + \epsilon$ for a selling auction and is $v - \epsilon$ for a buying auction, where v is the valuation of the buyers.

auction format		winning price	bidding strategy
First price	fixed FIFO	can be high	bid first at the highest possible price
	FIFO	$c + \epsilon$	everyone bids at $c + \epsilon$
	random	can be high	not considered ^a
Uniform price	FIFO	c if $D < \bar{D} - x_k$, $\leq c + \epsilon$ otherwise	bid at c , if can be under bidder; else, bid at $c + \epsilon$. In some cases, dynamic strategy.
	random	$\leq c + 2\epsilon$	bid at $c, c + \epsilon$, or $c + 2\epsilon$. May exist multiple equilibria

Table 2: Summary of market prices and bidding strategies in selling auctions

^aBecause the winning price can be high, we assume that no buyer would adopt this auction format. Hence, their optimal bidding strategies are not considered.

The non-sealed uniform-price FIFO rule is the most preferable to the buyer (seller) in a selling (buying) auction since it gives the lowest buying cost (highest selling revenue). The non-sealed first-price FIFO rule also yields the same revenue when demand (supply in buying auctions) is deterministic. The non-sealed uniform-price random rule yields a good revenue for the buyer (seller) in a selling (buying) auction and is also time insensitive without a progressive rule. Most optimal bidding strategies in all of the non-sealed auctions under these tie-breaking rules, however, are sensitive to bid submission time. If one does not want the bidders to play a game of speed, a non-sealed bid non-progressive auction with random tie-breaking rule may be used. Alternatively, our independent research also indicates that a sealed bid uniform-price auction with random tie-breaking should be implemented, where the winning price is bounded above by $c + 2\epsilon$ [10].

Under the assumption that there are enough consumers participating in an online buying auction of common product, the auction clears at a price that is little less than the actual value of the product being auctioned, i.e. $v - \epsilon$. As a result, many consumers may be discouraged to participate in the auction and choose to purchase the product directly from a store. On the other hand, when too many consumers choose to buy the product from a store, the auction becomes attractive again since the clearing price might be non-competitive and less than $v - \epsilon$.

There is also a number of bidders in online auctions today who are traders. They buy products at a cheap price and sell it at a higher price. The more the traders participate in auctions, the more competitive the online auctions are. On the other hand, the more competitive the online auctions, the less motivated the traders are to participate in the auctions. Online auctions of common products remain popular because it is a channel for sellers and buyers to sell or buy products quickly, despite the fact that the actual clearing price in an online auction depends on its competitiveness.

References

- [1] Cassady, R., *Auctions and Auctioneering*, Berkeley: University of California Press, 1967.

- [2] Dasgupta, P., and E. Maskin, "Efficient Auctions," *Quarterly Journal of Economics*, **115**: 341-388, 2000.
- [3] Klemperer, P. D., "Auction Theory: A Guide to the Literature," *Journal of Economic Surveys*, **13/3**: 227-86, 1999.
- [4] McKelvey, R. D., A. McLennan and T. Turocy, *Gambit Graphics User Interface* version 0.94, California Institute of Technology, 1997, <http://hss.caltech.edu/~gambit/>.
- [5] McKelvey, R. D. and T. R. Palfrey, "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior*, **10**: 6-38, 1995.
- [6] McKelvey, R. D., *A Lyapunov Function for Nash Equilibria*, 1991.
- [7] Press, W. H., B. P. Flannery and R. B. Friend, *Numerical Recipes in C : the Art of Scientific Computing, 2nd Edition*, Cambridge University Press, 1992.
- [8] Rothkopf, M. H. and R. M. Harstad, "On the role of discrete bid levels in oral auctions," *Eur. J. Oper. Res.*, **74**: 572-81, 1994.
- [9] Supatgiat, C. , Zhang, R. Q. and J. R. Birge, "Equilibrium Values in a Competitive Power Exchange Market," *Computational Economics*, **17**: 93-121, February 2001.
- [10] Supatgiat, C. , Birge, J. R. and R. Q. Zhang, "Optimal Bidding Strategies in Sealed Bid Online Auctions of Common Products with Quantity Uncertainty," *working paper*.
- [11] van Der Laan, G., A. J. J. Talman and L. Van der Heyden, "Simplicial Variable Dimension Algorithms for Solving the Nonlinear Complementary Problem on a Product of Unit Simplices Using a General Labelling," *Mathematics of Operations Research*, 377-397, 1987.
- [12] Vickrey W., "Counterspeculation, Auctions, and Competitive Sealed Tenders" *J. of Finance*, **16**: 8-37, 1961.
- [13] Yamey, B. S. "Why £2310000 for a Velazquez? An auction bidding rule," *J. of Political Economy*, **80**: 1323-1327, 1972.