# OPTIMAL MULTIPLE MODULE DESIGN - AN EXTENDED SIMPLEX-LIKE METHOD

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# Optimal Multiple Module Design - An Extended Simplex-like Method

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**Abstract**

Multiple Modular Design (MMD) is the method of designing a set of standard modules to meet demands for different functions. This work aims to find an optimal set of modules given the demands. In this paper, we start with Evans' nonlinear programming model of MMD. By exploring the special structure of the formulation, we develop several properties of optimal solutions. With these properties, we develop a heuristic algorithm, called The Extended Simplex-like Method. In this algorithm, the objective value monotonically decreases during each iteration until a KKT point of the problem is found. Conceptually, a global optimal solution can be found by enumerating all KKT points with a special property. The example and computational experience show that the Extended Simplex-like Method can solve the problem to similar accuracy but with much less computational effort than previous methods.

## 1. Introduction

Machine tool users have recently been demanding more customized machines to fit their specific needs due to shorter life cycles and higher levels of customization of their products. Competition in the machine tool industry results in highly price-sensitive customers requesting machines with multi-functionality. As a result of this market situation, two extreme strategies have emerged. The first one is to exploit the market for standard machinery. Japanese volume producers have used this strategy and achieved economies of scale. The other strategy is to concentrate on the market for high-priced and customized machine tools. German and Swiss manufacturer respond mostly to this market.

Machine tool builders may, however, be able to satisfy customer demand with lower cost through an intermediate strategy called *modularization* of machine tools. Building standardized modules that can then be quickly combined into individually customized machines offers manufacturers the advantages of both lower cost due to economies of scale and an enlarged customer base.

In general, a machine can be viewed as a combination of certain functions, including identical functions. Modularization of machine tools is to select functions to be included in different modules and then assemble the modules to customer demand. We illustrate modularization through the following simple example. In this example, there are three types of customer demands in the market where each consists of three functions. Demand type I requires 2 spindles, 2 motors and 2 tool supports; Demand type II consists of 2 spindles, 1 motor and 3 tool supports; Demand type III consists of 2 spindles, 2 motors and 1 tool support.

According to the first strategy, a machine tool (MT) company would have to design and build a standard machine tool with 2 spindles, 2 motors and 3 tool supports for all customers. While this strategy makes the design and production process simpler, the MT company has to provide extra functions that some customers may not need. According to the second strategy, a company designs and builds three individual types of machines customized for each customer. While the MT company is able to provide the exact machine to each customer, it incurs high design and production cost.

With the intermediate strategy of modularization, a company may design and build two types of modules, one with 2 spindles and 2 motors and the other with 1 tool support. The company then combines one type 1 module and two type 2 modules to meet demand type I, one type 1 module and three type 2 modules for demand type II, and one type 1 module and one type 2 module for demand type III. Since the MT company only needs to design and build 2 modules, it may save the cost associated with the design and over-supply of parts.

The research questions are how to design a set of modules that covers all desired functions and how to assemble the modules into machines that minimize over-supply. The problem to design modules for meeting customer demands is called the multiple modular design problem (MMD). A module (standard unit) is a group of functions; combinations of these modules are used to meet a variety of customer demands. A properly designed set of modules not only reduces the cost and time associated with the design and manufacturing of the final products, but also reduces the variability of the production of the final products. Clearly, if properly designed, the strategy brings benefits to both manufacturers and customers.

The mathematical framework of the MMD was first introduced by Evans [4] as an extension of the single modular design (MD) problem, in which only one module is allowed to meet the demands. Rutenberg and Shaftel [8] later developed an application framework of MMD for designing modules to meet multiple customer demands. They developed a heuristic procedure to search for optimal integer solutions; however, their heuristic search method does not provide much insight into the structure of the optimal solutions. Silverman [12] develops a decomposition approach for a general convex programming problem and applies the procedure to the MMD; however, since the MMD is, in general, not a convex programming problem, the decomposition procedure does not guarantee optimality for the MMD.

Shaftel and Thompson [9] develop an efficient simplex-like algorithm to solve the MD problem and extend the procedure to the MMD using a heuristic partition. The heuristic partition is similar to an intelligent enumeration; however, the heuristic partition does not even guarantee that solutions found satisfy the necessary conditions of optimality. Therefore, it sometimes fails to find an optimal solution for even trivial cases, such as the one shown in Thompson and Shaftel [10].

Tönshoff's Ph.D dissertation [14] studies MMD problem from a bundling (combination of modules) and pricing perspective. He creatively includes a new decision variable that is the price charged by manufacturer for a machine tool. His objective function is to maximize the profit given that customer has a reserved price for each possible bundle. He illustrates his model by a numerical example in German's machine tool industry; however, he simplifies the assumption on function selection and module section. He assumes that first a specific function appears at most once in a module and secondly a specific module appears at most once in a machine. Furthermore, he does not give any structure result on the optimal solution.

In this paper, we study a generalized MMD problem. In this problem, we allow multiple modules and identical functions in each module (e.g., more than one spindle can be included in a module). Assuming fixed costs associated with the design and production of modules, we seek a set of modules that minimizes the costs associated with production, design, and over-supply while meeting customer demands. We first derive an upper bound on the total number of modules. We then derive structural properties of the optimal solutions to the MMD problem and develop an Extended Simplex-like Method (ESM) for finding the optimal solutions. We show that the value of the objective function monotonically decreases in each iteration of ESM until a KKT point is found. A global optimal solution can be found by searching among a finite number of KKT points.

The algorithm reduces to the same simplex-like method proposed by Thompson and Shaftel [9] when it is applied to the MD problem. Our method also easily solves the trivial problem with which the heuristic partition method developed by Thompson and Shaftel [10] has trouble. Numerical examples show that the extended simplex-like method is more efficient than the methods proposed by both Silverman [12] and Thompson and Shaftel [10].

## 2. Problem Formulation

We consider a generalized MMD problem where considered costs include the cost for producing each function and the fixed cost associated with the design of the modules. The objective is to minimize the total cost including the production and the design of the modules. Before we present the model, we first introduce the following notation.

$m$ = the total number of functions

$n$ = the total number of demand types;

$p$ = the number of module types;

$a_i$ = the cost associated with function $i$ (material, labor, etc.);

$b_j$ = the number of customers of the same demand type $j$;

$$
\begin{aligned}
c_k &= \text{fixed cost for designing the } k\text{th module;} \\
r_{ij} &= \text{the number of functions } i \text{ that demand type } j \text{ requires;} \\
R &= \{r_{ij}\}_{m\times n}; \\
x_{ik} &= \text{the number of functions } i \text{ that goes into module } k; \\
X &= \{x_{ik}\}_{m\times p}; \\
y_{kj} &= \text{the number of modules } k \text{ used to meet customer demand of type } j; \\
Y &= \{y_{kj}\}_{p\times n}.
\end{aligned}
$$

Note that $c_k$ is the fixed cost associated with designing the $k$th module regardless of the functions in the modules. We make this assumption because in many instances, each module may require a common base platform, which significantly simplifies the problem. Of course, these costs are allowed to be identical. With the notation, we can present our mathematical model as the following:

$$
\begin{aligned}
\text{(P1) Minimize}_{x_{ij},y_{jk},p} \quad & \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n} a_i x_{ik} b_j y_{kj} + \sum_{k=1}^{p} c_k \\
\text{Subject to} \quad & \sum_{k=1}^{p} x_{ik} y_{kj} \geq r_{ij}, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \\
& x_{ik}, y_{kj} \geq 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \quad k = 1, \cdots, p, \\
& p \text{ is integer.}
\end{aligned}
$$

Note that the decision variables are $x_{ik}$ and $y_{kj}$ as well as $p$. Here, as in most standard MMD models, we allow $x_{ik}$ and $y_{kj}$ to be continuous variables. This assumption is valid for large $r_{ij}$ because we can then re-scale $x$ so that $x$ and $y$ become integers. For example, let $R = \begin{bmatrix} 24 & 36 \\ 48 & 72 \end{bmatrix}$, we have one solution $x = [8/3, 16/3]$ and $y = [9, 27/2]^t$. We can rescale $x$ and $y$ to obtain an integer solution, $x = [4, 8]$ and $y = [6, 9]^t$.

One way to attack this problem is to solve a series of subproblems for $p$, $p = 1, 2, \cdots$, and find the solution that results in the lowest cost. As we fix $p$, $\sum_{k=1}^{p} c_k$ becomes a constant. We then seek to solve $x_{ik}$ and $y_{kj}$; however, this approach has two obstacles. First, solving (P1) for each given $p$ is not trivial. We can show that the objective function is not quasiconcave in $X$ and $Y$ and that the feasible region is not convex when $p > 1$. There is also no existing technique to solve these problems efficiently and there exists a duality gap when $p$ is less than the rank of $R$ as we will show later. Additionally, the optimal value as a function of $p$ is, in general, not convex in $p$. To see this, consider the following example. Let the demand matrix be $R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $a_i = b_j = 1$ for all $i$ and $j$, $c_k = 0.5$ for all $k$. For $p = 1$, one optimal solution is $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ with objective value of 9.5. For $p = 2$, $X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an optimal solution with objective value of 9. When $p = 3$, it is obvious that we should design three individual modules for all three demand types and there will be no over-supply of functions. The resulting optimal value is, therefore, 7.5. The non-convexity is due to the fact that from $p = 1$ to $p = 2$, the objective value decreases by 0.5 which is less than the decrease of 1.5 from $p = 2$ to $p = 3$.

Fortunately, as we will show later, $p$ can be bounded from above by the total number of demand types $n$ or the total number of functions $m$. If we can find efficient algorithms to solve Problem (P1) for a given $p$, we can find the optimal solution by solving a finite number of subproblems; therefore, we will proceed to develop an algorithm to solve (P1) for a fixed $p$.

Let $x'_{ik} = a_i x_{ik}$, $y'_{kj} = b_j y_{kj}$, $r'_{ij} = a_i b_j r_{ij}$. The $\sum_{k=1}^{p} c_k$ term can be dropped from the objective function because it is a constant for a fixed $p$. With this transformation, a standard MMD problem formulation with a fixed number $p$ of modules is obtained:

$$\text{(TP) Minimize}_{x_{ij}, y_{jk}} \quad \sum_{k=1}^{p} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ik} y_{kj} \tag{2.1}$$

$$\text{subject to} \quad \sum_{k=1}^{p} x_{ik} y_{kj} \geq r_{ij}, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \tag{2.2}$$

$$x_{ik}, y_{kj} \geq 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \quad k = 1, \cdots, p. \tag{2.3}$$

To see that there exits a duality gap, we examine the Lagrangian dual of Problem (TP):

$$
\begin{aligned}
L(R, x_{ik}, y_{kj}, \lambda_{ij}) &= \max_{\lambda_{ij}} \left\{ \min_{x_{ik}, y_{kj}} \sum_{ijk} x_{ik} y_{kj} - \sum_{ij} \lambda_{ij} (\sum_{k} x_{ik} y_{kj} - r_{ij}) \right\} \\
&= \max_{\lambda_{ij}} \left\{ \min_{x_{ik}, y_{kj}} \sum_{ij} (1 - \lambda_{ij}) \sum_{k} x_{ik} y_{kj} + \sum_{ij} \lambda_{ij} r_{ij} \right\} \\
&= \max_{\lambda_{ij}} \begin{cases} -\infty, & \text{if any } \lambda_{ij} > 1 \\ \sum_{ij} r_{ij}, & \text{if all } \lambda_{ij} \leq 1 \end{cases} \\
&= \sum_{ij} r_{ij}.
\end{aligned}
$$

On the other hand, adding all the constraints associated with $r_{ij}$ in Problem (TP) when the rank of $R > p$, we have

$$
\sum_{ij} \sum_{k} x_{ik} y_{kj} \quad > \quad \sum_{ij} r_{ij}.
$$

As we will see from Theorem 2 in the next section, only when the rank of $R \leq p$, is the optimal objective value of (TP) equal $\sum_{ij} r_{ij}$.

## 3. Structural Analysis

We now focus on solving the standard MMD problem (TP). In this section, we derive some structural results that will help us develop a procedure for solving Problem (TP).

**Theorem 1** *A minimum is attained in the multiple modular problem (TP).*

*Proof.* Since $X$ and $Y$ are finite dimensional, the result follows by sequential compactness because the objective function is continuous, as long as we can show that there is always an equal-objective-value solution in a closed and bounded region. To show this, suppose we have a sequence of feasible solutions, $(x^l, y^l)$, with decreasing objective values tending to the infimum of objective in (TP), but where the sequence has no limit points. We will show that there is an equivalent (same objective value) sequence $(x'^l, y'^l)$ which belongs to $B_\infty(M, n+m)$, the unit infinity norm ball of "radius" $M$ in $R^{m+n}$, where $M = z^0 > 1$, the objective value of $(x^0, y^0)$. We construct $(x'^l, y'^l)$ from $(x^l, y^l)$ at each step. Suppose $x^l_{i-ma_k} = max_k \; x^l_{ik} > M$. Note this means that $y^l_{kj} < M / x^l_{i-max_k}$ for all $j$ since the overall objective must decrease. An equivalent solution is $x'^l_{ik} = x^l_{ik}(M/x^l_{i-max_k}) < M$ for all $i$ and $y'^l_{kj} = y^l_{kj}(x^l_{i-max_k}/M) < 1$ for all $j$ and $k$. Let all others such that $x'^l_{ik} = x^l_{ik} < M$ and $y'^l_{kj} = y^l_{kj}$. Now, we have no $x'^l_{ik} > M$. If there still exists some $y'^l_{kj} > M$ for some $j$ and $k$, then we can follow an equivalent procedure to update $(x'^l, y'^l)$ such that each component is nonnegative and $\leq M$. The result is an equivalent sequence in the bounded region; This region is closed in finite dimensions with our continuous objective, we must have limit points that correspond to attained minima.

$\diamond$

**Theorem 2** *For any given p, Problem (TP) is equivalent to the following problem (TPP); i.e., the objective values of the two problems at the optimal solutions coincide.*

$$\begin{aligned} \textit{(TPP) Minimize} \quad & \sum_{ij}(\delta_{ij} + r_{ij}) && (3.1)\\ \textit{subject to} \quad & rank(R + \Delta) = p, && (3.2)\\ & \Delta \geq 0, && (3.3) \end{aligned}$$

*where* $\Delta = (\delta_{ij})_{m \times n}$.

*Proof:* By introducing slack variables $\delta_{ij} \geq 0$, the problem TP is transformed to the following equivalent problem (PP):

$$\begin{aligned} \text{(PP) Minimize} \quad & \sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ik}y_{kj} && (3.4)\\ \text{subject to} \quad & \sum_{k=1}^{p} x_{ik}y_{kj} - \delta_{ij} = r_{ij}, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, && (3.5)\\ & x_{ik}, y_{kj}, r_{ij} \geq 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n, \quad k = 1, \cdots, p, && (3.6)\\ & \delta_{ij} \geq 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n. && (3.7) \end{aligned}$$

For any $X^*$ and $Y^*$ optimal to (TP), $\Delta^* = X^*Y^* - R$ is a feasible solution to TPP. On the other hand, for any solution $\Delta^*$ optimal to TPP, there exists $(X, Y) \geq (0, 0)$ such that $XY = \Delta^* + R$, $rank(X) = rank(Y) = p$, and $(X, Y)$ is feasible to (TP) with objective value $\sum_{i,j,k} x_{ik}y_{kj} = \sum_{ij}(r_{ij} + \delta^*_{ij})$.

$\diamond$

Since we can add any module to any given $p$ modules to create a feasible design for Problem (TP) with $p+1$ modules, the following is true.

**Lemma 1** *Given a demand matrix $R$, the optimal objective of (P1) net the fixed cost associated with the design, $\sum_{k=1}^{p} c_k$, decreases as $p$ increases.*

**Lemma 2** *At an optimal solution, the number of modules, $p$, is bounded from above by the rank of the demand matrix $R$.*

*proof*: For any $R \geq 0$ with rank $q$, there exists $X \geq 0$ with rank $q$ and $Y \geq 0$ with rank $q$ such that $X \times Y = R$ and $\sum_{k=1}^{q} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ik} y_{kj} = \sum_{ij} r_{ij}$. Note that with this design, no function is over-supplied and adding more modules only increases the fixed design costs. Therefore, $rank(R)$ is an upper bound of $p$ at an optimal solution.

**Theorem 3** *For any given p, there are at least p tight constraints in (3.5), corresponding to each row and column of R, at an optimal solution to Problem (TPP).*

*Proof:* Without loss of generality, we consider the first row. Suppose that at an optimal solution $\delta_{ij}^*$ to (TPP), there exists $l > n-p$ such that $\delta_{1j}^* > 0$ for all $j \leq l$. We now examine the following problem that is equivalent to Problem (TPP) but with variables $\delta_{1j}$ for $j = 1, \cdots, n$ and $\delta_{ij} = \delta_{ij}^*$ for $i \geq 2$.

$$\text{Minimize}_{\delta_{1j}} \quad \sum_j \delta_{1j} + \sum_{i \geq 2, j} \delta_{ij}^* \tag{3.8}$$

$$\text{subject to} \quad \sum_{j=i}^{p+i} ((r_{1j}) + (\delta_{1j})) A_{i,j} = 0, \quad i = 1, \cdots n-p, \tag{3.9}$$

$$\delta_{1j} \geq 0, \quad j = 1, \cdots, n, \tag{3.10}$$

where $A_{i,j}$ is the cofactor of the element $r_{1j}^*$ of the determinant of $A_i$, a $p+1$ square submatrix of $R^* = R + \Delta^*$ consisting of rows $1, \cdots, p+1$ and columns $i, \cdots, i+p$ of $R^*$. Since Problem (3.8) - (3.10) is a linear program, there exists an optimal solution at an extreme point $\hat{\delta_{1j}}$. Assume this extreme point solution has $\hat{\delta_{1k}} \neq 0$, for $k \in K = \{1, \cdots, n-p\}$. For all $j \notin K, \hat{\delta_{1j}} = 0$, i.e., at least $p$ of the $\hat{\delta_{1\cdot}}$'s are 0 and at most $n-p$ of them are greater than 0. Since $\delta_{1j}^*$ is feasible to Problem (3.8)-(3.10), we have $\sum_k \hat{\delta_{1k}} \leq \sum_j \delta_{1j}^*$. Therefore,

$(\hat{\delta_{1j}}, \delta^*_{ij})$, $i \geq 2$, is at least as good a solution as $\delta^*_{ij}$. The same argument can be applied to all other rows and columns. Since Problem (TPP) is equivalent to Problem (TP), the theorem holds.

◇

We now examine the KKT conditions for Problem (TP):

$$\sum_{j=1}^{n} y_{kj} - \sum_{j=1}^{n} \lambda_{ij} y_{kj} = 0, \quad \forall x_{ik} \neq 0; \tag{3.11}$$

$$\sum_{i=1}^{m} x_{ik} - \sum_{i=1}^{m} \lambda_{ij} x_{ik} = 0, \quad \forall y_{kj} \neq 0; \tag{3.12}$$

$$\lambda_{ij} \geq 0, \quad \forall (i,j); \tag{3.13}$$

$$\lambda_{ij}\Big(\sum_{k} x_{ik} y_{kj} - r_{ij}\Big) = 0, \quad \forall (i,j). \tag{3.14}$$

**Lemma 3** *The KKT equations for Problem (TP) are a necessary condition for optimality.*

*Proof:* Without loss of generality, we assume that the demand matrix has no zero rows and no zero columns. At any feasible solution $\bar{x}$ and $\bar{y}$, there exists at least one $(i,j)$ such that $\sum_k \bar{x}_{ik}\bar{y}_{kj} \geq r_{ij} > 0$ in each row and column. That is, for each $i$, at least one $\bar{x}_{ik} > 0$ and for each $j$, at least one $\bar{y}_{kj} > 0$.

We write the constraints of (TP) as $ga(X,Y) = R - XY \leq 0$, $gb(X,Y) = -X \leq 0$ and $gc(X,Y) = -Y \leq 0$. Let $IJ = \{ij : ga_{ij}(\bar{x},\bar{y}) = r_{ij} - \sum_k \bar{x}_{ik}\bar{y}_{kj} = 0\}$, $IK = \{ik : gb_{ik}(\bar{x},\bar{y}) = -\bar{x}_{ik} = 0\}$ and $KJ = \{kj : gc_{kj}(\bar{x},\bar{y}) = -\bar{y}_{kj} = 0\}$. Let $d = (dx, dy)^t$ where $dx_{ik}$ and $dy_{kj}$ are the directions of $x_{ik}$ and $y_{kj}$, respectively. Then we have

$$\nabla ga^t_{ij}(\bar{x},\bar{y})d = -\textstyle\sum_k (dx_{ik}\bar{y}_{kj} + dy_{kj}\bar{x}_{ik}),$$

$$\nabla gb^t_{ik}(\bar{x},\bar{y})d = -dx_{ik},$$

$$\nabla gc^t_{kj}(\bar{x},\bar{y})d = -dy_{kj}.$$

It is straightforward to show that

$$G_o = \{d : \nabla ga^t_{ij}(\bar{x},\bar{y})d < 0, \nabla gb^t_{ik}(\bar{x},\bar{y})d < 0, \nabla gc^t_{kj}(\bar{x},\bar{y})d < 0\}$$

is not empty and that the closure of

$$G_o = G' = \{d : \nabla ga^t_{ij}(\bar{x},\bar{y})d \leq 0, \nabla gb^t_{ik}(\bar{x},\bar{y})d \leq 0, \nabla gc^t_{kj}(\bar{x},\bar{y})d \leq 0\}.$$

Therefore, the cone of tangents is equal to $G'$, which means that the Abadie constraint qualification [1] and hence, Lemma 3 hold.

$\diamond$

With Theorems 2 and 3, we can develop an algorithm to solve Problems (TP) and (TPP) simultaneously as described in the next section. In the next section, we will always assume that $p$ in problem (TP) is less than the rank of $R$ because when $p$ is greater than or equal to the rank of $R$, the solution is trivial by Lemma 2.

## 4. Algorithm

Before presenting our algorithm, we first define a *forest-basis* (FB) and its properties. These properties enable us to find a feasible solution easily and update our feasible solutions by changing from one forest-basis to another forest-basis. The algorithm stops at a forest-basis from which a KKT point can be easily found.

**Definition 1** *A forest-basis (FB) for problem (TP) is defined as a minimum set of entries with the property that a unique rank p matrix $R'$ is determined when the values of entries in FB are given. A feasible forest-basis (FFB) is an FB such that given the values of entries $r'_{ij}$, $\forall(i,j) \in FB$, a rank p matrix $R' \geq R$ exist.*

Figure 1 is an example of FB of a matrix whose rank is greater than $p$.

**Lemma 4** *Given the values for the entries of an FFB to problem (TP), a unique matrix $R'$ with rank p and $R' \geq R$ exists such that the sum of all entries' values in $R'$ is minimum among all possible $R' \geq R$.*

*Proof:* We prove this result by constructing such a minimum sum matrix $R'$. Given the value in an FFB, for each entry not in this FFB, there exists a sub-matrix with $p+1$ rows and $p+1$ columns. The only entry with unknown value in that sub-matrix is the entry not in the FFB. Since our target matrix $R'$ has rank $p$, the determinant of that sub-matrix of $R'$ is 0 and the value of an unknown entry can be computed as follows.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $r'_{11}$ | $r'_{12}$ | $\cdots$ | $\cdots$ | $r'_{1,p+1}$ | | | | | |
| $r'_{12}$ | $r'_{22}$ | $\cdots$ | $\cdots$ | $r'_{2,p+1}$ | | | | | |
| | | $\cdots$ | | | | | | | |
| $r'_{p,1}$ | $r'_{p,2}$ | $\cdots$ | $\cdots$ | $r'_{p,p+1}$ | | | | | |
| | $r'_{p+1,2}$ | $r'_{p+1,3}$ | $\cdots$ | $r'_{p+1,p+1}$ | $r'_{p+1,p+2}$ | | | | |
| | | | $\cdots$ | | | | | | |
| | $r'_{2p,2}$ | $r'_{2p,3}$ | $\cdots$ | $r'_{2p,p+1}$ | $r'_{2p,p+2}$ | | | | |
| | | $r'_{2p+1,3}$ | $\cdots$ | $r'_{2p+1,p+1}$ | $r'_{2p+1,p+2}$ | $r'_{2p+1,p+3}$ | | | |
| | | | | | $\cdots$ | | $\vdots$ | | |
| | | | | | | | $\vdots$ | $\cdots$ | |
| | | | | | | | | | $\cdots$ |

Figure 1: The forest basis denoted for the $p$ modular problem

- Case 1: If the $p$ rows or columns in the FFB are independent, the unknown-value entry takes the value uniquely determined by setting the determinant of the sub-matrix to 0.

- Case 2: If the $p$ rows or columns in the FFB are dependent, then the unknown-value entry takes its lowest feasible value corresponding to $R$.

Now, if the $R'$ constructed by this method does not have a minimum sum, there are only two possible scenarios.

- Scenario 1: The value of the unknown entry in Case 1 is less than the value determined by our method. If this is true, the rank of $R' > p$.

- Scenario 2: The value of the unknown entry in Case 2 is less than the value determined by our method. If so, we would have $R' \ngeq R$.

Thus, the lemma is proved by contradiction.

$\diamond$

By Theorem 3, there are at least $p$ binding constraints in each row and column of $R$ at an optimal solution. The binding constraints form a forest, $F$, of $R$. It is straightforward to show that there exists a $FFB$ that contains $F$, i.e., $F \in FFB$ because a $FFB$ contains at least $p$ entries in each row and column.

**Definition 2** *A forest-basis-restricted problem (FBP) for (TP) is the following problem associated with a forest $F$ of $R$ and an $FFB$ associated with an $R'$ such that $F \in FFB$.*

$$\begin{aligned}
f = \textit{minimize}_{x,y} \quad & \sum_k \sum_i \sum_j x_{ik} y_{kj} && (4.1)\\
\textit{subject to} \quad & \sum_k x_{ik} y_{kj} = r'_{ij} = r_{ij}, \quad \forall (i,j) \in F, && (4.2)\\
& \sum_k x_{ik} y_{kj} = r'_{ij} \geq r_{ij}, \quad \forall (i,j) \in FFB \cap (i,j) \notin F, && (4.3)\\
& \sum_k x_{ik} y_{kj} \geq r_{ij}, \quad \forall (i,j) \notin FFB, && (4.4)\\
& x_{ik}, y_{kj} \geq 0, \quad \forall i, \ \forall j \ \textit{ and } \forall k. && (4.5)
\end{aligned}$$

**Lemma 5** *The optimal objective value to problem (3.15) - (3.19) is the sum of all $R'$ entries determined according to the method in Lemma 4 given the values of the entries in the FFB.*

*Proof:* The lemma follows from Lemma 4 and Theorem 2.

$\diamond$

By Lemma 5, any solution $X$ and $Y$ such that $X \times Y = R'$ solves the FBP. In other words, it is straightforward to find an optimal solution to the (FBP) problem, although our aim is to find an optimal solution to (TP). The algorithm that we develop uses the solutions that satisfy the KKT condition of the FFB problem at an optimal solution $(X, Y)$ to find a descent direction; thus, we discuss the solution of the KKT system of (FFB). Let

$$\begin{aligned}
h_{ij}(X,Y) &= \sum_k x_{ik} y_{kj}, \quad \forall (i,j) \in FFB,\\
g_{ij}(X,Y) &= \sum_k x_{ik} y_{kj}, \quad \forall (i,j) \notin FFB,\\
w1_{ik}(X,Y) &= x_{ik}, \quad \forall (i,k),\\
w2_{kj}(X,Y) &= y_{kj}, \quad \forall (k,j).
\end{aligned}$$

Then the KKT system of FBP becomes

$$\nabla f - \sum_{\forall ij \in FFB} \mu_{ij} \nabla h_{ij} - \sum_{\forall kl \notin FFB} \upsilon_{kl} \nabla g_{kl} - \tau 1_{ik} \nabla w 1_{ik} - \tau 2_{kj} \nabla w 2_{kj} = 0; \tag{4.6}$$

$$\upsilon_{kl} g_{kl} = 0 \quad \forall (k,l) \notin FFB; \tag{4.7}$$

$$\tau 1_{ik} w 1_{ik} = 0; \tag{4.8}$$

$$\tau 2_{kj} w 2_{kj} = 0; \tag{4.9}$$

$$\tau 1_{ik} \geq 0, \tau 2_{kj} \geq 0, \upsilon_{kl} \geq 0. \tag{4.10}$$

**Lemma 6** *The KKT system (4.6)-(4.10) at an optimal solution $(X, Y)$ to an FFB has a non-zero solution of $(\mu, \upsilon, \tau 1, \tau 2)$.*

*Proof:* We prove the lemma by constructing such a solution. Let $\upsilon = 0, \tau 1 = 0, \tau 2 = 0$. Then system (4.6) -(4.10) becomes

$$\nabla f - \sum_{\forall ij} \mu_{ij} \nabla h_{ij} = 0. \tag{4.11}$$

All we need to show is that there exists $\mu$, $\mu \neq 0$, that satisfies (4.11). If $\nabla f$ linearly depends on $\nabla h_{ij}$'s, then by $\nabla f \neq 0$, Lemma 6 follows. Indeed, $\nabla f$ is linearly dependent on $\nabla h_{ij}$'s as shown in claims 1 and 2.

**Claim 1** *$\nabla h_{ij}$'s are linearly independent .*

*Proof:* We prove Claim 1 by induction. First we claim that it is true for any $p+1$ by $p+1$ matrix $R$. By the definition of a forest base, only one entry of the $p+1$ by $p+1$ matrix $R$ is not in an FFB. Without loss of generality, let this entry be $(k, l)$. We want to show that $\sum_{ij|ij \neq kl} \alpha_{ij} \nabla h_{ij} = 0$ if and only if $\alpha_{ij} = 0, \forall ij \neq kl$. Since,

$$\frac{\partial h_{\cdot l}}{\partial y_{\cdot l}} = \begin{bmatrix} x_{11} & \cdots & x_{i1|i \neq k} & \cdots & x_{p+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{1p} & \cdots & x_{ip|i \neq k} & \cdots & x_{p+1,p} \end{bmatrix},$$

$$\frac{\partial h_{\cdot j|j \neq l}}{\partial y_{\cdot l}} = 0,$$

we have

$$\sum_{ij|ij \neq kl} \alpha_{ij} \nabla h_{ij} = 0$$

and

$$\sum_{i|i \neq k} x_{i\cdot}\alpha_{il} = 0.$$

Note that $X$ is a rank $p$ matrix; its $p$ rows are linearly independent and $\alpha_{il} = 0$, for all $i \neq k$. We now consider the equality constraint in the $i$th row where $i \neq k$. Since

$$\frac{\partial h_{i\cdot}}{\partial x_{i\cdot}} = \begin{bmatrix} y_{11} & \cdots & y_{1l} & \cdots & y_{1,p+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{p1} & \cdots & y_{pl} & \cdots & y_{p,p+1} \end{bmatrix},$$
$$\frac{\partial h_{s\cdot|s\neq i}}{\partial x_{i\cdot}} = 0,$$

we have $\sum_{ij|ij\neq kl} \alpha_{ij} \nabla h_{ij} = 0$, and hence, $\sum_j y_{\cdot j}\alpha_{ij} = 0$. Furthermore, the facts that $\alpha_{il} = 0$ and the solution $Y$ is rank $p$ ensure that $\sum_{j\neq l} y_{\cdot j}\alpha_{ij} = 0, \alpha_{ij} = 0,\ \forall j$. For $i = k$, since

$$\frac{\partial h_{k\cdot}}{\partial x_{k\cdot}} = \begin{bmatrix} y_{11} & \cdots & y_{1j|j\neq l} & \cdots & y_{1,p+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{p1} & \cdots & y_{pj|j\neq l} & \cdots & y_{p,p+1} \end{bmatrix},$$
$$\frac{\partial h_{s\cdot|s\neq k}}{\partial x_{k\cdot}} = 0,$$

we have

$$\begin{aligned} \sum_{ij|ij\neq kl} \alpha_{ij}\nabla h_{ij} &= 0, \\ \alpha_{kj} &= 0\ \forall j \neq l. \end{aligned}$$

because $Y$ has rank $p$. Thus we proved that $\alpha_{ij} = 0, \forall ij \neq kl$.

Claim 1 holds for any $p+1$ by $p+1$ matrix $R$. Suppose that Claim 1 is true for any $m$ by $n$ matrix $R$ where $m \geq p+1$ and $n \geq p+1$. We next show that Claim 1 is true for any $m$ by $n+1$ matrix and any $m+1$ by $n$ matrix. Without loss of generality, we consider an $m$ by $n+1$ matrix. Let $FFB'$ be a forest basis of an $m$ by $n$ matrix and $FFB''$ be a forest basis of an $m$ by $n+1$ matrix. By the property of a forest basis, $p$ of the entries in the $n+1$st column are in $FFB''$. Without loss of generosity, let them be in the set $B(n+1) = \{(s, n+1), \cdots, (s+p-1, n+1)\}$. For any $FFB''$, there exists an $FFB'$ and $B(n+1)$ such that $FFB'' = FFB' \cup B(n+1)$. We need to show that $\sum_{ij|ij \in FFB''} \alpha_{ij} \nabla h_{ij} = 0$ if and only if $\alpha_{ij} = 0, \forall ij \in FFB''$.

For the constraints in the $n+1$st column, we have:

$$\frac{\partial h_{ij \in B(n+1)}}{\partial y_{\cdot n+1}} = \begin{bmatrix} x_{s1} & \cdots & x_{i1} & \cdots & x_{s+p-1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{sp} & \cdots & x_{ip} & \cdots & x_{s+p-1,p} \end{bmatrix},$$

$$\frac{\partial h_{ij \notin B(n+1)}}{\partial y_{\cdot n+1}} = 0;$$

since $\sum_{ij \in FFB''} \alpha_{ij} \nabla h_{ij} = 0$, $\sum_{ij \in B(n+1)} x_{i \cdot} \alpha_{i,n+1} = 0$ and $\alpha_{i,n+1} = 0 \forall i = s, \cdots, s+p-1$. Therefore, $\alpha_{ij} = 0$, $\forall ij \in B(n+1)$ because the rank of $X$=$p$. Furthermore, since $\sum_{ij \in FFB''} \alpha_{ij} \nabla h_{ij} = \sum_{ij \in FFB'} \alpha_{ij} \nabla h_{ij} + \sum_{ij \in B(n+1)} \alpha_{ij} \nabla h_{ij} = 0$, we have $\sum_{ij \in FFB'} \alpha_{ij} \nabla h_{ij} = 0$. Thus $\alpha_{ij} = 0$, $\forall ij \in FFB'$ by induction and Claim 1 is true for an $m$ by $n+1$ matrix.

$\diamond$

**Claim 2** *$\nabla g_{kl}$'s, $\forall kl \notin FFB$, are linearly dependent on $\nabla h_{ij}$'s.*

*Proof:* Note that the value of every entry not in an FFB is determined by the values of the entries in the FFB. The entries not in the FFB form two mutually exclusive and collectively exhaustive classes. The first class includes all entries whose values are determined directly by the values of entries in the FFB, i.e., there is a $p+1$ by $p+1$ square sub-matrix $R''$ where a first class entry is the only entry not in the FFB. Without loss of generality, let the first class entry be $(p+1, p+1)$ in the $p+1$ by $p+1$ matrix $R''$ which corresponds to the function $g_{(p+1,p+1)}$. The other entries in $R''$ correspond to functions $h_{ij}$'s, where $(i,j) \neq (p+1, p+1)$. Let $X', Y'$ be the sub-matrices of the solution $X, Y$ such that $X' \times Y' = R''$. Since $X'$ is $p+1$ by $p$ and rank $p$, its $p+1$st row linearly depends on the other $p$ linearly independent rows. Hence, there is a unique non-zero vector $\gamma = (\gamma_1, \cdots, \gamma_p)^t$ such that $x'_{(p+1,\cdot)} = \sum_{i=1}^{p} \gamma_i x'_{i\cdot}$. Similarly, there is a unique non-zero $\beta$ such that $y'_{(\cdot,p+1)} = \sum_{j=1}^{p} \beta_j y'_{\cdot j}$. With these, we shall see that $\nabla g_{(p+1,p+1)} - \sum_{i=1}^{p} \gamma_i \nabla h_{(i,p+1)} - \sum_{j=1}^{p} \beta_j \nabla h_{(p+1,j)} + \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_i \beta_j \nabla h_{ij} = 0$. That is,

$$\frac{\partial g_{(p+1,p+1)}}{\partial x_{(p+1,\cdot)}} - \sum_{i=1}^{p} \gamma_i \frac{\partial h_{(i,p+1)}}{\partial x_{(p+1,\cdot)}} - \sum_{j=1}^{p} \beta_j \frac{\partial h_{(p+1,j)}}{\partial x_{(p+1,\cdot)}} + \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_i \beta_j \frac{\partial h_{ij}}{\partial x_{(p+1,\cdot)}}$$

$$= \quad \frac{\partial g_{(p+1,p+1)}}{\partial x_{(p+1,\cdot)}} - \sum_{j=1}^{p} \beta_j \frac{\partial h_{(p+1,j)}}{\partial x_{(p+1,\cdot)}} = y'_{(\cdot,p+1)} - \sum_{j=1}^{p} \beta_j y'_{\cdot j} = 0.$$

Similarly,

$$\frac{\partial g_{(p+1,p+1)}}{\partial y_{(\cdot,p+1)}} - \sum_{i=1}^{p} \gamma_i \frac{\partial h_{(i,p+1)}}{\partial y_{(\cdot,p+1)}} - \sum_{j=1}^{p} \beta_j \frac{\partial h_{(p+1,j)}}{\partial y_{(\cdot,p+1)}} + \sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_i \beta_j \frac{\partial h_{ij}}{\partial y_{(\cdot,p+1)}}$$

$$= \quad \frac{\partial g_{(p+1,p+1)}}{\partial y_{(\cdot,p+1)}} - \sum_{i=1}^{p} \gamma_i \frac{\partial h_{(i,p+1)}}{\partial y_{(\cdot,p+1)}} = x'_{(p+1,\cdot)} - \sum_{i=1}^{p} \gamma_i x'_{i,\cdot} = 0;$$

$\forall r \neq p+1$,

$$\frac{\partial g_{(p+1,p+1)}}{\partial x_{(r,\cdot)}} - \sum_{i=1}^{p} \gamma_i \frac{\partial h_{(i,p+1)}}{\partial x_{(r,\cdot)}} - \sum_{j=1}^{p} \beta_j \frac{\partial h_{(p+1,j)}}{\partial x_{(r,\cdot)}} + \sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_i \beta_j \frac{\partial h_{ij}}{\partial x_{(r,\cdot)}}$$

$$= \quad -\gamma_r \frac{\partial h_{(r,p+1)}}{\partial x_{(r,\cdot)}} + \gamma_r \sum_{j=1}^{p} \beta_j \frac{\partial h_{(r,j)}}{\partial x_{(r,\cdot)}} = -\gamma_r (y'_{(\cdot,p+1)} - \sum_{j=1}^{p} \beta_j y'_{\cdot j}) = 0;$$

$\forall s \neq p+1$,

$$\frac{\partial g_{(p+1,p+1)}}{\partial y_{(\cdot,s)}} - \sum_{i=1}^{p} \gamma_i \frac{\partial h_{(i,p+1)}}{\partial y_{(\cdot,s)}} - \sum_{j=1}^{p} \beta_j \frac{\partial h_{(p+1,j)}}{\partial y_{(\cdot,s)}} + \sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_i \beta_j \frac{\partial h_{ij}}{\partial y_{(\cdot,s)}}$$

$$= \quad -\beta_s \frac{\partial h_{(p+1,s)}}{\partial y_{(\cdot,s)}} + \beta_s \sum_{j=1}^{p} \beta_j \frac{\partial h_{(i,s)}}{\partial y_{(\cdot,s)}} = -\beta_s (x'_{(p+1,\cdot)} - \sum_{i=1}^{p} \gamma_i x'_{i,\cdot}) = 0.$$

Therefore, $\nabla g_{(p+1,p+1)} = \sum_{i=1}^{p} \gamma_i \nabla h_{(i,p+1)} + \sum_{j=1}^{p} \beta_j \nabla h_{(p+1,j)} - \sum_{i=1}^{p}\sum_{j=1}^{p} \gamma_i \beta_j \nabla h_{ij}$.

Next we consider the second class. The value of the first second-class entry $(k,l)$ is determined by a $p+1$ by $p+1$ matrix where that second class entry is the only unknown entry. Using similar arguments, $\nabla g_{kl}$ depends linearly on $\nabla h_{ij}$'s and $\nabla g_{pq}$ where $(p,q)$ belongs to the first-class entry. Hence, $\nabla g_{kl}$ depends linearly on $\nabla h_{ij}$'s. The same argument is true for all other second-class entries. Therefore, Claim 2 is true.

$\diamond$

Since $\nabla f = \sum_{\forall ij \in FFB} \nabla h_{ij} + \sum_{\forall kl \notin FFB} \nabla g_{kl}$ $\nabla f$ depends linearly on $\nabla h_{ij}$ by Claim 2. Furthermore, $\nabla f \neq 0$ because $X, Y \geq 0$ and $X \neq 0, Y \neq 0$ for all $R \neq 0$; Lemma 6 then follows by Claim 1.

$\diamond$

Next, we show that the solution of KKT system to an FFB gives a direction of changing forest-basis. By updating the forest basis, we can find better feasible solutions until we find a local optimum to Problem (TP).

**Theorem 4** *Given an optimal solution, $(X, Y)$, to an FBP of $R$ and $R'$ associated with the $FFB$; let $(X, Y, \lambda_{ij})$, $\forall(i,j)$, solve the KKT equation of the FBP (4.12-4.15); If for an entry $(s,t)$ in this $FFB$, over a sufficiently small* {**but positive? Prof. Birge, why we need it is positive. It is negative when we reduce** $r'_{st}$} *range of $\delta_{st}$ change of $r'_{st}$, the FB is still feasible, then: (a) if $\lambda_{st} < 0$, increasing $r'_{st}$ decreases the objective value $f$; (b) if $\lambda_{st} > 0$, decreasing $r'_{st}$ decreases the objective value $f$.*

*Proof:* Since there is a solution to the KKT system of any FBP, the feasible solution, $(X, Y, \lambda_{ij})$, $\forall(i,j)$, satisfies the following system of equations.

$$\sum_{j=1}^{n} y_{kj} - \sum_{j=1}^{n} \lambda_{ij} y_{kj} - \tau_{ik} = 0, \quad \forall(i,k); \tag{4.12}$$

$$\sum_{i=1}^{m} x_{ik} - \sum_{i=1}^{m} \lambda_{ij} x_{ik} - \tau_{kj} = 0, \quad \forall(k,j); \tag{4.13}$$

$$\lambda_{ij} \geq 0 \text{ and } \lambda_{ij}(\sum_k x_{ik} y_{kj} - r_{ij}) = 0, \ \forall(i,j) \notin FFB; \tag{4.14}$$

$$\tau_{ik}, \tau_{kj} \geq 0 \text{ and } \tau_{ik} x_{ik} = 0, \tau_{kj} y_{kj} = 0, \ \forall(i,k), \ \forall(k,j). \tag{4.15}$$

If $\exists(s,t) \in FFB$ satisfying the condition of the Theorem, with a small change of $\delta_{st}$ for $r'_{st}$ such that $r''_{st} = \delta_{st} + r'_{st}$, $r''_{ij} = r'_{ij}$ for all $ij \neq st$ and $ij \in FFB$, there exists $x_{ik} + \Delta x_{ik}$ and $y_{kj} + \Delta y_{kj}$ that are feasible with $R''$ replacing $R'$ and are optimal to $R''$ now by Lemma 4. Since the objective function and all the constraint functions are continuous and differentiable, the following can be obtained by Taylor's theorem.

$$f(X + \Delta X, Y + \Delta Y)$$

$$= f(X,Y) + \sum_{ik} \Delta x_{ik} \sum_j y_{kj} + \sum_{kj} \Delta y_{kj} \sum_i x_{ik} + \bigcirc(\Delta X, \Delta Y); \tag{4.16}$$

$$r'_{st} + \delta_{st} = \sum_k (x_{sk} + \Delta x_{sk})(y_{kt} + \Delta y_{kt}); \tag{4.17}$$

$$r'_{ij} = \sum_k (x_{ik} + \Delta x_{ik})(y_{kj} + \Delta y_{kj}), \ \forall(i,j) \in FFB; \tag{4.18}$$

$$r_{ij} = \sum_k (x_{ik} + \Delta x_{ik})(y_{kj} + \Delta y_{kj}), \ \forall \text{ binding}(i,j) \notin FFB; \tag{4.19}$$

$$\Delta x_{ik} = 0, \ \forall x_{ik} = 0, \tag{4.20}$$

$$\Delta y_{kj} = 0, \ \forall y_{kj} = 0. \tag{4.21}$$

The above Taylor expansion of the constraint functions are multiplied by their corresponding multipliers. They are then added to the Taylor expansion of the objective function to form:

$$
\begin{aligned}
&\Delta f = && (4.22)\\
&f(X+\Delta X, Y+\Delta Y) - f(X,Y)\\
&= \sum_{ik} \Delta x_{ik} \sum_j y_{kj} + \sum_{kj} \Delta y_{kj} \sum_i x_{ik} + \bigcirc(\Delta X, \Delta Y)\\
&+\lambda_{st}\delta_{st} - \lambda_{st} \sum_k \Delta x_{sk} y_{kt} - \lambda_{st} \sum_k \Delta y_{kt} x_{sk} - \bigcirc(\Delta x_{sk}, \Delta y_{kt})\\
&- \sum_{ij\neq st} \lambda_{ij} \sum_k \Delta x_{ik} y_{kj} - \sum_{ij\neq st} \lambda_{ij} \sum_k \Delta y_{kj} x_{ik} - \bigcirc(\Delta x_{ik}, \Delta y_{kj})\\
&- \sum_{ik,\forall x_{ik}=0} \tau_{ik} \Delta x_{ik} - \sum_{kj,\forall y_{kj}=0} \tau_{kj} \Delta y_{kj}. && (4.23)
\end{aligned}
$$

Since the non-binding constraint multipliers are zero, we have:

$$
\begin{aligned}
\Delta f = \quad & \sum_{\forall ik} \Delta x_{ik} (\sum_j y_{kj} - \sum_j \lambda_{ij} y_{kj} - \tau_{ik})\\
& + \sum_{\forall kj} \Delta y_{kj} (\sum_i x_{ik} - \sum_i \lambda_{ij} x_{ik} - \tau_{kj}) + \lambda_{st}\delta_{st} + \bigcirc(\Delta X, \Delta Y) \qquad (4.24)
\end{aligned}
$$

Note that $(X, Y, \lambda_{ij}, \tau_{ik}, \tau_{kj})$ satisfies the KKT equations. Hence,

$$
\begin{aligned}
\sum_{\forall ik} \Delta x_{ik} (\sum_j y_{kj} - \sum_j \lambda_{ij} y_{kj} - \tau_{ik}) = 0,\\
\sum_{\forall kj} \Delta y_{kj} (\sum_i x_{ik} - \sum_i \lambda_{ij} x_{ik} - \tau_{kj}) = 0.
\end{aligned}
$$

Since $\Delta x_{ik} = \delta_{st}(\frac{dx_{ik}}{dr'_{st}}) + O(\delta^2_{st})$ and $\Delta y_{kj} = \delta_{st}(\frac{dy_{kj}}{dr'_{st}}) + O(\delta^2_{st})$, higher orders of $\Delta x_{ik}$ and $\Delta y_{kj}$ are also higher orders of $\delta_{st}$. Therefore, $\Delta f = \lambda_{st}\delta_{st} + O(\delta_{st})$.

We will also see that the optimal objective value of an FBP as a function of $\delta_{st}$ is continuous and differentiable within its feasible region. Note that the value of each $(k,l) \notin FFB$ is determined by setting the determinant of a $p+1$ by $p+1$ sub-matrix of $R'$ to be zero. Therefore, $\psi_{kl}(\delta_{st})$ , the value of entry $(k,l)$ as a function of $\delta_{st}$, is a polynomial function of $\delta_{st}$. Let $\omega = \{\delta_{st} | \delta_{st} + r'_{st} \geq r_{st},\ \ \psi_{kl}(\delta_{st}) \geq r_{kl}, \forall (k,l) \notin FFB\}$. Keeping $FFB$ feasible means $\delta_{st} \in \omega$. Furthermore, $f(\delta_{st}) = \sum_{ij \in FFB} r'_{ij} + \sum_{\forall kl \notin FFB} \psi_{kl}(\delta_{st}) + \delta_{st}, \forall \delta_{st} \in \omega$. Therefore, $f(\delta_{st})$ is continuous and differentiable for $\delta_{st} \in \omega$, and Theorem 4 holds. $\diamond$

**Theorem 5** *If $(X', Y', \lambda_{ij})$ is a KKT point to the forest-base restricted problem of $R'$ with basis $FFB$, it is also a KKT point to problem (TP) if and only if $\lambda_{ij} \geq 0, \forall (i,j) \in FFB$.*

*Proof.* Theorem 5 follows directly from the KKT equation system of (TP) and the KKT equation system of an FBP. ⋄

**Lemma 7** *For an optimal solution $X', Y'$ to (TP) with at least p binding constraints in each row and column of R, there exists a basis FFB of $R' = X'Y'$ such that an optimal solution to the FBP in Definition 2 is identical to the optimal solution of (TP).*

*Proof:* The lemma follows directly from Definition 2 and Lemma 5.

⋄

Conceptually, there is a finite number of forest-bases (FB) for a given matrix $R$; therefore, an optimal solution can be found by comparing the solutions of all feasible FB problems. Since the values of some entries in an FB can change, however, it is very difficult to list all FB's.

The heuristic algorithm developed below can quickly lead to a KKT point to problem (TP). The algorithm involves changes of forest bases. It is similar to the simplex method and is an extension of the simplex-like method by Shaftel and Thompson [9]. We name the algorithm, *Extended Simplex-like Method.*

The steps of the algorithm are described as follows:

(0) Initialization. Find a feasible forest basis $T$ to $R^* \geq R$.

(1) Solve (X, Y) for a forest-base $T$ restricted feasible solution to $R^*$.

(2) Solve the KKT equations to obtain $\lambda_{ij}$'s. Let $(k,l) = argmax_{ij \in T}\{|\lambda_{ij}\}$.

(a) If $\lambda_{kl} < 0$, increase $r^*_{kl}$ to the point where either (i) the objective value stops decreasing or (ii) some cell not in $T$ becomes infeasible. If (i) is true, go to (1); otherwise, let the first infeasible cell substitute cell $(k,l)$ in the forest-basis $T$ and then go to (1).

(b)If $\lambda_{kl} > 0$ and $r^*_{kl} > r_{kl}$, reduce $r^*_{kl}$ to decrease the objective value. While reducing $r^*_{kl}$, (i) if $r_{kj}* = r_{kj}$ then go to (1); (ii) if the objective value starts to increase, go to (1); (iii) if some cells not in $T$ become infeasible, substitute cell $(k,l)$ in the $T$ by the first infeasible cell and then go to (1).

(c) If all cells satisfy the local optimal condition, stop; compute the local optimum $X, Y$.

## 5. Numerical Example

In this section, we demonstrate the algorithm by some numerical examples. We first consider the example in Thompson and Shaftel [10], where their heuristic partition method fails to find an optimal solution. The demand matrix in that example is given by $R = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$. Since the rank of R is 2, by Theorem 2, it is easy to show that the optimal objective value for $p = 2$ is the sum of the values of all entries in R.

We now illustrate our algorithm using the example in Evans [3]. This example is also used by Silverman [12] to illustrate his decomposition method. The demand matrix of the example is given by $R = \begin{bmatrix} 15 & 23 & 44 \\ 13 & 13 & 0 \\ 15 & 17 & 35 \\ 34 & 12 & 22 \end{bmatrix}$. Consider $p = 2$, the algorithm starts with a feasible $R^*$. Since the solution to $p = 1$ is a feasible solution to $p = 2$, we can start with the solution to $p = 1$: $X = \begin{bmatrix} 31.06 & 23 \\ 17.56 & 13 \\ 24.71 & 18.29 \\ 34 & 25.17 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 & 0.9420 \\ 0 & 1.000 & 0.6410 \end{bmatrix}$ Note that the above solution is an optimal solution for $p = 1$ because the problem with $p = 1$ can be converted to an equivalent convex problem.

Iteration 1: Step 1: Solve $(X, Y)$ for $R^* = \begin{bmatrix} 31.06 & 23 & 44 \\ 17.56 & 13 & \hat{24.87} \\ 24.71 & 18.29 & 35 \\ 34 & \hat{25.17} & 48.16 \end{bmatrix} = XY$. All entries except the ones with a hat form an initial forest-basis.

Step 2: Given $(X, Y)$ we solve for KKT conditions for the Lagrange multipliers corresponding to each cell (i,j) and we obtain $\Lambda = \begin{bmatrix} 1.3258 & 1.2217 & 0.6541 \\ 1.9420 & 1.6410 & 0.0000 \\ 1.9432 & 1.6418 & -0.0013 \\ -0.4696 & 0.0000 & 2.5602 \end{bmatrix}$.

Step 3: change the value of entries in the FFB. Note that some $\lambda_{ij} > 0$. We can reduce the corresponding cell's value. We reduce $r^*_{43}$ first until cell (4,2) becomes infeasible.

Step 4: Update the FFB. The basis is updated by pivoting cell (4,2) into the basis and cell (4,3) out of the basis.

Iteration 2: After the first iteration we have: $R^* = \begin{bmatrix} 31.06 & 23 & 44 \\ 17.56 & 13 & 24.\hat{8}739 \\ 24.71 & 18.29 & 35 \\ 34 & 12 & 40.\hat{5}606 \end{bmatrix}$, the corresponding $X = \begin{bmatrix} 31.06 & 23 \\ 17.56 & 13 \\ 24.71 & 18.29 \\ 34 & 12 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0 & 0.9893 \\ 0 & 1.000 & 0.5771 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1327.4480 & 774.7613 & -1339. \\ 1.9893 & 1.5771 & 0. \\ -1668.3841 & -972.8074 & 1668. \\ 1.9893 & 1.5771 & 0. \end{bmatrix}$

Since $\lambda_{33} > 0$, reducing $r^*_{33}$ will reduce objective value.

Iteration 3: After the second iteration, we have $R^* = \begin{bmatrix} 31.\hat{3}156 & 23 & 44 \\ 17.56 & 13 & 24.\hat{8}772 \\ 24.71 & 18.29 & 35 \\ 34 & 12 & 22 \end{bmatrix}$, the corresponding $X = \begin{bmatrix} 31.3156 & 44 \\ 17.56 & 24.8772 \\ 24.71 & 35 \\ 34 & 22 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0.0273 & 0.000 \\ 0 & 0.5033 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0.000 & 37.6844 & -17.4 \\ 0.9458 & 2.9868 & 0.0 \\ 2.2871 & -46.2159 & 24.7 \\ 1.0136 & 0.5007 & 1.2 \end{bmatrix}$

Since $\lambda_{31} > 0$ and $r^*_{31} > r_{31}$, we can reduce $r^*_{31}$.

Iteration 4: After the third iteration, we have: $R^* = \begin{bmatrix} 19.\hat{1}918 & 23 & 44 \\ 17.56 & 13 & 24.\hat{6}019 \\ 15 & 18.29 & 35 \\ 34 & 12 & 22 \end{bmatrix}$, the corresponding $X = \begin{bmatrix} 19.1918 & 44 \\ 17.56 & 24.6019 \\ 24.71 & 35 \\ 34 & 22 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0.0205 & 0.000 \\ 0 & 0.5138 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0.000 & 49.8082 & -24.0 \\ 0.9601 & 2.9463 & 0.0 \\ 2.2695 & -60.9597 & 32.8 \\ 1.0250 & -0.2206 & 1.6 \end{bmatrix}$

Since $\lambda_{21} > 0$ and $r^*_{21} > r_{31}$, we can reduce $r^*_{21}$.

Iteration 5: After the fourth iteration, we have: $R^* = \begin{bmatrix} 19.\hat{1}918 & 23 & 44 \\ 13 & 13 & 24.\hat{7}837 \\ 15 & 18.29 & 35 \\ 34 & 12 & 22 \end{bmatrix}$, the corresponding $X = \begin{bmatrix} 19.1918 & 44 \\ 13 & 24.7837 \\ 15 & 35 \\ 34 & 22 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0.0205 & 0.000 \\ 0 & 0.5138 & 1 \end{bmatrix}$, and $\Lambda = \begin{bmatrix} 0.000 & 49.8082 & -24. \\ 0.9601 & 2.9463 & 0. \\ 2.2744 & -61.2007 & 32. \\ 1.0175 & 0.1468 & 1. \end{bmatrix}$

Since $\lambda_{32} < 0$ and $r^*_{21} > r_{31}$, we can increase $r^*_{21}$.

Iteration 6: After the fifth iteration, we have: $R^* = \begin{bmatrix} 15 & 23 & 44 \\ 13 & 13 & 24.\hat{7}043 \\ 15 & 18.\hat{3}533 & 35 \\ 34 & 12 & 22 \end{bmatrix}$, the corresponding $X = \begin{bmatrix} 15 & 44 \\ 13 & 24.7043 \\ 15 & 35 \\ 34 & 22 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 0.0189 & 0.000 \\ 0 & 0.5163 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1.0034 & 0.8174 & 1.0943 \\ 0.9635 & 2.9369 & 0.0000 \\ 1.0189 & 0.0000 & 1.5163 \\ 1.0041 & 0.7812 & 1.1130 \end{bmatrix}$.

After the sixth iteration, the above solution is a KKT point to the original problem. Furthermore, we have verified that the local optimality condition holds at this solution. The objective value is 269.0577. This is the best solution found in Silverman's paper [12], however, it takes 56 iterations for their algorithm to arrive at this solution. This solution is obtained in 6 iterations with ESM.

## 6. Conclusion

In this paper we consider a general Multiple Modular Design problem. We prove the existence of the optimal solution to MMD problem and develop an efficient algorithm using the properties of the optimal solution. These properties enable us to easily solve some MMD problems for which the other methods fail or have long solution times. Future research directions are to find a good integer solution of MMD given a continuous solution, to include random demand distribution, and to look at more complex modularization costs.

## 7. Appendix 1, Disscussion of the degeneracy of the extended simplex-like method

A key requirement in Theorem 4 is a form of nondegeneracy for FFB solutions to generate a decent directions to a local optimality. In general, it is difficult to show that degeneracy cannot happen in ESM. In some special cases, we can show that there is no degeneracy in the method.

**Lemma 8** *No degeneracy happens using the Extended Simplex-like Method to solve for an optimal module given any 2 by 2 matrix with strictly positive entries .*

Let $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} > 0$. Without loss of generosity, we assume $FFB_1 = \{(1,1),(1,2),(2,1)\}$. By the algorithm, we obtain a solution $y_1 = 1, y_2 = r_{12}/r_{11}, x_1 = r_{11}, x_2 = r_{21}$. We solve the following K.K.T corresponding to $FFB_1$ for $\lambda$:

$$
\begin{aligned}
1 + r_{12}/r_{11} - \lambda_{11} - \lambda_{12} r_{12}/r_{11} = 0; \\
1 + r_{12}/r_{11} - \lambda_{21} = 0; \\
r_{11} + r_{21} - \lambda_{11} r_{11} - \lambda_{21} r_{21} = 0; \\
r_{11} + r_{21} - \lambda_{12} r_{11} = 0.
\end{aligned}
$$

We have $\lambda_{11} = 1 - r_{12}r_{21}/r_{11}^2, \lambda_{12} = 1 + r_{21}/r_{11}, \lambda_{21} = 1 + r_{12}/r_{11}$. If $\lambda_{11}$ is greater than 0, we obtain a local optimum and there is nothing to prove. When $\lambda_{11}$ is less than 0, i.e., $r_{11}^2 < r_{12}r_{21}$, we show that either we can reduce the objective value with $FFB_1$ by increasing $r_{11}$, or we change to another $FFB_2$ and stop.

Case I: $\omega = \{\delta_{11} | \delta_{11} + r_{11} > r_{11}, r_{12} * r_{21}/(\delta_{11} + r_{11}) \geq r_{22}\} \neq \phi$. In this case, by Theorem 4, we can reduce the objective value by increasing $r_{11}$ by some $\delta_{11} > 0$.

Case II: $\omega = \{\delta_{11} | \delta_{11} + r_{11} > r_{11}, r_{12} * r_{21}/(\delta_{11} + r_{11}) \geq r_{22}\} = \phi$; that is, $r_{22} \geq r_{12}r_{21}/r_{11}$. By the ESM, we move to $FFB_2 = \{(1,2),(2,1),(2,2)\}$. We obtain $y_1 = 1, x_2 = r_{21}, y_2 = r_{22}/r_{21}, x_1 = r_{12}r_{21}/r_{22}$ and $\lambda_{12} = 1 + r_{21}/r_{22}, \lambda_{21} = 1 + r_{12}/r_{22}, \lambda_{22} = 1 - r_{12}r_{21}/r_{22}^2$. This is actually an optimal solution to the problem because

$$
\lambda_{22} \;\geq\; 1 - r_{12}r_{21}/(r_{12}r_{21}/r_{11})^2
$$

$$\begin{aligned} &\geq 1 - r_{11}^2/(r_{12}r_{21}) \\ &> 0. \end{aligned}$$

$\diamond$

# 1 References


1. Bazarra M. S. and Shetty C. M. 1979. *Nonlinear programming: theory and algorithm.* U.S, John Wiley & Sons.

2. Charnes, A. and Kirby, M. 1965. "Modular Design , Generalized Inverses, and Convex Programming", *Operations Research* **13**, 836-847.

3. Evans, D. 1963. "Modular Design: A Special Case in Nonlinear Programming", *Operations Research* **11**, 637-647.

4. Evans, D. 1970, "A Note on Modular Design - A Special Case in Nonlinear Programming",*Operations Research* **18**, 562-564.

5. Fiacco A. V. 1983. *Introduction to sensitivity and stability analysis in nonlinear programming.* Academic Press.

6. OReilly, G. P. 1975. "The Modular Design Problem with Production and Inventory Considerations", Unpublished Ph.D Dissertation, Columbia University.

7. Passy, U. 1970. "Modular Design: An Application of Structured Geometric Programming", *Operations Research* **18**, 441-453.

8. Rutenberg, D. P. and Shaftel, T. L. 1971. "Product Design: Subassemblies for Multiple Markets",*Management Science* **18**, b22-b231.

9. Shaftel, T. L. and Thompson G. L. 1977. "A Simplex-Like Algorithm for the Continuous Modular Design Problem", *Operations Research* **25**, 788-805.

10. Shaftel, T. L. and Thompson, G. L. 1983. "The Continuous Multiple-Modular Design Problem", *Naval Research Logistics Quarterly* **30**, 199-215.

11. Shaftel, T. L. 1971. "An Integer Approach to Modular Design", *Operations Research* **19**, 130-134.

12. Silverman, G. J. 1972. "Primal Decomposition of Mathematical Programs by Resource Allocation: II. Computational Algorithm with and Application to the Modular Design Problem", *Operations Research* **20**, 58-74.

13. Smeers, Y. 1974. "A Monotonic Passy-Like Algorithm for the Modular Design Problem", *Mathematical Programming* **7**, 46-55.

14. Tönshoff Nils 1997. *Modular Machine Tools: Bundling and Pricing Strategies under Demand Uncertainty.* Germany, Deutscher Universitäts Verlag.

15. Zangwill, W. I. 1968. "Minimum Concave Cost Flows in Certain Networks", *Management Science* **14**, 429-450.