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Abstract

We address how a firm should optimally allocate its capacity investments between dedicated systems (DMS) and reconfigurable systems (RMS). Motivated by our work with machine tool makers, and auto companies who buy their tools, our model addresses the case of a firm making a single product for which demand is random. Furthermore, in stochastic time intervals, the firm introduces a new generation of products. Dedicated capacity is generally cheaper but only functional for one product generation while reconfigurable capacity can be reconfigured for future generations as well albeit at the expense of potentially higher initial investments per unit capacity, maintenance and unit production costs. Given the stochastic nature of demand, and new product introduction intervals, the firm has to decide every period how many reconfigurable and dedicated modules to buy or salvage.

We first characterize the structure of the optimal investment policy when DMS and RMS module sizes are identical. In this case, we are able to use multimodularity properties of value functions (Hajek (1985)), to show that the optimal policy structure of our model is of ISD (Invest/Stay Put/Disinvest) type (Eberly and Van Mieghen (1997)). We are also able to provide comparative statics results on how the thresholds defining the ISD structure are affected by changes in problem parameters. Interestingly, the amount of reconfigurable capacity is not necessarily monotonic in the probability of new product introductions or relative cost of dedicated capacity.

The ISD policy is unfortunately sub-optimal when DMS and RMS capacity have different module sizes. We provide numerical examples that display characteristics of the optimal policy with different module sizes. We are able to fully characterize the structure of the optimal investment policy under the irreversible investment assumption. For the general case, we provide structural results that significantly decrease the computational load for computing the optimal investment policy. We also show that the optimal policy has an ISD-like structure with perturbations caused by the fact that capacity is lumpy and not continuous as assumed in previous work.

Our results indicate that in most situations, firms should keep a portfolio of dedicated and reconfigurable machine tools and the mix should be driven by the relative costs of each as well as the forecasts on how often new products will be introduced and the expected level of demand. In many firms we

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are familiar with, these investment decisions are largely made by intuition and managers' faith (or lack thereof) in the "advantages" of reconfigurable systems. Thus, our model provides managers with important intuition into which variables managers should take into account (and how) in determining this allocation.

1 Introduction

Motivated by our work with the U.S. automotive industry and the machine tool industry which supply automotive companies, we consider sequential capacity planning decisions for a manufacturer who plans to satisfy uncertain demand of a single product in a marketplace. Due to intensified competition in the marketplace, uncertain changes in customer taste, and unpredictable changes in environmental and safety regulations, the manufacturer needs to introduce new versions of this product periodically. The manufacturer can invest in dedicated manufacturing system (DMS) and/or reconfigurable manufacturing systems (RMS). DMS equipment is generally cheaper per unit capacity but is only functional for one product generation. On the other hand, RMS equipment will last more than one generation albeit at the expense of potentially higher initial investments, maintenance and production costs per unit capacity. DMS and RMS capacity can be purchased in fixed module sizes; however, the module sizes of DMS and RMS capacity can be non-identical. In this paper, we address how the manufacturer should optimally add a portfolio of DMS and RMS capacity to the existing capacity to satisfy uncertain demand over a planning horizon.

The situations described above abound in the U.S. automotive industry. For example, consider a manufacturer who is building capacity for V-6 engines for a small sport utility. The manufacturer may only think of offering the SUV with a V-6 engine while the price of gas is relatively low, and CAFE requirements remain as they are now. However, sometime in the future, CAFE requirements may change or consumers may require more efficient engines due to increases in fuel prices. This may force the manufacturer to offer a more fuel-efficient V6 by perhaps changing valve angles or to start offering a V-4 engine. If the firm buys dedicated equipment, the cost to reconfigure would be prohibitive; however the cost of dedicated equipment is generally low. On the other hand, tooling companies now offer reconfigurable equipment which can be adjusted in relatively short time and expense but which has a very significant initial capital expense. Our paper focuses on how firms should allocate their capacity between these two types of capacity.

Our paper is focused on determining the optimal investment in a portfolio of DMS and RMS capacity. As such, it is related to a large body of work that exists in the capacity expansion and flexible manufacturing literature. However, it is different in several respects. First, unlike most papers in the flexible manufacturing literature, the reconfigurable machines we consider can only produce one product at a time. That is, although both flexible and reconfigurable machines can be set up to produce more than one product type, what differentiates them is one of time scale. Whereas a flexible machine may be set up to produce a different type of product every day or every week, the reconfigurable machines that machine tool manufacturers are producing will be set up only when the facility is experiencing a model changeover. Whereas flexible equipment can switch very quickly between previously determined types of products, reconfigurable equipment is designed to be able to be set up less quickly but over a wider range of possible configurations that might be required by changing customer tastes (e.g, a changeover time of 5-10 minutes for flexible equipment versus 3-7 days for reconfigurable equipment). Also, in general, the investment costs per unit capacity of FMS are higher than that for RMS. Based on a case study for high volume plants in the U.S. automotive industry, the initial investment cost for FMS (mainly CNCs) capable of switching quickly between predefined models is about 30 % higher than that for RMS (Spicer (2002). Thus, in many environments where demand is high enough to support a plant producing high volumes of a single product, manufacturers are considering adding reconfigurability to make model changeovers less costly and time consuming but not full flexibility to produce multiple products every week on the same lines.

Our paper is related to a literature on capacity investment and selection of flexible equipment. For excellent reviews of general capacity expansion problems, see Luss (1982), Freidenfelds (1981). In a recent paper, Van Mieghem (2002) provides a review of the literature on capacity portfolio investment and hedging. Gerwin and Kolodny (1992) provide thorough definitions and management issues of flexibilities in advanced manufacturing systems. For literature reviews on manufacturing flexibility, see Sethi and Sethi (1990), Gerwin and Kolodny (1992), Son (1992), and more recently De Toni and Tonchia (1998). Reeve and Sullivan (1990), Partovi (1990), and Primrose (1990) discuss several equipment selection methodologies and their advantages and disadvantages. On empirical work on selection of FMS capacity, see Jaikumar (1986), Womack et al. (1991), and Tombak and De Meyer (1988). Monahan and Smunt (1999) develop an analytical model for determining optimal acquisition of FMS capacity. For related issues in adoption of new technology under competition, see Reinganum (1981), Fudenberg and Tirole (1985), and Mamer and McCardle (1987).

Real options methodology has also been applied to evaluate the value of flexibility. Pindyck (1988) considers a homogeneous capacity expansion problem where the capacity can be acquired in infinitesimal sizes. The acquired capacity is, however, irreversible, i.e., it cannot be disposed once it has been acquired. On the other hand, Dixit (1989) considers the optimal entry (investment) and exit (disposal) time for a firm producing a single product with fixed demand where the price of the product follows a geometric Brownian motion process. Kulatilaka (1988) formulates stochastic dynamic programming models to evaluate option values of FMS capacity in a multiple product environment.

The traditional work on equipment selection focuses on selecting the best production system among the available alternatives. Until recently, few papers have analytically addressed the issue of determining optimal portfolio of manufacturing systems. Fine and Freund (1990) and Van Mieghem (1998) determine the optimal level of dedicated and flexible equipment in a two-product and random but stationary demand environment by using a 2-stage stochastic programming modeling approach. He and Pindyck (1992) consider the same issue under the Real Options framework. Li and Tirupati (1994) formulate mathematical programming models to determine the optimal capacity profiles of dedicated and flexible equipment to satisfy multiple products over a planning horizon. Since their models are very complicated, numerical solution approaches are the main discussion.

The papers closest to the present work are by Eberly and Van Mieghem (1997), Dixit (1997), and Harrison and Van Mieghem (1999). Eberly and Van Mieghem (1997) show that the ISD (Invest/Stay put/Disinvest) policy is optimal under a general set of assumptions. In short, the ISD policy is a policy where the state space is partitioned into several sub-regions including one continuation region, where if the current capacity is in this region, it is optimal to continue without investing or salvaging the existing capacity. The remaining state space is partitioned into several sub-regions where each sub-region is associated with the decision to invest/salvage to a boundary of the continuation region. (More details on the ISD policy in 2-dimensional space are elaborated in Section 3.) Independently, Dixit (1997) presents a model to determine an optimal mix of capital and labor resources in a stochastic stationary demand process over an infinite planning horizon and also shows that the optimal policy is also of ISD type. Harrison and Van Mieghem (1999) use the multidimensional newsvendor solution approach to show that optimal investment decision is of ISD type under independent and identically distributedas demand processes for each period. The authors show that the optimal dynamic investment path over time is to invest in a capacity portfolio at the beginning of the planning horizon and keep the capacity portfolio unchanged in later periods.

These last papers consider the problem in a continuous decision space where capacity can be purchased in arbitrary sizes. We, however, consider the problem in a discrete decision space where demand over time is a non-stationary stochastic process with discrete distribution and capacity can be purchased in modules of fixed sizes. The first contribution of this paper is to show that under such lumpy capacity, the ISD policy remains optimal when the two types of capacity have identical module sizes. We also show how when this assumption is violated, the policy is ISD-like with perturbations (however, the number of perturbations can be many depending on the problem). We also provide structural results that partially characterize the optimal policy structure. These structural results enable the load for computing the optimal policy to be significantly reduced compared to solving the original MDP formulation of the problem. Finally, when investments are irreversible, we are able to fully characterize optimal structure under lumpy, non-identical sized capacity.

A second contribution of the paper is provided by our explicit focus on the difference between DMS and RMS. We provide comparative statics and are able to answer questions like: 1) How does cheaper RMS capacity change firms' investment strategies or 2) Does an increase in the likelihood of new product introduction or in the pace of new product introduction necessarily lead firms to invest in more RMS capacity? For example, under stationary and identical demand and cost data assumptions, we are able to show how the pace of new product introduction plays a key role in such investment decisions dividing the policy region into three so that the firm invests only in DMS modules if this pace is low enough, only in

RMS modules if this pace is high enough. Thus, we believe that we also are able to provide new managerial insights specific to our problem.

The paper is organized as follows: Section 2 provides a general statement of the problem and a stylized model where DMS and RMS capacity have non-identical but fixed module sizes. Section 3 considers a special case where the module sizes of DMS and RMS are identical. Comparative statics and some numerical examples are provided in Section 4. Section 5 provides some results (but not a full characterization) of the optimal policy structure in the general case where module sizes of DMS and RMS capacity are non-identical. Furthermore, we provide a full characterization of the optimal policy under the irreversible investment environment. (The formal definition of irreversible investment environment will be provided in the next section.) The paper concludes in Section 6.

2 Problem Statement

Consider a manufacturer planning to satisfy demand for several generations of a product over the next T periods. At any point in time, the manufacturer produces only one generation (version) of the product (e.g., auto manufacturers producing the 2003 model year car no longer produce the 2002 model year car). While producing this generation, the manufacturer, however, works on the design of the next generation. Due to the usual uncertainties in new product design and development and changes in customer tastes , the firm is never completely sure when the next generation is going to be introduced.

We assume that at the beginning of period t, the manufacturer knows the whole history of product introduction times, demand levels for all periods $0, \dots, t-1$, and the product generation to be produced in period t. At that point in time, the manufacturer makes a decision whether to invest or disinvest in more capacity. Capacity can be in two forms: dedicated which can only be used with the current generation product, and reconfigurable which can be used in all future generations as well (during the planning horizon). After the manufacturer completes the capacity adjustment, he realizes the demand for period t. Whether a new product will be introduced in period $t+1$ is a function of the whole past history of new product introductions as well as demand in periods $0, \dots, t$. Before the beginning of period $t+1$ (alternatively, at the end of period t), the manufacturer is also made aware whether a new product will be introduced in period $t+1$.

The above description of system dynamics is described by the notation we will now introduce. Let $g^t = (g_0, g_1, \dots, g_t)$ denote the history for product generation sold in each period until period t where g_i represents the product generation in period i (e.g., $g_3 = 2$ denotes that in period 3, the second product generation was on sale), while $x^t = (x_0, x_1, \dots, x_t)$ denotes the history for demand level up to period t where x_i represents the demand level in period i. Let $\mathscr{H}_t^b$ denote the set of all possible realizations of history (which consists of demand and product generation for each period) up to period t before the manufacturer adjusts his capacity in the beginning of period of t, i.e.,

$$\mathscr{H}_t^b = \{h_t^b | h_t^b = (g^t, x^{t-1})\},$$

and define $\mathscr{H}_t^a$ as the set of all possible realizations of history in period t after the manufacturer has adjusted his capacity and realized his demand, i.e.,

$$\mathscr{H}_t^a = \{h_t^a | h_t^a = (g^t, x^t)\}.$$

Denote $\mathscr{H}^b = \bigcup_{t=0,\ldots,T} \mathscr{H}_t^b$, and $\mathscr{H}^a = \bigcup_{t=0,\ldots,T-1} \mathscr{H}_t^a$.

Finally, we define $p : \mathscr{H}^a \mapsto [0,1]$ to be a mapping from the history space to the interval $[0,1]$. Since the probability of a new product introduction in period $t+1$ is conditioned on g^t and x^t, we let $p(h_t^a)$ denote the probability of a new product generation in period $t+1$ given history $h_t^a = (g^t, x^t)$ of demands and new products up to and including period t. Demand level in period t, $X_t \in \mathbb{N}$, is random variable depending on history $h_t^b \in \mathscr{H}_t^b$. We assume that X_t is uniformly bounded for all t almost surely.

The firm can invest in both dedicated and reconfigurable modules to satisfy demand over time. A dedicated module can produce just one product generation and each dedicated module can produce $\kappa_D \in \mathbb{N}$ units per period. In period t, these modules can be acquired with an acquisition cost of $c_{D,t}$ dollars per module and can be sold/disposed with a salvage value/disposal cost of $s_{D,t}$ dollars per module if the module is sold/disposed while the current product is still being produced. If the next generation product has already been introduced, we assume that the dedicated modules have a salvage value/disposal cost $\tilde{s}_{D,t}$ per module and that these modules are immediately salvaged. Additionally, there is a maintenance cost for each dedicated module of $m_{D,t}$ dollars per period per module in period t. On the other hand, we assume that a reconfigurable module can produce multiple product generations and each reconfigurable module can produce $\kappa_R \in \mathbb{N}$ units per period with an acquisition cost of $c_{R,t}$ dollars per module and a salvage value/disposal cost of $s_{R,t}$ dollars per module at period t. There is a maintenance cost for each reconfigurable module of $m_{R,t}$ dollars per period per module in period t. The maintenance costs of both production systems are incurred regardless of whether we plan to use the production modules to produce any product in a given period. These costs would include annual maintenance costs and overhead costs, for example. We also assume that $c_{D,t} > s_{D,t'}$, $c_{R,t} > s_{R,t'}$ for all $t' \geq t$. We assume that any new capacity purchased comes online in the same period it is purchased. Finally, an investment environment is called *reversible* if we allow to salvage/dispose our existing capacity at any decision epoch, i.e., when $-\infty < \tilde{s}_{D,t} < s_{D,t}$ and $-\infty < s_{R,t}$ for all t. On the other hand, an investment environment is called *irreversible* if we cannot salvage/dispose any acquired capacity, except when the existing DMS modules become obsolete due to new product introductions, i.e., when $-\infty < \tilde{s}_{D,t}$ and $s_{D,t} = s_{R,t} = -\infty$ for all $0 \leq t < T$ and $s_{D,T} = \tilde{s}_{D,T} = s_{R,T} = 0$. In this paper, we assume that the investment environment is reversible, except where otherwise explicitly indicated.

Define $\pi_D, \pi_R, \phi : \mathscr{H}^b \mapsto \mathbb{R}_+$ to be mappings from the history space to a positive number. There is a profit (sales price - production costs) of $\pi_{D,t} = \pi_D(h_t^b)$ ($\pi_{R,t} = \pi_R(h_t^b)$) dollars for each unit of demand satisfied by DMS (RMS), and a penalty of $\phi_t = \phi(h_t^b)$ dollars for each unit of unsatisfied demand at period t given $h_t^b \in \mathscr{H}_t^b$. We assume the $\pi_{D,t} \geq \pi_{R,t} \geq 0$ for all t and h_t^b, i.e., the production cost per part for RMS modules is higher than that for DMS modules. Denote $\gamma_t(i,j;x)$ as the profit function in period t given that we have i DMS modules and j RMS modules and the realized demand for period t equals x units. Since $\pi_{D,t} \geq \pi_{R,t}$, we can write

$$\gamma_t(i,j;x) = \begin{cases} \pi_{D,t}x & , x \leq \kappa_D i \\ \pi_{D,t}\kappa_D i + \pi_{R,t}(x - \kappa_D i) & , \kappa_D i < x \leq \kappa_D i + \kappa_R j \\ \pi_{D,t}\kappa_D i + \pi_{R,t}\kappa_R j - \phi_t[x - (\kappa_D i + \kappa_R j)] & , x > \kappa_D i + \kappa_R j \end{cases} . \tag{2.1}$$

Let δ be the discount factor per period and $V_t(i,j;h_t^b)$ be the optimal expected net present value of having i dedicated modules and j reconfigurable modules in period t given history $h_t^b \in \mathscr{H}_t^b$. Thus, we can write the dynamic programming formulation for our model as :

$$V_t(i,j;h_t^b) = \max_{k,l \in \mathbb{N}} \left\{ \begin{array}{l} \alpha_t(k,l;h_t^b) - (m_{D,t}k + m_{R,t}l) \\ -(\beta_{D,t}(k-i) + \beta_{R,t}(l-j)) + W_{t+1}(k,l;h_t^b) \end{array} \right\}, \tag{F1}$$

where

$$\alpha_t(k,l;h_t^b) = \mathsf{E}[\gamma_t(k,l;X_t)|h_t^b]$$

is the one-period expected profit (and penalty) in period t obtained from i DMS modules and j RMS modules given history $h_t^b \in \mathscr{H}_t^b$ where E is the expected value operator,

$$W_{t+1}(k,l;h_t^b) = \delta\mathsf{E}[p(H_t^a)\{V_{t+1}(0,l;H_{t+1,N}^b) + k\tilde{s}_{D,t+1}\} + (1-p(H_t^a))V_{t+1}(k,l;H_{t+1,C}^b)|h_t^b]$$

is the expected profit-to-go function (conditioned on having k DMS modules and l RMS modules at the beginning of next period),

$$\beta_{D,t}(a) = \begin{cases} c_{D,t}a & , a \geq 0 \\ s_{D,t}a & , a < 0 \end{cases}, \qquad \beta_{R,t}(a) = \begin{cases} c_{R,t}a & , a \geq 0 \\ s_{R,t}a & , a < 0 \end{cases},$$

are, respectively, the cost function for investing (or salvaging) DMS and RMS modules in period t, $H_t^a = (h_t^b, X_t)$, $H_{t+1,N}^b = (H_t^a, g_{t+1} = g_t + 1)$ (the history in period $t+1$ when a new generation product is introduced in period $t+1$) and $H_{t+1,C}^b = (H_t^a, g_{t+1} = g_t)$ (the history in period $t+1$ when the product in period t and $t+1$ is the same). We further assume that at the end of the planning horizon T, we salvage all existing production modules, i.e., $V_T(i,j;h_T^b) = s_{D,T}i + s_{R,T}j$ for any $h_T^b \in \mathscr{H}_T^b$.

Eberly and Van Mieghem (1997), considered a general problem of determining optimal portfolio of m types of production capacity for satisfying demand of n products over time. Their main result (ISD policy is optimal) is based heavily on the assumption that the reward function is jointly concave in the capacities, leading to the joint concavity of the optimal value function, and the convexity of the decision space, i.e., it is allowed to invest/disinvest in arbitrary module sizes of any production capacity. On the other hand, we consider a problem where the decision space is $\mathbb{N}^2$. We will show that if the DMS and RMS have the same module sizes, the optimal policy for our problem is also of ISD type. (We do show that if this assumption is violated, the ISD policy is no longer optimal but the policy is ISD-like with perturbations in Section 5. This is critical as capacities for different manufacturing systems rarely come in identical sizes.) Due to the discrete spaces we work with, our proof uses a different approach than that of Eberly and Van Mieghem (1997) and we need to show that our value functions satisfy multimodularity properties (introduced by Hajek (1985) and further investigated by Altman et al. (2000)). We also explicitly focus on the effects of having a type of capacity (DMS) which becomes useless periodically whenever new products are introduced. We are able to derive comparative statics results showing how the ISD curves move as a function of problem parameters such as maintenance costs for DMS and RMS and probability of new product introduction. In the next section, we show that the ISD policy is also optimal for our problem when the module sizes for reconfigurable and dedicated capacity are identical.

3 Identical Module Sizes for Reconfigurable and Dedicated Systems

In this section, we focus on the case where $\kappa_D = \kappa_R = \kappa$, i.e., dedicated and reconfigurable modules come in the same sizes. Define $\hat{\alpha}_t(k, l; h_t^b) = \alpha_t(k, l; h_t^b)$ given that $\kappa_D = \kappa_R = \kappa$. We rewrite our formulation as

$$V_t(i, j; h_t^b) = \max_{k,l \in \mathbb{N}} \left\{ -(\beta_{D,t}(k - i) + \beta_{R,t}(l - j)) + \hat{W}_{t+1}(k, l; h_t^b) \right\}, \tag{F2}$$

where

$$\hat{W}_{t+1}(k, l; h_t^b) = \hat{\alpha}_t(k, l; h_t^b) - (m_{D,t}k + m_{R,t}l) + W_{t+1}(k, l; h_t^b),$$

is the expected profit-to-go after adjusting the existing capacity portfolio to k DMS and l RMS modules (each of size κ).

We will show that the optimal policy structure in period t given h_t^b is characterized by the ISD (Invest, Stay put, Disinvest) structure displayed in Figure 3.1. Essentially, this structure is described completely by four investment/disinvesment threshold levels (E, F, G, H in Figure 3.1 where $E = (k_t^{B,B}(h_t^b), l_t^{B,B}(h_t^b))$, $F = (k_t^{B,S}(h_t^b), l_t^{B,S}(h_t^b))$, $G = (k_t^{S,S}(h_t^b), l_t^{S,S}(h_t^b))$, and $H = (k_t^{S,B}(h_t^b), l_t^{S,B}(h_t^b)))$ and four switching curves connecting these threshold levels. (Note that by our notation, the first and the second component of the superscript represent the decision to buy (B) or sell (S) dedicated and reconfigurable modules, respectively.)

The optimal ISD structure can be described as follows:

Area 1 Invest in more production modules until the portfolio consists of $k_t^{B,B}(h_t^b)$ dedicated modules and $l_t^{B,B}(h_t^b)$ reconfigurable modules (i.e., go to point E).

Area 2 Invest in only dedicated modules until the portfolio reaches line EF.

Area 3 Invest in more dedicated modules and disinvest some reconfigurable modules until the portfolio consists of $k_t^{B,S}(h_t^b)$ dedicated and $l_t^{B,S}(h_t^b)$ reconfigurable modules (i.e., go to point F).

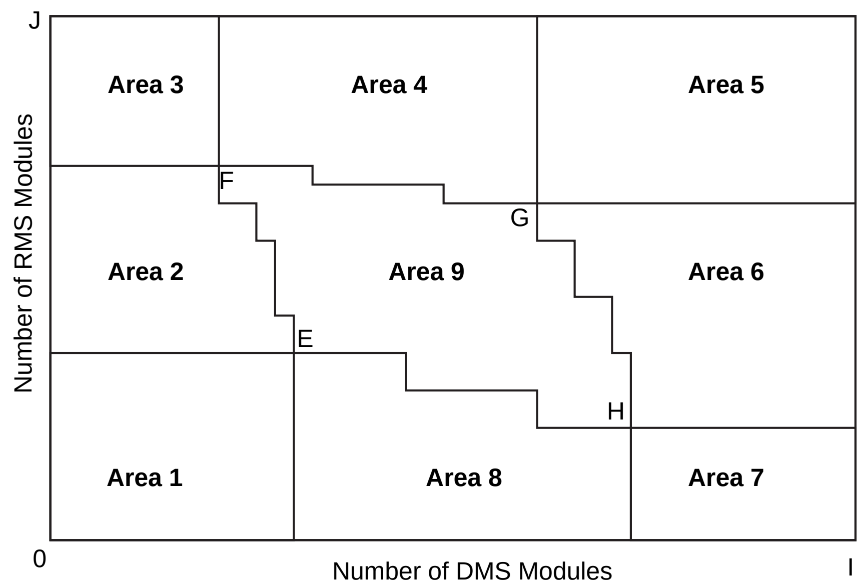

Figure 3.1: Optimal policy structure for Formulation F2

Area 4 Salvage only reconfigurable modules until the portfolio reaches line FG.

Area 5 Salvage modules of both dedicated and reconfigurable capacity until the portfolio consists of $k_t^{S,S}(h_t^b)$ dedicated and $l_t^{S,S}(h_t^b)$ reconfigurable modules (i.e., go to point G).

Area 6 Salvage dedicated modules until the portfolio reaches line GH.

Area 7 Invest in more reconfigurable modules and salvage some dedicated modules until the portfolio consists of $k_t^{S,B}(h_t^b)$ dedicated and $l_t^{S,B}(h_t^b)$ reconfigurable modules (i.e., go to point H).

Area 8 Invest in only reconfigurable modules until the portfolio reaches line EH.

Area 9 Do nothing.

The fact that we assume capacity comes in lumpy sizes makes it more complicated to prove an ISD-policy is optimal than in the continuous case. To show that the optimal investment decisions satisfy the structure shown in Figure 3.1, we first need to show that for each t, $V_t(i, j; h_t^b)$ is anti-multimodular in $(i, j) \in \mathbb{N}^2$ for given $h_t^b \in \mathscr{H}_t^b$. (We define multimodularity and anti-multimodularity in Appendix A.) From the definition of anti-multimodular functions, this is equivalent to showing the following conditions:
Condition: For all $0 \leq t \leq T$ and $h_t^b \in \mathscr{H}_t^b$,

1. $V_t(i+1, j+1; h_t^b) - V_t(i, j+1; h_t^b) \leq V_t(i+1, j; h_t^b) - V_t(i, j; h_t^b)$ for all i and j.

2. $V_t(i, j+1; h_t^b) - V_t(i, j; h_t^b) \leq V_t(i+1, j; h_t^b) - V_t(i+1, j-1; h_t^b)$ for all i and j.

3. $V_t(i+1, j; h_t^b) - V_t(i, j; h_t^b) \leq V_t(i, j+1; h_t^b) - V_t(i-1, j+1; h_t^b)$ for all i and j.

Note also that Conditions 1-3 imply that V_t is discretely concave in i for a given j and is discretely concave in j for a given i.

Before proceeding to the proof of these conditions, we first state the following results. Define $\hat{\gamma}_t(i,j;x) = \gamma_t(i,j;x)$ given that $\kappa_D = \kappa_R = \kappa$.

Lemma 3.1 *$\hat{\gamma}_t(i,j;x)$ is anti-multimodular in $(i,j) \in \mathbb{N}^2$ for given t and $x \in \mathbb{N}$, i.e.,*

1. $\hat{\gamma}_t(i+1,j+1;x) - \hat{\gamma}_t(i,j+1;x) \leq \hat{\gamma}_t(i+1,j;x) - \hat{\gamma}_t(i,j;x)$ *for all i and j.*

2. $\hat{\gamma}_t(i,j+1;x) - \hat{\gamma}_t(i,j;x) \leq \hat{\gamma}_t(i+1,j;x) - \hat{\gamma}_t(i+1,j-1;x)$ *for all i and j.*

3. $\hat{\gamma}_t(i+1,j;x) - \hat{\gamma}_t(i,j;x) \leq \hat{\gamma}_t(i,j+1;x) - \hat{\gamma}_t(i-1,j+1;x)$ *for all i and j.*

Proof: See Appendix B. □

Proposition 3.1 *$\hat{\alpha}_t(i,j;h_t^b)$ is anti-multimodular in $(i,j) \in \mathbb{N}^2$ for given $h_t^b \in \mathscr{H}_t^b$, i.e.,*

1. $\hat{\alpha}_t(i+1,j+1;h_t^b) - \hat{\alpha}_t(i,j+1;h_t^b) \leq \hat{\alpha}_t(i+1,j;h_t^b) - \hat{\alpha}_t(i,j;h_t^b)$ *for all i and j.*

2. $\hat{\alpha}_t(i,j+1;h_t^b) - \hat{\alpha}_t(i,j;h_t^b) \leq \hat{\alpha}_t(i+1,j;h_t^b) - \hat{\alpha}_t(i+1,j-1;h_t^b)$ *for all i and j.*

3. $\hat{\alpha}_t(i+1,j;h_t^b) - \hat{\alpha}_t(i,j;h_t^b) \leq \hat{\alpha}_t(i,j+1;h_t^b) - \hat{\alpha}_t(i-1,j+1;h_t^b)$ *for all i and j.*

Proof: : By definition, $\hat{\alpha}_t(i,j;h_t^b) = \mathsf{E}[\hat{\gamma}_t(i,j;X_t)|h_t^b]$. Since E is a linear operator, the results then follow from Lemma 3.1 □

We next prove a technical lemma which will be required to show that the value function is anti-multimodular.

Lemma 3.2 *For a given t, if $V_t(i,j;h_t^b)$ is anti-multimodular in i and j for all $h_t^b \in \mathscr{H}_t^b$, then, $\hat{W}_t(i,j;h_{t-1}^b)$ is also anti-multimodular in i and j for all $h_{t-1}^b \in \mathscr{H}_{t-1}^b$, i.e.,*

1. $\hat{W}_t(i+1,j+1;h_{t-1}^b) - \hat{W}_t(i,j+1;h_{t-1}^b) \leq \hat{W}_t(i+1,j;h_{t-1}^b) - \hat{W}_t(i,j;h_{t-1}^b)$ *for all i and j.*

2. $\hat{W}_t(i,j+1;h_{t-1}^b) - \hat{W}_t(i,j;h_{t-1}^b) \leq \hat{W}_t(i+1,j;h_{t-1}^b) - \hat{W}_t(i+1,j-1;h_{t-1}^b)$ *for all i and j.*

3. $\hat{W}_t(i+1,j;h_{t-1}^b) - \hat{W}_t(i,j;h_{t-1}^b) \leq \hat{W}_t(i,j+1;h_{t-1}^b) - \hat{W}_t(i-1,j+1;h_{t-1}^b)$ *for all i and j.*

Proof: We only show the proof for the first case as the others are entirely similar.

$$
\begin{aligned}
&\hat{W}_t(i+1,j;h_{t-1}^b) - \hat{W}_t(i,j;h_{t-1}^b) \\
&= \hat{\alpha}_{t-1}(i+1,j;h_{t-1}^b) - \hat{\alpha}_{t-1}(i,j;h_{t-1}^b) - m_{D,t-1} + \delta\mathsf{E}[p(H_{t-1}^a)\tilde{s}_{D,t}|h_{t-1}^b] \\
&\quad + \delta\mathsf{E}[(1-p(H_{t-1}^a))\{V_t(i+1,j;H_{t,C}^b) - V_t(i,j;H_{t,C}^b)\}|h_{t-1}^b] \\
&\geq \hat{\alpha}_{t-1}(i+1,j+1;h_{t-1}^b) - \hat{\alpha}_{t-1}(i,j+1;h_{t-1}^b) - m_{D,t-1} + \delta E[p(H_{t-1}^a)\tilde{s}_{D,t}|h_{t-1}^b] \\
&\quad + \delta\mathsf{E}[(1-p(H_{t-1}^a))\{V_t(i+1,j+1;H_{t,C}^b) - V_t(i,j+1;H_{t,C}^b)\}|h_{t-1}^b] \\
&= \hat{W}_t(i+1,j+1;h_{t-1}^b) - \hat{W}_t(i,j+1;h_{t-1}^b).
\end{aligned}
$$

The inequality follows from Condition 1 and Proposition 3.1 statement 1. □

Theorem 3.1 *For all* $0 \le t \le T, V_t(i,j;h_t^b)$ *is anti-multimodular in* (i,j) *for all* $h_t^b \in \mathscr{H}_t^b$*, and the optimal policy is of the ISD type.*

Proof: We prove the result by induction on t. First consider the case $t = T$. By the boundary condition, $V_T(i,j;h_T^b) = s_{D,T}i + s_{R,T}j$ which is obviously anti-multimodular. Suppose the assertion is true for $t = t'+1$ for all $h_{t'+1}^b \in \mathscr{H}_{t'+1}^b$.

Consider the case $t = t'$. By Lemma 3.2, $\hat{W}_{t'+1}$ is anti-multimodular since $V_{t'+1}$ is anti-multimodular by the induction hypothesis. Additionally, $\beta_{D,t'}(k-i) + \beta_{R,t'}(l-j)$ is multimodular in $(k,l) \in \mathbb{N}^2$ given $(i,j) \in \mathbb{N}^2$ since it is a summation of two piecewise linear and convex functions. Define $F_{t'}(k,l;i,j;h_{t'}^b) = -(\beta_{D,t'}(k-i) + \beta_{R,t'}(l-j)) + \hat{W}_{t'+1}(k,l;h_{t'}^b)$. This implies that $F_{t'}$ is also anti-multimodular in (k,l) given (i,j). Let $\tilde{F}_{t'}$ be the piecewise affine interpolation of $F_{t'}$ on $\mathbb{R}_+^2$ given $(i,j) \in \mathbb{N}^2$. Note that $\tilde{F}_{t'}(y,z;i,j;h_{t'}^b) = -(\beta_{D,t'}(y-i) + \beta_{R,t'}(z-j)) + \tilde{W}_{t'+1}(y,z;h_{t'}^b)$ for $(y,z) \in \mathbb{R}_+^2$ and $(i,j) \in \mathbb{N}^2$ where $\tilde{W}_{t'+1}$ is the piecewise affine interpolation of $\hat{W}_{t'+1}$. Define $\tilde{G}_{t'}(y,z;u,v;h_{t'}^b) = -(\beta_{D,t'}(y-u) + \beta_{R,t'}(z-v)) + \tilde{W}_{t'+1}(y,z;h_{t'}^b)$ for $(y,z),(u,v) \in \mathbb{R}_+^2$. Clearly, $\tilde{G}_{t'}(y,z;i,j;h_{t'}^b) = \tilde{F}_{t'}(y,z;i,j;h_{t'}^b)$ for all $y,z \in \mathbb{R}_+$ and $i,j \in \mathbb{N}$. This implies that $\tilde{G}_{t'}$ is an extension of $\tilde{F}_{t'}$ from $\mathbb{R}_+^2 \times \mathbb{N}^2$ to $\mathbb{R}_+^2 \times \mathbb{R}_+^2$. Moreover, $\tilde{G}_{t'}(y,z;u,v;h_{t'}^b)$ is jointly concave in (y,z) and (u,v) since $-(\beta_{D,t'}(y-u) + \beta_{R,t'}(z-v))$ is jointly concave in (y,z) and (u,v) and $\tilde{W}_{t'+1}(y,z;h_{t'}^b)$ is concave in (y,z) by Corollary A.1. By Proposition B-4 from Heyman and Sobel (1984), $\tilde{G}_{t'}^*(u,v;h_{t'}^b) = \sup_{y,z\in\mathbb{R}_+}\{\tilde{G}_{t'}(y,z;u,v;h_{t'}^b)\}$ is concave in $(u,v) \in \mathbb{R}_+^2$. By Theorem 2 from Eberly and Van Mieghem (1997), the concavity of this value function ensures that the ISD policy is optimal for $\tilde{G}_{t'}^*(u,v;h_{t'}^b)$. Finally, by Corollary A.2,

$$V_{t'}(i,j;h_{t'}^b) = \sup_{k,l\in\mathbb{N}}\{F_{t'}(k,l;i,j;h_{t'}^b\} = \sup_{y,z\in\mathbb{R}_+}\{\tilde{G}_{t'}(y,z;i,j;h_{t'}^b)\}$$

is anti-multimodular and the optimal policy structure of $V_{t'}$ is of ISD type. This implies that the assertion is true at $t = t'$ for all $h_{t'}^b \in \mathscr{H}_{t'}^b$. Hence, the result is true for all t. □

We note that our results are not direct extensions of either Eberly and Van Mieghem (1997) or Dixit (1997). In particular, concavity preservation of the value functions which are required by Eberly and Van Mieghem (1997) can not be directly used in our discrete space environment and in fact in Section 5, we will show examples of how the ISD policy is no longer optimal when the module sizes for dedicated and reconfigurable systems are different. (This issue does not come up in Eberly and Van Mieghem (1997) as they assume capacity can be bought in any amount in a continuous space.)

We conclude this section by restricting our attention to the special case where demand in each period is IID (and not dependent on product generation). The profit and penalty functions, the investment/disinvestment costs, and the maintenance costs are time-invariant. Finally, the chance of product change is geometric. Let X denote the (random) demand and let p denote the probability that a new product generation is introduced in any given period. Furthermore, we assume $c_{D,t} = c_D, c_{R,t} = c_R, s_{D,t} = s_D, \tilde{s}_{D,t} = \tilde{s}_D, s_{R,t} = s_R, m_{D,t} = m_D$, and $m_{R,t} = m_R$ for all t.

In this case, we show in the following theorem that there are two thresholds such that i) it is optimal to exclusively invest in DMS modules if p is less than a lower threshold which is independent of the distribution of demand, X, and ii) it is optimal to exclusively invest in RMS modules if p is higher than an upper threshold which depends on the distribution of demand X. That is, one would never invest in both types of modules when p is either too low or too high although if at time zero, the user has a DMS (or RMS) module, it is possible that it is optimal to keep it. This is a very interesting result as it provides managers with the important insight that if the pace of change is slow, it is probably best to invest in dedicated machines alone, while if it is really fast, one would best invest in reconfigurable systems only. However, in between, it is optimal to invest in a portfolio that includes both kinds of systems.

Theorem 3.2 *Consider Formulation F2 in an infinite horizon setting where demand in each period is IID, all problem parameters are time-invariant, and the chance of product change is geometric. Define*

$$p_* = \frac{(1-\delta)(c_R - c_D) + (m_R - m_D)}{\delta(c_D - \tilde{s}_D)},$$
$$p^* = p_* + \frac{(\pi_D - \pi_R)\mathsf{E}\min(X,\kappa)}{\delta(c_D - \tilde{s}_D)},$$

where κ is the module size of each production module. Then,

1. *It is optimal to invest only in DMS modules if $p < p_*$ regardless of the distribution of the demand X.*

2. *It is optimal to invest only in RMS modules if $p > p^*$.*

3. *If $p < p^*$, it is optimal to invest in at least one DMS module whenever it is optimal to invest in any capacity given that we currently have no capacity.*

Proof: See Appendix C. □

As an example for Theorem 3.2, consider the following problem parameters: $c_D = 1, s_D = 0.1, \tilde{s}_D = 0, m_D = 0.1, \pi_D = 0.6, c_R = 1.5, s_R = 0.3, m_R = 0.15, \pi_R = 0.5, \phi = 0.5$, and $\kappa = 1$. Assume that demand is IID and uniformly distributed from 1 to 10 units in each period. We further assume that we have no capacity in the beginning of the planning horizon. We then vary p and record the optimal investment level in DMS and RMS modules. The results are summarized in Table 3.1.

As can be seen in Table 3.1, as p increases, the optimal investment level in DMS modules decreases while the optimal investment level in RMS modules increases. (Although as we show in the next section, such monotonic behavior is not guaranteed under all conditions). Note that based on this parameter set, it can be computed that $p_* = 0.286$ and $p^* = 0.429$. Note that the region in which it is optimal to exclusively invest in DMS is actually $p \leq 0.35$, this is because the threshold provided by p_* is not tight. However, as shown in Theorem 3.2, the upper threshold is tight, i.e., whenever $p < p^*$, it is optimal to invest in at least one DMS module. Again, Table 3.1 confirms this result.

p	Opt. # of DMS Modules	Opt. # of RMS Modules
0.25	5	0
0.35	5	0
0.40	3	1
0.41	2	2
0.42	1	3
0.43	0	4

Table 3.1: The Effect of p on the Optimal Investment Levels in DMS and RMS Modules.

An interesting special case of Theorem 3.2 is when $\pi_D = \pi_R = \pi$. In this case, it is clear that the upper and lower thresholds coincide. Therefore, we can immediately state the result in the following

Corollary 3.1 *Consider Formulation F2 in an infinite horizon setting where demand in each period is IID, all problem parameters are time-invariant, the chance of product change is geometric and $\pi_D = \pi_R = \pi$. Then, $p_* = p^*$ and it is optimal to invest only in DMS modules if $p \leq p^*$. Otherwise, it is optimal to invest only in RMS modules. Furthermore, the optimal capacity choice is independent of the demand distribution.*

Having determined what the optimal policy structure is like, we next explore how optimal policy structure changes with problem parameters.

4 Comparative Statics and Computational Examples

In this section, we explore how the optimal policy changes as a function of problem parameters. In particular, we focus on a few parameters that have non-monotonic behavior. It would be natural, for example, to conjecture that an increase DMS capital investment costs, or a decrease in RMS capital investment costs would result in an immediate expansion of RMS capacity and a decrease in DMS capacity. We show that this is not always the case. Similarly, one might expect that a higher value of $p(h_t^a)$ would result in larger RMS and smaller DMS capacity, but we show that this is also not necessarily true without further assumptions. For example, one interesting insight is that if the next generation product's arrival is imminent, the firm may prefer to wait it out and invest in a new DMS in some situations, while if the next generation product's time of arrival is uncertain but generally expected in the next few periods, this may make it more likely that an RMS will be the firm's choice.

We start by analyzing the parameters that behave in completely intuitive ways. First we show that in Formulation F2, i.e., when DMS and RMS modules have the same module sizes, an increase in the maintenance cost of dedicated modules leads to a decision to invest in more reconfigurable modules and fewer dedicated modules. In this case for which the optimal policy is characterized in Theorem 3.1 and graphically shown in Figure 3.1, the points E, F, G and H, and the switching curves defined around those points in Figure 3.1 move in the northwest direction as the maintenance cost of dedicated modules increases. Similarly, maintenance costs for reconfigurable systems, $m_{R,t}$ and unit profit obtained by processing jobs on dedicated or reconfigurable systems ($\pi_{D,t}$ and $\pi_{R,t}$) result in intuitive behavior.

Before proceeding to show these results, we first prove that the value function defined in Formulation F2 is *supermodular*[1] (with respect to a space equipped with a binary relation defined below). Once we have established this result, we can immediately show the monotonicity property of the optimal solutions.

Assume a function $F(x, \tau)$ is jointly supermodular in x and τ where τ denotes a problem parameter and x is a decision variable. Let τ_1 and τ_2 be two instances of the problem parameters ordered under a binary relation $\preceq$, such that either $(\tau_1 \preceq \tau_2)$ or $(\tau_2 \preceq \tau_1)$ is true. Suppose $\tau_1 \preceq \tau_2$. Consider the problem $\max_{x \in X}\{F(x, \tau_i)\}, i = 1, 2$. Then, among (the possibly multiple) optimal solutions in both problem instances, there is at least one pair ordered such that $x_1^* \hat{\preceq} x_2^*$ where x_i^* is an optimal decision of problem instance i, (i = 1,2) and $\hat{\preceq}$ is a binary relation of the decision space X.

In our problem environment, we want to compare a point $x = (x_1, x_2)$ in a two-dimensional discrete space and another point $x' = (x_1', x_2')$ lying in the northwest direction of x. These points represent configurations of capacity portfolio where the first component represents the number of DMS modules and the second component represents the number of RMS modules that we currently have. Notice that these two points could not be ordered under the canonical binary relation of the real numbers. They could, however, be ordered under a binary relation where we use the reverse binary relation of the real numbers on the horizontal axis (number of DMS modules) and the canonical binary relation of the real numbers on the vertical axis (number of RMS modules). More precisely, define $(x_1, x_2) \preceq (x_1', x_2')$ if $x_1 \geq x_1'$ and $x_2 \leq x_2'$. For these reasons, we first redefine the binary relation of the state space of $V_t(i, j; h_t^b)$ in Formulation F2 as follows.

Let $\mathcal{S}_D = \mathbb{N}$ denote the set of possible number of DMS modules to own in each period. The binary relation $\preceq_D$ of $\mathcal{S}_D$ is defined to be the reverse binary relation of the real line, i.e., if $x_1, x_2 \in \mathcal{S}_D, x_1 \preceq_D x_2$, if $x_1 \geq x_2$. Let $\mathcal{S}_R = \mathbb{N}$ denote the set of possible number of RMS modules to own in each period. The binary relation $\preceq_R$ of $\mathcal{S}_R$ is defined to be the canonical binary relation of the real line, i.e., if $x_1, x_2 \in \mathcal{S}_R, x_1 \preceq_R x_2$ if $x_1 \leq x_2$. Define $\mathcal{S}_E = \mathcal{S}_D \times \mathcal{S}_R$, the space of capacity portfolio of DMS and RMS modules and $\preceq_E = \preceq_D \times \preceq_R$ as its associated binary relation. Thus, $(i_1, j_1) \preceq_E (i_2, j_2)$ if and only if $i_1 \geq i_2$ and $j_1 \leq j_2$. We next show that the investment/disinvestment cost function is a submodular function in the following technical lemma.

Lemma 4.1 *Define $f_t(i, j; k, l) = \beta_{D,t}(k - i) + \beta_{R,t}(l - j)$. Then f_t is jointly submodular in (i, j) and (k, l) on $\mathcal{S}_E \times \mathcal{S}_E$.*

Proof: By Corollary 2.6.1 from Topkis 1998, it suffices to show that $f_t(i, j; k, l)$ has decreasing differences

[1] For the complete treatment of this topic, please see Topkis (1998)

in each pair of its arguments. First, we show that f_t has *decreasing differences*[2] in (i,j) on $\mathcal{S}_E$, i.e.,

$$f_t(i,j+1;k,l) - f_t(i,j;k,l) \le f_t(i+1,j+1;k,l) - f_t(i+1,j;k,l).$$

Indeed,

$$\begin{aligned} f_t(i,j+1;k,l) - f_t(i,j;k,l) &= \beta_{R,t}(l-j-1) - \beta_{R,t}(l-j) \\ &= f_t(i+1,j+1;k,l) - f_t(i+1,j;k,l). \end{aligned}$$

By the same argument we can show that f_t has decreasing differences in (i,l) on $\mathcal{S}_E$, in (k,j) on $\mathcal{S}_E$, and in (k,l) on $\mathcal{S}_E$.

Next, we show that f_t has decreasing differences in (i,k) on $\mathcal{S}_D \times \mathcal{S}_D$, i.e.,

$$f_t(i,j;k,l) - f_t(i,j;k+1,l) \le f_t(i+1,j;k,l) - f_t(i+1,j;k+1,l)$$

Indeed,

$$\begin{aligned} f_t(i,j;k,l) - f_t(i,j;k+1,l) &= \beta_{D,t}(k-i) - \beta_{D,t}(k+1-i) \\ &\le \beta_{D,t}(k-i-1) - \beta_{D,t}(k-i) \\ &= f_t(i+1,j;k,l) - f_t(i+1,j;k+1,l), \end{aligned}$$

where the inequality follows from the fact that $\beta_{D,t}(a)$ is convex in a. By exactly the same argument, we can also show that f_t has decreasing differences in (j,l) on $\mathcal{S}_R \times \mathcal{S}_R$. Therefore, f_t is jointly submodular in (i,j) and (k,l) on $\mathcal{S}_E \times \mathcal{S}_E$. □

We are now ready to show that for any $h_t^b \in \mathscr{H}_t^b$, $V_t(i,j;h_t^b)$ defined in Formulation F2 is supermodular in (i,j) on $\mathcal{S}_E$.

Proposition 4.1 *For all $0 \le t \le T, V_t(i,j;h_t^b)$ (defined in Formulation F2) is supermodular in (i,j) on $\mathcal{S}_E$ for any history $h_t^b \in \mathscr{H}_t^b$.*

Proof: By Condition 1 for V_t,

$$V_t(i+1,j+1;h_t^b) - V_t(i+1,j;h_t^b) \le V_t(i,j+1;h_t^b) - V_t(i,j;h_t^b),$$

which implies that $V_t(i,j;h_t^b)$ has increasing differences in (i,j) on $\mathcal{S}_E$. Therefore, the result immediately follows from Corollary 2.6.1 from Topkis (1998). □

Denote $\mathbf{m_D} = \{m_{D,t}\}_{t=0}^T$ as the maintenance cost of DMS modules over the planning horizon. Define $\mathbf{m_D} \preceq \mathbf{m'_D}$ if and only if $m_{D,t} \le m'_{D,t}$ for all $0 \le t \le T$. Let $V_t(i,j;h_t^b;\mathbf{m_D})$ and $\hat{W}_{t+1}(i,j;h_t^b;\mathbf{m_D})$ be the parametric versions of $V_t(i,j;h_t^b)$ and $\hat{W}_{t+1}(i,j;h_t^b)$, respectively.

[2] Suppose that $x \in X$ and $t \in T$. A function $f(x,t)$ has *increasing differences* in (x,t) on $X \times T$ if and only if $f(x,t') - f(x,t)$ is increasing in x given $t \prec t'$ on the set of x such that $f(x,t') - f(x,t)$ is well-defined. A function $f(x,t)$ has decreasing differences in (x,t) on $X \times T$ if and only if $-f(x,t)$ has increasing differences in (x,t) on $X \times T$. Please, see Topkis (1998) for further details.

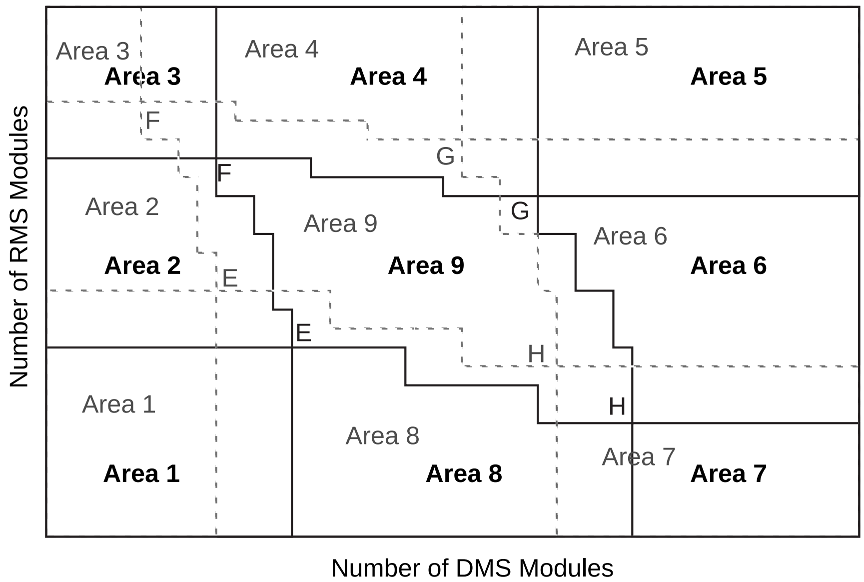

Figure 4.1: The shift in optimal policy regions with an increase in $\mathbf{m_D}$ or a decrease in $\mathbf{m_R}$ when module sizes for DMS and RMS are identical

Proposition 4.2 *For all* $0 \leq t \leq T$ *and* $h_t^b \in \mathscr{H}_t^b$, $V_t(i, j; h_t^b; \mathbf{m_D})$ *is supermodular in* $(i, j, \mathbf{m_D})$ *on* $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$.

Proof: See Appendix D. □

We now use Proposition 4.2 to show that when $\mathbf{m_D}$ increases, it is optimal to invest more (and disinvest less) in RMS modules and invest less (and disinvest more) in DMS modules. The shift in the optimal policy regions can be graphically displayed as in Figure 4.1 where the regions defined by the solid lines represent optimal decision regions when the maintenance cost of DMS modules is $\mathbf{m_D}$ and the regions defined by the dashed lines represent the optimal decision regions when the maintenance cost of DMS modules is $\mathbf{m'_D} \succeq \mathbf{m_D}$. We are also able to use the same arguments as in Proposition 4.2 to show the comparative statics of $\pi_{D,t}, \pi_{R,t}, \tilde{s}_{D,t}$ to the optimal policy structure. We state these general results in the following

Theorem 4.1 *For any period* $0 \leq t < T$, *an increase in* $\pi_{R,t}$ *and* $m_{D,t}$ *or a decrease in* $\tilde{s}_{D,t}, \pi_{D,t}$ *and* $m_{R,t}$ *causes the optimal number of dedicated modules to decrease and the optimal number of reconfigurable modules to increase.*

Proof: We prove only the effect of the changes in $\mathbf{m_D}$ to the shift of the optimal policy structure since the proof for the other parameters follows by exactly the same argument. Consider two problem instances of Formulation F2 where all parameters are identical except that in the first instance the maintenance cost of DMS modules is $\mathbf{m_D}$ while in the second instance, it is $\mathbf{m'_D} \succeq \mathbf{m_D}$. From the proof of Proposition 4.2, we showed that F_t is jointly supermodular in (k, l) on $\mathcal{S}_E$ and in $(i, j, \mathbf{m_D})$ on $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$. By Theorem

2.8.1 from Topkis (1998), $\arg\max_{(k,l)\in\mathcal{S}_E} F_t(k,l;i,j,h_t^b,\mathbf{m_D})$ is increasing in $(i,j,\mathbf{m_D})$ on $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$. The result, therefore, immediately follows. $\square$

Theorem 4.1 characterizes how the switching curves determining the structure of the optimal policy shift in a monotonic manner with changes in maintenance cost, or profit per part produced by either DMS or RMS modules. It is also interesting to characterize how changes in when the new product is expected affects the optimal policy. However, if we assume a very general probability function for when the next generation product will be introduced (where this probability could presumably depend on the whole history of product introductions and/or demand levels), it is easy to show that even if we compare two probability functions one of which stochastically dominates the other, the optimal policy switching curves do not move in a monotonic fashion. In fact, even if demand is constant from one period and product generation to another, and the probability that the next generation product is introduced is an IFR function of when the last generation product was introduced, changes in the probability function do not lead to monotonic behavior. This is very interesting as it indicates that even though we might expect a new product introduction sooner (in the stochastic sense), this may lead to buying fewer and not more reconfigurable modules. Appendix E exhibits a counterexample for the above situation.

It is possible to obtain monotonic behavior in how the switching curves move if we further restrict our demand probability, probability of new product introduction, profit, and penalty cost functions so that they are allowed to depend on past demand history but not on new product introduction history. That is, history h_t^b now equals x^{t-1}, the history of demands and does not contain g^{t-1}, the history of product introductions. In this case, Formulation F2 can be revised to become:

$$V_t(i,j;x^{t-1}) = \max_{k,l\in\mathbb{N}} \left\{ -(\beta_{D,t}(k-i) + \beta_{R,t}(l-j)) + \hat{W}_{t+1}(k,l;x^{t-1}) \right\}, \tag{F3}$$

with the boundary condition $V_T(i,j;x^{T-1}) = s_{D,T}i + s_{R,T}j$ for every history x^{T-1}.

Under these assumptions, we are able to show that if the probability of new product introduction (which depends on past history) stochastically increases so that the firm is more likely to introduce products more often on average, then the optimal investment policy shifts towards more reconfigurable and fewer dedicated modules. In order to prove this result, we first define a binary relation among conditional probability measures. Define $p \preceq_P p'$ if for all $t = 1,\ldots,T$, and all possible realizations of x^{t-1}, $p(x^{t-1}) \leq p'(x^{t-1})$. Let $V_t(i,j;x^{t-1};p)$ denote the parametric version for Formulation F3 given the conditional probability measure of new product arrival being p. We use the same argument as in the proof of Proposition 4.2 and Theorem 4.1 to show that if $p \preceq_P p'$, it becomes optimal to invest in more RMS modules and fewer DMS modules.

The proof once again proceeds by showing that $V_t(i,j;x^{t-1};p)$ is supermodular in (i,j,p) on $\mathcal{S} = \mathcal{S}_E \times [0,1]$ with respect to the binary relation $\preceq = \preceq_E \times \preceq_P$.

Proposition 4.3 *For all t and x^{t-1}, $V_t(i,j;x^{t-1};p)$ is supermodular in (i,j,p) on $\mathcal{S}$.*

Proof: The proof is omitted since it is similar to the proof of Proposition 4.2. □

We now use the results in Proposition 4.3 to show that if the probability of new production introduction increases such that $p \preceq_P p'$, it is then optimal to invest in more RMS modules and fewer DMS modules. We state this result in the following

Theorem 4.2 *Consider two instances of the system defined by Formulation F3 where all parameters are identical except in the first instance the conditional probability of new product introduction given past history is p and in the second instance it is p'. Suppose that $p \preceq_P p'$. Let $x^*(i,j;x^{t-1};p)$ and $x^*(i,j;x^{t-1};p')$ denote the optimal solutions in both instances. Then $x^*(i,j;x^{t-1};p')$ has more reconfigurable and fewer dedicated modules than $x^*(i,j;x^{t-1};p)$ for any i,j,t and x^{t-1}.*

Proof: The proof is similar to the proof of Theorem 4.1 and is omitted. □

The difference between Proposition 4.3 and the counterexamples shown in Figure E.1 is interesting. If new product introduction, and demand probabilities are allowed to depend on the time since the last product introduction, our results indicate that the behavior of the optimal policy is fairly complicated and an increase in the probability of a new product introduction may actually have the consequence that RMS becomes less desirable. Of course, in practice, the decision to introduce a new product is likely to depend on demand history just as demand is likely to depend on when the new product was introduced. This indicates that companies should take into account where products are in their life-cycle when making investment decisions.

Finally, we focus on the behavior of the optimal policy structure as a function of the investment costs. One might guess that the effect of the investment costs to the optimal policy will be similar to those in Theorem 4.1, i.e., *for any period t, an increase in $c_{D,t}$ or an decrease in $c_{R,t}$ causes the optimal number of dedicated modules to decrease and the optimal number of RMS modules to increase.*

Even when we restrict ourselves to the case where the investment/disinvestment costs are time-invariant, we can easily construct a counterexample to the above conjecture. Consider the following situation: Suppose that we plan to satisfy demand from period 0 to 9 where each unit of demand has to be satisfied. The remaining problem parameters are described as follows: $\pi_D = 0.6, \pi_R = 0.5, s_D = 0.1, c_R = 1.5, s_R = 0.1, m_D = 0.1, m_R = 0.1, \kappa = 1, \delta = 0.7$, and $\{d(t)\}_{t=0}^{9} = \{1,2,3,4,5,1,2,3,4,5\}$, where $d(t)$ is the demand in period t. We further assume that the chance of product change p is geometric and independent of the demand. In this example, we assume that $p = 0.1$. Suppose that the investment cost of DMS modules for the first problem instance is 1 per module while each DMS module for the second problem instance costs 20% more. Figure 4.2 displays the optimal decision regions at the beginning of the planning horizon for these problem instances where the left and right sub-figures display the optimal decision regions for the first and the second problem instances, respectively.

As can be seen in Figure 4.2, even though in almost all cases, it is optimal to have more RMS modules and fewer DMS modules in the second problem instance than in the first problem instance, it is optimal

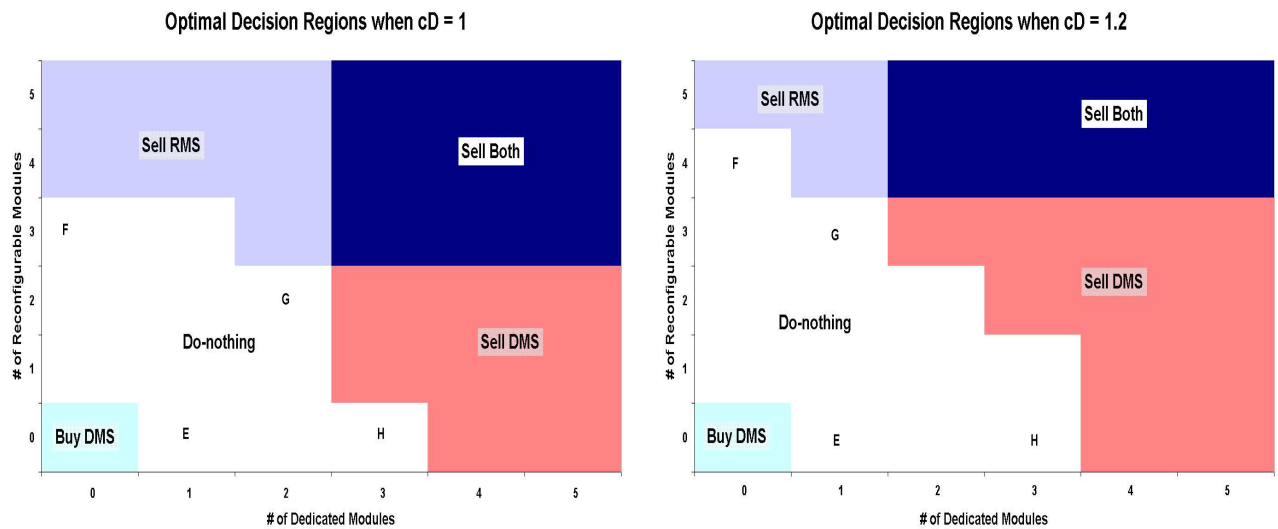

Figure 4.2: Optimal Decision Regions when $c_D = 1$ (Left) and $c_D = 1.2$ (Right) at time 0.

to have more DMS modules in the second problem instance if we have 3 DMS modules and one RMS module in the beginning of the planning horizon. In the first problem instance, it is optimal to salvage one excess DMS module whereas in the second instance, it is optimal to keep the excess DMS module idle in anticipation of satisfying future demand.

The reason for this phenomenon can be described as follows: in the first problem instance where the investment cost for DMS modules is lower, it is more economical for the firm to salvage excess capacity to immediately receive the salvage value and to prevent incurring the maintenance costs for the DMS. The firm can simply buy back more DMS capacity when demand increases. On the other hand, when the cost of a new DMS increases, this becomes more costly and the firm has a greater incentive to idle the unused capacity.

Counterexamples to the above conjecture are even more easily constructed if we allow the investment/disinvestment cost functions to be general functions of time; however, most of the counterexamples provide no further economic interpretation than the one that we have provided that when the capacity can be salvaged, lower capacity costs and high salvage value can lead to lower levels of capacity overall. It is, however, possible to show the monotonic shift of the optimal policy with respect to the investment costs if we further restrict our attention to the case where the investment costs are time-invariant under the irreversible investment environment (i.e., capacity can not be salvaged). To see this, let $V_t(i, j; h_t^b; c_D)$ and $\hat{W}_{t+1}(i, j; h_t^b; c_D)$ be the parametric versions for $V_t(i, j; h_t^b)$ and $\hat{W}_{t+1}(i, j; h_t^b)$, respectively, in the problem instance where the investment cost of dedicated modules is c_D. (Note that $V_t(i, j; h_t^b; c_D)$ is not supermodular in (i, j, c_D) on $\mathcal{S}_E \times \mathbb{R}_+$ in this case). We can still show that the change in c_D will have the a monotonic effect on investment decisions.

Proposition 4.4 *For all* $0 \leq t \leq T$ *and* $h_t^b \in \mathscr{H}_t^b$, $V_t(i, j; h_t^b; c_D) - c_D i$ *is supermodular in* (i, j, c_D) *on* $\mathcal{S}_E \times \mathbb{R}_+$.

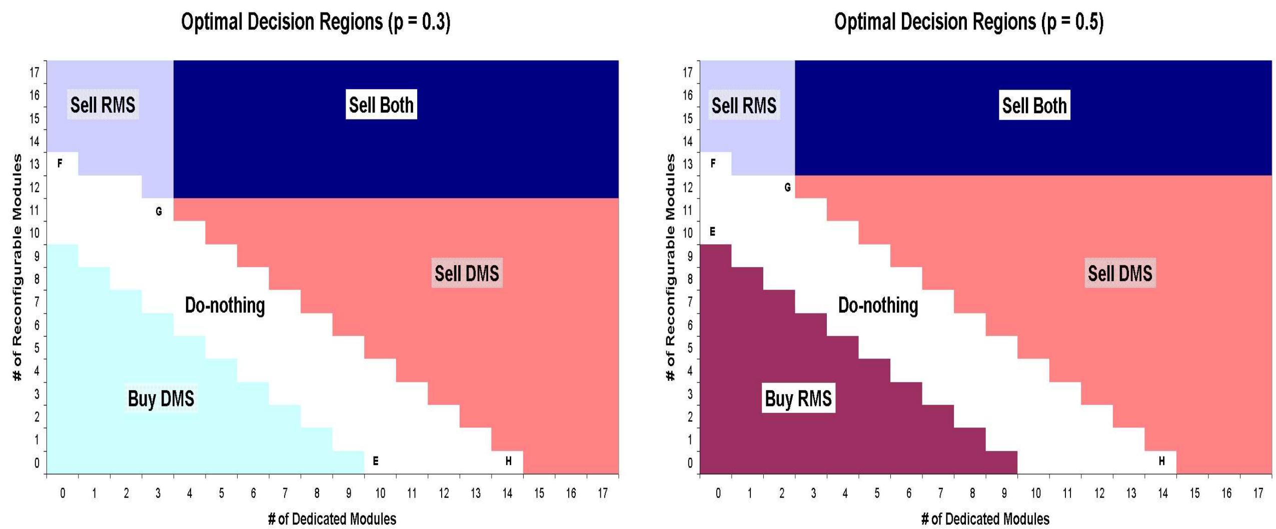

Figure 4.3: Optimal Decision Regions when $p = 0.3$ and $p = 0.5$.

Proof: The proof is omitted since it is similar to the proof of Proposition 4.2. □

Theorem 4.3 *Consider two problem instances of Formulation F2 under the irreversible investment environment where all parameters are identical except that in the first instance the investment cost of DMS modules is c_D while in the second instance, it is $c'_D \geq c_D$. It is optimal to own more reconfigurable and fewer dedicated modules in the second instance.*

Proof: The proof is similar to the proof of Theorem 4.1 and is omitted. □

To conclude this section, we now turn to some numerical examples to illustrate our analytical results. For simplicity, consider an example where demand is IID and the distribution of the arrival time of the next generation product is geometric. Furthermore, we assume that all problem parameters are time-invariant. Specifically, let $\pi_D = \pi_R = \pi = 2, \phi = 1.5, \delta = 0.7, c_D = 5, s_D = 0.5, \tilde{s}_D = 0, c_R = 8, s_R = 1.5, m_D = 1.5, m_R = 2, \kappa = 3$ and the demand is uniformly distributed from 0 to 50 units. Figure 4.3 shows the optimal decision regions when $p = 0.3$ (on the left side) and $p = 0.5$ (on the right side).

As can be seen, when $p = 0.3$, $E = (10, 0), F = (0, 13), G = (3, 11)$, and $H = (14, 0)$. Furthermore, since p is relatively low, it is optimal to invest in DMS modules, if the firm does not have sufficient modules. For example, if the firm initially has no capacity at all, it is optimal to invest in 10 DMS modules. Every time a new product generation is introduced, it is optimal to continue investing in only DMS modules. On the other hand, when p is increased from 0.3 to 0.5, the points E, F, G and H shift in such a way that the firm ends up with more RMS and fewer DMS modules as a consequence of Theorem 4.2. Specifically, $E = (0, 10), F = (0, 13), G = (2, 12)$, and $H = (14, 0)$. In this case, if the firm has no capacity, it is optimal to invest in 10 RMS modules and make no more investments in the future. The optimal policy structures of these examples directly follow from Corollary 3.1, which implies that if $p \geq p^* = 0.4$, it is optimal to invest only in RMS modules; otherwise, invest only in DMS modules.

5 Optimal Policy Structure when Module Capacities are not Identical

We have so far only characterized the structure of the optimal investment/disinvestment policy in RMS and DMS in the case where both DMS and RMS have the same module sizes. In this section, we provide some aspects of the optimal policy structure of the model where DMS and RMS have different module sizes. In fact, one of the advantages of the reconfigurable machines that machine makers are designing is that they can be bought in smaller module sizes than DMS. We therefore assume throughout this section that $\kappa_D \geq \kappa_R$. We further assume that $\frac{c_{R,t}}{\kappa_R} \geq \frac{c_{D,t}}{\kappa_D}$ and $\frac{m_{R,t}}{\kappa_R} \geq \frac{m_{D,t}}{\kappa_D}$ for all period t to reflect the fact that the investment cost and maintenance cost per unit of capacity of reconfigurable capacity are higher than those of dedicated capacity.

First, we focus on the special case of Formulation F1 where demand is IID and independent of product generations, the chance of product change is geometric, the investment/disinvestment costs and maintenance costs are time-invariant, and the module size of DMS modules is an integer multiple of the module size of RMS modules, i.e., $\kappa_D = \hat{k}\kappa_R$ where $\hat{k}$ is a positive integer greater than 1. Our aim is to see if some of the results of Theorem 3.2 carry over to this more complex case. Indeed, we find that *i*) when the chance of product change is above an upper threshold as defined in Theorem 5.1, it is optimal to invest only in RMS modules, but *ii*) when the chance of product change is below the lower threshold as defined in the same theorem, then it is optimal to not invest more than $\frac{\kappa_D}{\kappa_R}$ in RMS modules. This result is interesting in that when the chance of product change is lower than the lower threshold, it could be optimal to invest in both DMS and RMS modules (instead of investing in only DMS modules as shown in Theorem 3.2). Before proceeding, denote $\bar{c}_D = \frac{c_D}{\kappa_D/\kappa_R}$, $\bar{s}_D = \frac{\tilde{s}_D}{\kappa_D/\kappa_R}$ and $\bar{m}_D = \frac{m_D}{\kappa_D/\kappa_R}$ as the normalized costs per κ_R units capacity of DMS modules. We formally state our results in the following

Theorem 5.1 *Consider Formulation F1 under an infinite horizon setting where we assume that demand X is IID (and independent of product generation), all problem parameters are time-invariant, the chance of product change is geometric, and $\kappa_D = \hat{k}\kappa_R$ where $\hat{k} \in \mathbb{N}$ and $\hat{k} \geq 2$. Define*

$$\begin{aligned} p_* &= \frac{(1-\delta)(c_R - \bar{c}_D) + (m_R - \bar{m}_D)}{\delta(\bar{c}_D - \bar{s}_D)}, \\ p^* &= p_* + (\pi_D - \pi_R)\frac{\mathsf{E}\min(X, \kappa_R)}{\delta(\bar{c}_D - \bar{s}_D)}. \end{aligned}$$

Then,

1. *It is optimal to invest only in RMS modules if $p > p^*$.*
2. *Suppose that $p < p_*$. Then, starting from no existing capacity, the number of RMS modules it is optimal to invest in is strictly less than $\hat{k}$.*

Proof: The proof is similar to the proof of Theorem 3.2. □

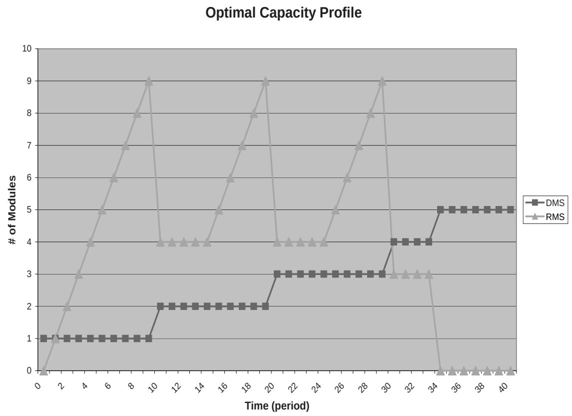

Figure 5.1: Optimal Capacity Profile when $c_R = 5$

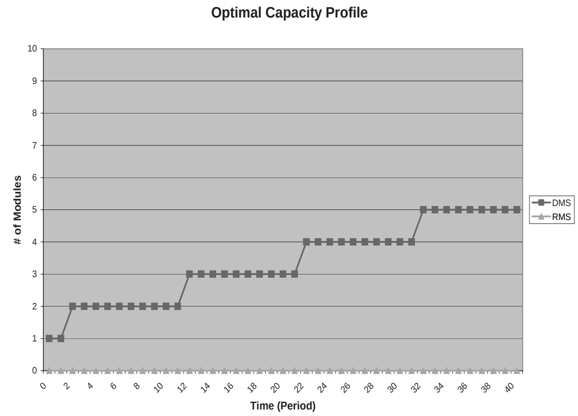

Figure 5.2: Optimal Capacity Profile when $c_R = 10$

In the case where demand is non-stationary, the optimal policy becomes more complicated (even when demand is deterministic). Consider an example of Formulation F1 where we plan to satisfy demand for 40 periods and demand is deterministic and increasing over time. Assume that demand in period 0 is 10 units and increases by a unit every period until period 40, i.e., demand in period t is $10 + t$ units. New product generations will also be deterministically introduced in period 10, 20, and 30. Suppose that we have no capacity in the beginning of the planning horizon and we sell the remaining capacity for salvage value at the end of the planning horizon. Assume that $\kappa_D = 10, \kappa_R = 1, c_D = 20, s_D = 5, c_R = 5, s_R = 2, m_D = m_R = \tilde{s}_D = 0, \pi_D = \pi_R = \pi = 1, \phi = 2.5$, and $\delta = 0.8$.

Figure 5.1 displays the optimal capacity profile over time for this problem instance. In this case, it is optimal to own both DMS and RMS modules at the same time due to the difference in module sizes. In the beginning of the planning horizon, it is optimal to immediately purchase a DMS module since demand in period 0 is 10 units and the capital cost per module of DMS is low relative to the profit and the penalty costs. From period 1 - 9, it is optimal to invest in one RMS module per period to manufacture the incremental demand that occurs during these periods since investing in RMS modules allows us to gradually make capital outlay, while satisfying demand during these periods even thought the capital cost per module of RMS is significantly higher than that of DMS. In period 10, the second product generation is introduced and the existing DMS becomes obsolete. In this case, it is optimal to immediately salvage 5 RMS modules and invest in 2 new DMS modules. Note that after the investment/disinvestment, demand is 20 units while the total capacity is 24 units (20 of which come from 2 DMS modules and the remaining come from 4 RMS modules). It is optimal not to salvage all RMS modules at this time because we have to keep some RMS modules in anticipation of increasing demand in the future. It is optimal to start to invest in new RMS modules again from period 15-19. Once again, the third product generation arrives in period 20, and the optimal investment action repeats itself from period 20 - 29. In period 30, the fourth product generation arrives and there is no more new product generation anticipated. In this case, it is optimal to gradually salvage the remaining RMS modules and invest in only new DMS modules which can be seen during period 30 - 40.

In comparison to the previous problem instance, Figure 5.2 exhibits the optimal capacity profile when c_R increases to 10. In this case, it is optimal to invest only in DMS module since the capital cost of RMS modules becomes prohibitively expensive. Notice also that it is optimal to invest in more DMS modules sooner than in the previous case to minimize the effect of penalty costs.

Next, we demonstrate that when we allow the module sizes of DMS and RMS modules to be general, the optimal policy becomes an ISD-like policy with several local perturbations. For example, consider a situation where demand in each period is a discrete uniform random variable between 0 to 60 units, the time until a new product introduction is geometrically distributed and independent of the demand levels, and we are planning our capacity with an infinite planning horizon. The following are the parameters for this problem instance:

$$\pi_D = \pi_R = \pi = 0.977, \phi = 2.724, p = 0.040, \delta = 0.924, \kappa_D = 4, \kappa_R = 3,$$
$$c_D = 18.8, c_R = 21.3, s_D = 5, s_R = 9, \tilde{s}_D = m_D = m_R = 0.0.$$

Denote (i^*, j^*) as the optimal capacity portfolio when starting the period with capacity portfolio (i, j) where i and j represent the numbers of existing DMS and RMS modules, respectively. Table 5.1 displays the optimal capacity portfolio (i^*, j^*) for given existing capacity portfolio (i, j). For example, if we have no capacity (i.e., capacity portfolio $(0, 0)$), it is optimal to make capacity adjustment to capacity portfolio $(13, 0)$. On the other hand, suppose that we have two RMS modules but no DMS module (i.e., capacity portfolio $(0, 2)$). In this case, it is optimal to make capacity adjustment to capacity portfolio $(10, 4)$, i.e., it is optimal to purchase 10 more DMS modules and two more RMS modules. Unlike the ISD policy structure, we see that the investment level in new RMS modules is not monotone in the number of existing RMS modules. This non-monotonicity is clearly due to the difference in module sizes for RMS and DMS.

Table 5.1 also indicates that the optimal policy for this problem instance possesses several features of the ISD policy. For example, if the current capacity portfolio lies in the northeast direction of capacity portfolio $(1, 18)$ (roughly corresponding to point G in Figure 3.1), it is optimal to sell existing capacity modules until we reach capacity portfolio $(1, 18)$. Similarly, the points $(13, 0), (0, 19)$, and $(15, 0)$ correspond to points E, F and H respectively in the description of the ISD policy in Figure 3.1. In Table 5.1, we denote these capacity portfolios with subscripts E, F, G, and H, respectively. Furthermore, the investment region lies roughly below curve FEH, the disinvestment region lies roughly above curve FGH and the continuation (i.e., no investment or disinvestment) region lies between curves FEH and FGH.

The optimal policy, however, has several perturbations that cause the violation of the pure ISD policy . More specifically, capacity portfolios $(1, 16), (4, 12), (7, 8)$, and $(10, 4)$ are optimal for some capacity portfolios that lie in their southwest direction. For example, capacity portfolio $(4, 12)$ is the optimal capacity portfolio for starting capacity portfolios (i, j) such that $0 \leq i \leq 4$ and $10 \leq j \leq 12$. In other words, these capacity portfolios act as "attractors" which attract the optimal portfolio after investment to a point other than E thus violating the ISD policy [3]. In Table 5.1, we denote these attractor optimal capacity portfolios with subscript A.

In the general case where we put no restriction on the module sizes of DMS and RMS, we can show further structural results despite the overall nonmonotonicity of the investment levels in the number of existing DMS or RMS modules. Before proceeding to show the structural results, we show that $\gamma_t(i,j;x)$ is supermodular in (i,j) on $\mathcal{S}_E$ for a given x and consequently $\alpha_t(i,j;h_t^b)$ is supermodular in (i,j) on $\mathcal{S}_E$ for a given $h_t^b \in \mathscr{H}_t^b$ for all t. From Eqn. 2.1, it is obvious that $\gamma_t(i,j;x)$ increases in i for given j and x and increases in j for given i and x for all t.

Proposition 5.1 *For all* $0 \le t \le T$,

1. $\gamma_t(i,j;x)$ *is supermodular in* (i,j) *on* $\mathcal{S}_E$ *for given* x.
2. $\alpha_t(i,j;h_t^b)$ *is supermodular in* (i,j) *on* $\mathcal{S}_E$ *for given* $h_t^b \in \mathscr{H}_t^b$.
3. $V_t(i,j;h_t^b)$ *is supermodular in* (i,j) *on* $\mathcal{S}_E$ *for given* $h_t^b \in \mathscr{H}_t^b$.

Proof: The proof is entirely similar to the proof in Lemma 3.1 statement 1 and Proposition 4.1 and is omitted for space. □

Theorem 5.2 *Consider two capacity portfolios* (i_1,j_1) *and* (i_2,j_2) *where* $i_1 \ge i_2$ *and* $j_1 \le j_2$. *Let* (i_1^*,j_1^*) *and* (i_2^*,j_2^*) *be the optimal portfolios after the capacity adjustment process for starting capacity portfolios* (i_1,j_1) *and* (i_2,j_2), *respectively. Then,* $i_1^* \ge i_2^*$ *and* $j_1^* \le j_2^*$.

Proof: The proof is entirely similar to the proof of Theorem 4.1 and is omitted for space. □

This result is helpful in decreasing the number of computations required to solve for the optimal solution as it limits the space of optimal actions in different states. Next, consider a capacity portfolio (i,j) consisting of i DMS modules and j RMS modules. Let (i^*,j^*) denote the optimal capacity portfolio when the starting capacity portfolio is (i,j), i.e., it is the optimal solution of Formulation F1. (If (i^*,j^*) is not unique, then choose a unique (i^*,j^*) using any tie-breaking rule.) Another interesting characteristic of the optimal policy is that if (i^*,j^*) is the optimal portfolio given starting state (i,j), then (i^*,j^*) is also the optimal portfolio of capacity given all starting states (k,l) where (k,l) is any integer point in the smallest rectangle containing (i,j) and (i^*,j^*). (Note that if $i=i^*$ or $j=j^*$, then we get a line and not a rectangle.) We formally state this result in the following

[3]We thank Jan Van Mieghem for suggesting the terminology which made this discussion clearer

Optimal Capacity Portfolio (i^*, j^*)

20	(0, 19)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)
19	$(\mathbf{0}, \mathbf{19})_{\mathbf{F}}$	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)
18	(0, 18)	$(\mathbf{1}, \mathbf{18})_{\mathbf{G}}$	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)	(1, 18)
17	(0, 17)	(1, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)	(2, 17)
16	(1, 16)	$(\mathbf{1}, \mathbf{16})_{\mathbf{A}}$	(2, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)	(3, 16)
15	(1, 16)	(1, 16)	(2, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)	(3, 15)
14	(1, 16)	(1, 16)	(2, 15)	(3, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)	(4, 14)
13	(3, 13)	(3, 13)	(3, 13)	(3, 13)	(4, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)	(5, 13)
12	(4, 12)	(4, 12)	(4, 12)	(4, 12)	$(\mathbf{4}, \mathbf{12})_{\mathbf{A}}$	(5, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)	(6, 12)
11	(4, 12)	(4, 12)	(4, 12)	(4, 12)	(4, 12)	(5, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)	(6, 11)
10	(4, 12)	(4, 12)	(4, 12)	(4, 12)	(4, 12)	(5, 11)	(6, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)	(7, 10)
9	(6, 9)	(6, 9)	(6, 9)	(6, 9)	(6, 9)	(6, 9)	(6, 9)	(7, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)	(8, 9)
8	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	$(\mathbf{7}, \mathbf{8})_{\mathbf{A}}$	(8, 8)	(9, 8)	(9, 8)	(9, 8)	(9, 8)	(9, 8)	(9, 8)	(9, 8)	(9, 8)
7	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(8, 7)	(9, 7)	(9, 7)	(9, 7)	(9, 7)	(9, 7)	(9, 7)	(9, 7)	(9, 7)
6	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(7, 8)	(8, 7)	(9, 6)	(10, 6)	(10, 6)	(10, 6)	(10, 6)	(10, 6)	(10, 6)	(10, 6)
5	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(9, 5)	(10, 5)	(11, 5)	(11, 5)	(11, 5)	(11, 5)	(11, 5)	(11, 5)
4	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	$(\mathbf{10}, \mathbf{4})_{\mathbf{A}}$	(11, 4)	(12, 4)	(12, 4)	(12, 4)	(12, 4)	(12, 4)
3	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(11, 3)	(12, 3)	(12, 3)	(12, 3)	(12, 3)	(12, 3)
2	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(10, 4)	(11, 3)	(12, 2)	(13, 2)	(13, 2)	(13, 2)	(13, 2)
1	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(12, 1)	(13, 1)	(14, 1)	(14, 1)	(14, 1)
0	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	(13, 0)	$(\mathbf{13}, \mathbf{0})_{\mathbf{E}}$	(14, 0)	$(\mathbf{15}, \mathbf{0})_{\mathbf{H}}$	(15, 0)
(i, j)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Number of DMS Modules (i)

Number of RMS Modules (j)

Table 5.1: Optimal Capacity Portfolio (i^*, j^*) for Given Existing Capacity Portfolio (i, j)

Average Relative Computational Time of sVIA over mVIA

DMS Module Size (κ_D)									
10	12.42		7.21				3.30		2.00
9	14.09	9.68		7.07	6.01		4.58	3.62	
8	16.34		9.41		7.21		5.94		
7	17.36	13.25	10.56	9.94	7.77	7.85			
6	18.80				8.82				
5	19.56	15.89	12.66	11.64					
4	28.65		15.58						
3	32.07	23.45							
2	43.55								
	1	2	3	4	5	6	7	8	9

RMS Module Size (κ_R)

Table 5.2: Comparison of Average Relative Computational Time of sVIA and mVIA for Various Module Sizes of DMS and RMS Capacity

Theorem 5.3 *Suppose* (i^*, j^*) *is the optimal capacity portfolio given starting capacity portfolio* (i, j) *for a given history* h_t^b *in Formulation F1. Then,* (i^*, j^*) *is also the optimal capacity portfolio given all starting states* (k, l) *for all* $\min\{i, i^*\} \leq k \leq \max\{i, i^*\}$ *and* $\min\{j, j^*\} \leq l \leq \max\{j, j^*\}$.

Proof: The result can be easily shown and the proof is omitted for space. □

Although the general Formulation F1 does not possess an elegant optimal policy structure such as the ISD policy, we can use Theorem 5.2 and Theorem 5.3 to reduce the action space in the value iteration algorithm. To get an empirical estimation of the range of benefits possible from the use of these results to reduce computational load, we randomly generated more than 3,000 examples with κ_D and κ_R ranging from 1 to 10 and $\kappa_D > \kappa_R$. We found that using Theorem 5.2 and Theorem 5.3 can speed up the value iteration algorithm anywhere from 1.5 times to 214 times in our numerical experiments.

Table 5.2 displays the average relative computational time of the standard Value Iteration Algorithm (sVIA) of MDP and our modified Value Iteration Algorithm (mVIA) using the results from Theorem 5.2 and Theorem 5.3. From Table 5.2, mVIA outperforms sVIA in all cases especially in the problem instances with high number of capacity states (when κ_D and κ_R are small), i.e., when $\kappa_D = 2$ and $\kappa_R = 1$, on average mVIA generates the optimal solutions on average more than 43 times faster than using sVIA. Notice that when κ_R increases with fixed κ_D the advantage of our algorithm degraded since the number of states of the problem instance is reduced. We believe that in the problem where demand over time is a more general process (dependent and non-stationary), our algorithm will perform even better due to significant increase in problem state space.

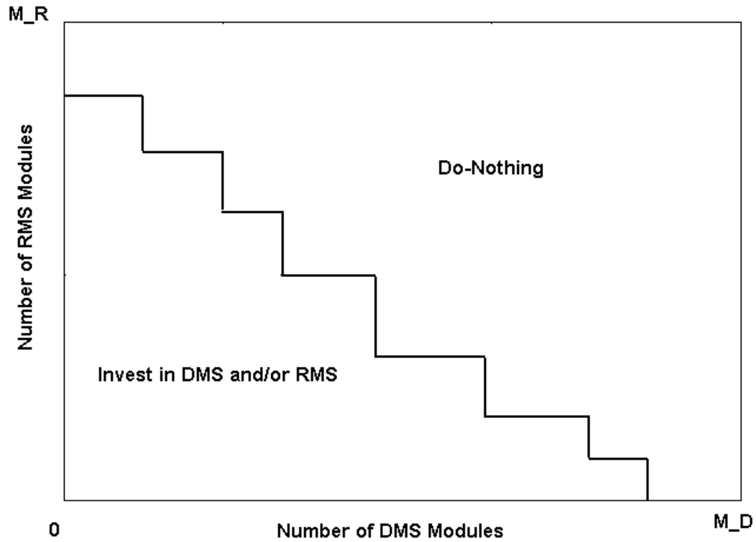

Figure 5.3: Optimal Policy Structure under Irreversible Investment Environment for Non-Identical Module Size Model

However, we are unable to characterize the optimal policy structure beyond Theorem 5.2 and Theorem 5.3 at this point for this most general case.

If, however, we focus our attention to the irreversible investment environment, where once a unit of capacity is bought, it is assumed that it can not be salvaged, we are able to characterize the structure of the optimal policy fully. For DMS, this assumption means that the equipment can only be replaced when a new generation product is introduced. For RMS, the assumption means that any bought equipment will be used until the end of the planning horizon. In this case, the optimal policy structure can be described by a downward slope curve separating investment and do-nothing regions where investment region is on the left side of the curve and do-nothing region is on the right side of the curve. This optimal policy structure is displayed in Figure 5.3. Before proceeding to prove this result, we show that $\gamma_t(i,j;x)$ and $\alpha_t(i,j;h_t^b)$ are discrete concave in i for a given j and in j for a given i in the following

Lemma 5.1 *For given t and x:*

1. $\gamma_t(i+1,j;x) - \gamma_t(i,j;x) \leq \gamma_t(i,j;x) - \gamma_t(i-1,j;x)$.

2. $\gamma_t(i,j+1;x) - \gamma_t(i,j;x) \leq \gamma_t(i,j;x) - \gamma_t(i,j-1;x)$.

Proof: We prove only 1) since the proof of 2) is entirely similar. First note that we can determine γ_t by solving the following linear programming formulation.

$$
\begin{array}{rclcccccccc}
\gamma_t(i,j;x) & = & \max_{y_D,y_R,z} & \pi_{D,t}y_D & + & \pi_{R,t}y_R & - & \phi_t z & & \\
& & \text{s.t.} & y_D & + & y_R & + & z & = & x \\
& & & y_D & & & & & \leq & \kappa_D i \\
& & & & & y_R & & & \leq & \kappa_R j \\
& & & y_D, & & y_R, & & z & \geq & 0
\end{array}
$$

where y_D and y_R represent, respectively, the number of parts manufactured by DMS and RMS modules and z represents the units of unsatisfied demand. Let $(y^*_{D,t}(i+1,j;x), y^*_{R,t}(i+1,j;x), z^*_t(i+1,j;x))$ and $(y^*_{D,t}(i-1,j;x), y^*_{R,t}(i-1,j;x), z^*_t(i-1,j;x))$ be the optimal solution for $\gamma_t(i+1,j;x)$ and $\gamma_t(i-1,j;x)$, respectively. Clearly,

$$
\frac{1}{2}(y^*_{D,t}(i+1,j;x), y^*_{R,t}(i+1,j;x), z^*_t(i+1,j;x)) + \frac{1}{2}(y^*_{D,t}(i-1,j;x), y^*_{R,t}(i-1,j;x), z^*_t(i-1,j;x))
$$

is a feasible solution to the following system of linear inequality:

$$
\begin{array}{ccccccc}
y_D & + & y_R & + & z & = & x \\
y_D & & & & & \leq & \kappa_D i \\
& & y_R & & & \leq & \kappa_R j \\
y_D, & & y_R, & & z & \geq & 0
\end{array}
$$

Then, it follows that

$$
\begin{aligned}
\gamma_t(i,j;x) \geq & \frac{1}{2}\{\pi_{D,t}(y^*_{D,t}(i+1,j;x) + y^*_{D,t}(i-1,j;x)) + \pi_{R,t}(y^*_{R,t}(i+1,j;x) + y^*_{R,t}(i-1,j;x)) \\
& - \phi_t(z^*_t(i+1,j;x) + z^*_t(i-1,j;x))\} \\
= & \frac{1}{2}\{\gamma_t(i+1,j;x) + \gamma_t(i-1,j;x)\},
\end{aligned}
$$

where the inequality follows from the optimality of $\gamma_t(i,j;x)$. The result then follows. □

Proposition 5.2 *For given t and $h^b_t \in \mathscr{H}^b_t$:*

1. $\alpha_t(i+1,j;h^b_t) - \alpha_t(i,j;h^b_t) \leq \alpha_t(i,j;h^b_t) - \alpha_t(i-1,j;h^b_t)$.

2. $\alpha_t(i,j+1;h^b_t) - \alpha_t(i,j;h^b_t) \leq \alpha_t(i,j;h^b_t) - \alpha_t(i,j-1;h^b_t)$.

Proof: Along with Lemma 5.1, the proof is entirely similar to the proof in Proposition 5.1 statement 2 and is omitted for space. □

Remember that we assume earlier that demand is uniformly bounded almost surely, i.e., we assume that there exists an constant $\bar{X}$ such that $X_t \leq \bar{X}$ almost surely for any given history h^b_t. Therefore the maximum numbers of DMS and RMS modules to hold are $M_D = \lceil \frac{\bar{X}}{\kappa_D} \rceil$ and $M_R = \lceil \frac{\bar{X}}{\kappa_R} \rceil$, respectively, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. We finally formalize the result in the following

Theorem 5.4 *Consider Formulation F1 under the irreversible investment environment. For any given history* h_t^b*, the optimal policy is characterized by a non-increasing threshold curve* $\tau(i, h_t^b)$ *for* $0 \le i \le M_D$*, such that for a given capacity portfolio* (i, j)*, if* $j \ge \tau(i, h_t^b)$*, then the optimal policy is to do nothing (i.e., no new investment) and if* $j < \tau(i, h_t^b)$*, then the optimal policy is to invest in RMS and/or DMS modules. Furthermore, define*

$$\begin{aligned} \mathscr{T}_1(h_t^b) &= \{(i, \tau(i, h_t^b)) | 0 \le i \le M_D\}, \\ \mathscr{T}_2(h_t^b) &= \{(i, j) | 1 \le i \le M_D, \tau(i, h_t^b) < j < \tau(i-1, h_t^b)\}, \\ \mathscr{T}(h_t^b) &= \mathscr{T}_1(h_t^b) \cup \mathscr{T}_2(h_t^b). \end{aligned}$$

Then, if it is optimal for capacity portfolio (i, j) *to be adjusted to capacity portfolio* (i^*, j^*) *for given* h_b^t*, then* $(i^*, j^*) \in \mathscr{T}(h_t^b)$*.*

Proof: Consider a capacity portfolio (i, j) where it is optimal not to make any further investment adjustments given h_t^b. Suppose that at capacity portfolio (k, l) where $k \ge i$ and $l \ge j$ where at least one inequality is strict, it is optimal to invest to capacity portfolio (k^*, l^*) where $k^* \ge k$ and $l^* \ge l$ where at least one inequality is strict. However, this obviously leads to a contradiction since the marginal benefits of adding capacity when one is already at capacity portfolio (k, l) will be less than that at capacity portfolio (i, j). To see this, by Proposition 5.1 statement 2 and Proposition 5.2, it follows that

$$\begin{aligned} &\alpha_t(k^*, l^*; h_t^b) - (m_{D,t}k^* + m_{R,t}l^*) - (\alpha_t(k, l; h_t^b) - (m_{D,t}k + m_{R,t}l)) \\ &\le \alpha_t(i + (k^* - k), j + (l^* - l); h_t^b) - (m_{D,t}(i + (k^* - k)) + m_{R,t}(j + (l^* - l))) \\ &\quad - (\alpha_t(i, j; h_t^b) - (m_{D,t}i + m_{R,t}j)), \end{aligned}$$

for given t and h_t^b. Note that the left hand side of the inequality represents the marginal benefits from investing at capacity portfolio (k, l) while the right hand side of the inequality represents the marginal benefits of the same investment at capacity portfolio (i, j).

The above automatically leads to the definition of the threshold τ for any given history h_t^b. For each $0 \le i \le M_D$, define $\tau(i, h_t^b)$ be the smallest l such that it is optimal to make no further capacity adjustment at capacity portfolio (i, l) for $0 \le l \le M_R$ and given h_t^b. If no such capacity portfolio exists, define $\tau(i, h_t^b) = M_D + 1$. The fact that $\tau(i, h_t^b)$ is a non-increasing function in i also follows from the above argument.

Finally, we prove that if it is optimal to make a capacity adjustment at capacity portfolio (i, j), it is optimal to make the adjustment to a capacity portfolio in $\mathscr{T}(h_t^b)$. We again prove this result by contradiction. There are two cases to consider. First, suppose that it is optimal to adjust capacity portfolio (i, j) to $(k, l) \notin \mathscr{T}(h_t^b), (i, j) \ne (k, l)$ and $l > \tau(k; h_t^b)$. There are two subcases to consider:

Subcase 1: $l > j$. It follows from Theorem 5.3 that it is as well optimal to adjust capacity portfolio $(k, l-1)$ to capacity portfolio (k, l). However, we previously showed that for capacity portfolio (k, l') where

$\tau(k, h_t^b) \leq l' \leq M_R$, it is optimal not to make further capacity adjustment. Since $\tau(k, h_t^b) \leq l-1$, making capacity adjustment at capacity portfolio $(k, l-1)$ leads to a contradiction to the definition of $\tau(k; h_t^b)$.

Subcase 2: $l = j$ and $k > i$. Again, it follows from Theorem 5.3 that it is optimal, as well, to adjust capacity portfolio $(k-1, l)$ to capacity portfolio (k, l). There are two subcases to consider: *a)* $l \geq \tau(k-1, h_t^b)$. Similar to the argument in *Subcase 1*, this leads to a contradiction to the definition of $\tau(k-1; h_t^b)$. *b)* $l < \tau(k-1, h_t^b)$. In this case, it follows that $(k, l) \in \mathscr{T}_2(h_t^b)$. However, this contradicts with the hypothesis that $(k, l) \notin \mathscr{T}(h_t^b)$.

In the other case, consider a state (i, j) that violates the theorem in that it is optimal to adjust capacity portfolio (i, j) to $(k, l) \notin \mathscr{T}(h_t^b)$ and $k < \tau(k, h_t^b)$, whereas in state (k, l), it is optimal to adjust it to capacity portfolio (k^*, l^*) in $\mathscr{T}(h_t^b)$. This implies that the decision at capacity portfolio (i, j) to invest to capacity portfolio (k^*, l^*) provides higher value than to invest to capacity portfolio (k, l) which also leads to a contradiction. Therefore, $(k, l) \in \mathscr{T}(h_t^b)$. □

Finally, we conclude this section with a numerical example for the non-identical module size model under irreversible investment environment. Consider a situation where demand in each period is a discrete uniform random variable between 0 to 30 units, the time until a new product introduction is geometrically distributed and independent of the demand levels, and we are planning our capacity with an infinite planning horizon. The following are the parameters for this problem instance:

$$\pi_D = \pi_R = \pi = 1.0, \phi = 1.5, p = 0.2, \delta = 0.8$$
$$\kappa_D = 5, \kappa_R = 1, c_D = 7.5, c_R = 3, m_D = m_R = 0.0.$$

Note that in this case $M_D = 6$ and $M_R = 30$.

Denote (i^*, j^*) as the optimal capacity portfolio of capacity portfolio (i, j) after the capacity adjustment process where i and j represent the numbers of existing DMS and RMS modules, respectively. Table 5.3 displays the optimal capacity portfolio (i^*, j^*) for given existing capacity portfolio (i, j). For example, if we have no capacity (i.e., capacity portfolio (0,0)), it is optimal to make capacity adjustment to capacity portfolio (5,0). On the other hand, suppose that we have one RMS module but no DMS module (i.e., capacity portfolio (0,1)). In this case, it is optimal to make capacity adjustment to capacity portfolio (4,3), i.e., it is optimal to purchase four more DMS modules and two more RMS modules.

According Theorem 5.4, for $0 \leq i \leq M_D$, $\tau(i) = \min\{l | (i^*, l^*) = (i, l), 0 \leq l \leq M_R\}$. From Table 5.3, it follows that $\{\tau(i)\}_{i=0}^{M_D} = \{23, 18, 13, 8, 3, 0, 0\}$ which means that $(0, 23), (1, 18), (2, 13), (3, 8), (4, 3), (5, 0)$ and $(6, 0)$ are in $\mathscr{T}_1$. (Note that points (4,3), (3,8), (2,13) and (0, 23) also violate the pure ISD structure as in Figure 3.1). Table 5.3 displays these capacity portfolios in **bold** face (these states constitute $\mathscr{T}_1$). Furthermore, Table 5.3 displays capacity portfolios with $\mathscr{T}_2$ subscription as the capacity portfolios in $\mathscr{T}_2$. Define the investment region as the set of capacity portfolio (i, j) such that $(i^*, j^*) \neq (i, j)$. For example, capacity portfolio $(i, j) = (1, 2)$ is in the investment region since $(i^*, j^*) = (4, 3) \neq (i, j)$. Table 5.3 displays these capacity portfolios in *italics*. From Table 5.3, the capacity portfolios in $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ constitute a

Optimal Capacity Portfolio (i^*, j^*)

Number of RMS Modules (j)

25	(0, 25)	(1, 25)	(2, 25)	(3, 25)	(4, 25)	(5, 25)	(6, 25)
24	(0, 24)	(1, 24)	(2, 24)	(3, 24)	(4, 24)	(5, 24)	(6, 24)
23	$(\mathbf{0}, \mathbf{23})_{\mathscr{T}_1}$	(1, 23)	(2, 23)	(3, 23)	(4, 23)	(5, 23)	(6, 23)
22	*(0, 23)*	$(1, 22)_{\mathscr{T}_2}$	(2, 22)	(3, 22)	(4, 22)	(5, 22)	(6, 22)
21	*(0, 23)*	$(1, 21)_{\mathscr{T}_2}$	(2, 21)	(3, 21)	(4, 21)	(5, 21)	(6, 21)
20	*(1, 20)*	$(1, 20)_{\mathscr{T}_2}$	(2, 20)	(3, 20)	(4, 20)	(5, 20)	(6, 20)
19	*(1, 19)*	$(1, 19)_{\mathscr{T}_2}$	(2, 19)	(3, 19)	(4, 19)	(5, 19)	(6, 19)
18	*(1, 18)*	$(\mathbf{1}, \mathbf{18})_{\mathscr{T}_1}$	(2, 18)	(3, 18)	(4, 18)	(5, 18)	(6, 18)
17	*(1, 18)*	*(1, 18)*	$(2, 17)_{\mathscr{T}_2}$	(3, 17)	(4, 17)	(5, 17)	(6, 17)
16	*(1, 18)*	*(1, 18)*	$(2, 16)_{\mathscr{T}_2}$	(3, 16)	(4, 16)	(5, 16)	(6, 16)
15	*(2, 15)*	*(2, 15)*	$(2, 15)_{\mathscr{T}_2}$	(3, 15)	(4, 15)	(5, 15)	(6, 15)
14	*(2, 14)*	*(2, 14)*	$(2, 14)_{\mathscr{T}_2}$	(3, 14)	(4, 14)	(5, 14)	(6, 14)
13	*(2, 13)*	*(2, 13)*	$(\mathbf{2}, \mathbf{13})_{\mathscr{T}_1}$	(3, 13)	(4, 13)	(5, 13)	(6, 13)
12	*(2, 13)*	*(2, 13)*	*(2, 13)*	$(3, 12)_{\mathscr{T}_2}$	(4, 12)	(5, 12)	(6, 12)
11	*(2, 13)*	*(2, 13)*	*(2, 13)*	$(3, 11)_{\mathscr{T}_2}$	(4, 11)	(5, 11)	(6, 11)
10	*(3, 10)*	*(3, 10)*	*(3, 10)*	$(3, 10)_{\mathscr{T}_2}$	(4, 10)	(5, 10)	(6, 10)
9	*(3, 9)*	*(3, 9)*	*(3, 9)*	$(3, 9)_{\mathscr{T}_2}$	(4, 9)	(5, 9)	(6, 9)
8	*(3, 8)*	*(3, 8)*	*(3, 8)*	$(\mathbf{3}, \mathbf{8})_{\mathscr{T}_1}$	(4, 8)	(5, 8)	(6, 8)
7	*(3, 8)*	*(3, 8)*	*(3, 8)*	*(3, 8)*	$(4, 7)_{\mathscr{T}_2}$	(5, 7)	(6, 7)
6	*(3, 8)*	*(3, 8)*	*(3, 8)*	*(3, 8)*	$(4, 6)_{\mathscr{T}_2}$	(5, 6)	(6, 6)
5	*(4, 5)*	*(4, 5)*	*(4, 5)*	*(4, 5)*	$(4, 5)_{\mathscr{T}_2}$	(5, 5)	(6, 5)
4	*(4, 4)*	*(4, 4)*	*(4, 4)*	*(4, 4)*	$(4, 4)_{\mathscr{T}_2}$	(5, 4)	(6, 4)
3	*(4, 3)*	*(4, 3)*	*(4, 3)*	*(4, 3)*	$(\mathbf{4}, \mathbf{3})_{\mathscr{T}_1}$	(5, 3)	(6, 3)
2	*(4, 3)*	*(4, 3)*	*(4, 3)*	*(4, 3)*	*(4, 3)*	$(5, 2)_{\mathscr{T}_2}$	(6, 2)
1	*(4, 3)*	*(4, 3)*	*(4, 3)*	*(4, 3)*	*(4, 3)*	$(5, 1)_{\mathscr{T}_2}$	(6, 1)
0	*(5, 0)*	*(5, 0)*	*(5, 0)*	*(5, 0)*	*(5, 0)*	$(\mathbf{5}, \mathbf{0})_{\mathscr{T}_1}$	$(\mathbf{6}, \mathbf{0})_{\mathscr{T}_1}$
	0	1	2	3	4	5	6

Number of DMS Modules (i)

Table 5.3: Optimal Capacity Portfolio (i^*, j^*) for Given Existing Capacity Portfolio (i, j) with Annual Demand = 30 units

threshold dividing the region where an investment is optimal from the do-nothing region.

Therefore, under the irreversible investment assumption, we are able to fully characterize the structure of the optimal investment policy even when DMS and RMS are assumed to have different unit capacities. However, the policy is fairly complicated. Further research is needed to characterize the structure when investments can be reversible.

6 Conclusion and Further Research

In this paper, we considered the question of how to optimally invest in a capacity portfolio of dedicated and reconfigurable capacity modules to satisfy uncertain demand where demand is stochastic and new products are introduced periodically making the dedicated capacity obsolete.

We showed that the ISD policy (first introduced concurrently by Eberly and Van Mieghem (1997) and Dixit (1997)) continues to hold even when capacity comes in discrete rather than continuous capacity increments when capacity sizes for DMS and RMS are identical. We were also able to use the main difference between DMS and RMS capacity (that one periodically becomes obsolete while the other can be reconfigured) to obtain comparative statics results for certain problem parameters and show their effects on the ISD policy structure. In particular, we obtained results that describe under what assumptions a faster new product introduction pace leads to a preference for higher amounts of reconfigurable capacity investments, and showed that this is not always the case.

However, we also showed that the elegant pure ISD policy is guaranteed to be optimal only when DMS and RMS modules have identical module sizes, an assumption that is not likely to hold in practice, and that when this assumption is violated, it leads to an ISD-like structure with some perturbations. We were able to obtain some structural results that significantly improve the speed for computing optimal policies. Furthermore, under the irreversible investments assumption, we were able to fully characterize the optimal policy which still has an elegant but more complicated structure.

Our research could be extended in several directions. First, our model assumes a maintenance cost for the modules that is not a function of the age of the modules. This extension, which would lead to a more sophisticated life-cycle model of RMS and DMS equipment would require a much more complicated model and computational algorithm, since the state space we have to keep track of becomes much larger. Furthermore, our initial model only considered an environment where each module (RMS or DMS) could be used to make only a single product at a time. Though this assumption is valid in a lot of high-volume automotive plants, it is not likely to be valid in many other environments. Further research is needed to characterize optimal capacity investments in an environment where the firm makes multiple products at a time with new products also being introduced periodically.

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Appendix A: Multimodular Functions and Their Piecewise Affine Interpolations

In this section, we briefly introduce the definition and main results of multimodular functions based on the works of Hajek (1985) and Altman et al. (2000). Later, we develop the Multimodularity Preservation theorem which is analogous to the Convexity Preservation theorem from Heyman and Sobel (1984).

Let $\mathscr{F} = \{f_0, f_1, \ldots, f_m\} = \{-e_1, d_2, d_3, \ldots, d_m, e_m\}$ where e_i is an m-dimensional vector where $e_i(j) = 0, i \neq j$ and $e_i(i) = 1$ and $d_i = e_{i-1} - e_i$.

Definition A.1 (Hajek (1985)) *A function F on $\mathbb{Z}^m$ is multimodular with respect to $\mathscr{F}$ if for all $u \in \mathbb{Z}^m$,*

$$F(u+v) + F(u+w) \geq F(u) + F(u+v+w)$$

whenever $v, w \in \mathscr{F}$ and $v \neq w$.

Intuitively, one way to linearly interpolate a function defined on $\mathbb{Z}^2$ to $\mathbb{R}^2$ is first to partition $\mathbb{Z}^2$ to a space of 3 adjacent points in a particular way and for each element of the newly defined space, find a hyperplane $\hat{H}$ on $\mathbb{R}^3$ that passes through all 3 points. Afterwards, for a point z in the simplex generated by these 3 points, we assign the value of that point to be $\hat{H}(z)$. An idea to partition $\mathbb{R}^n$ to a space of simplexes is formalized by Hajek(1985) through the notion of *atoms*.

Definition A.2 (Hajek (1985)) *Let $\{v_0, \ldots, v_m\} \subset \mathbb{Z}^m$. A simplex a induced by $\{v_0, \ldots, v_m\}$ is called an atom if and only if for some permutation σ of $(0, 1, \ldots, m)$, $\{v_0, \ldots, v_m\}$ can be ordered as follows:*

$$\begin{aligned} v_1 &= v_0 + f_{\sigma(1)} \\ v_2 &= v_1 + f_{\sigma(2)} \\ &\vdots \\ v_m &= v_{m-1} + f_{\sigma(m)} \\ v_0 &= v_m + f_{\sigma(0)}, \end{aligned}$$

where $f_{\sigma(i)} \in \mathscr{F}$.

We call $\{v_0, \ldots, v_m\}$ the set of the extreme points of a. Define $\mathcal{A}^m$ and $\mathcal{A}^m_+$ to be the set of all atoms in $\mathbb{R}^m$ and $\mathbb{R}^m_+$, respectively. Note that $\mathcal{A}^m$ and $\mathcal{A}^m_+$ are triangulations of $\mathbb{R}^m$ and $\mathbb{R}^m_+$, respectively.

Then if F is a function on $\mathbb{Z}^m$, denote $\tilde{F}$ on $\mathbb{R}^m$, the *piecewise affine interpolation* of F. The interpolation is performed by determining a unique hyperplane $\hat{H}_a$ of dimension m that passes through points $(u_0, F(u_0)), (u_1, F(u_1)), \ldots, (u_m, F(u_m))$ where u_i is an extreme point of atom a for each atom $a \in \mathbb{R}^m$. Then, for each $z \in a$, define $\tilde{F}(z) = \hat{H}_a(z)$. Note also that $\tilde{F}$ is uniquely defined.

Theorem A.1 (Altman et al. (2000)) *F on $\mathbb{N}^m$ is multimodular if and only if $\tilde{F}$ is convex.*

It is immediate from the above theorem that if $-F$ is multimodular then $\tilde{F}$ is concave. We call a function F to be *anti-multimodular* if and only if $-F$ is multimodular. We state the result in the following

Corollary A.1 *F on $\mathbb{N}^m$ is anti-multimodular if and only if $\tilde{F}$ is concave.*

In the next theorem, we show that the infimum operator preserves multimodularity. This result is analogous to the Convexity Preservation under infimum operator from Heyman and Sobel (1984).

Theorem A.2 (Multimodularity Preservation) *Suppose $F(u, v)$ is multimodular in $u \in \mathbb{N}^m$ for given $v \in \mathbb{N}^n$ and F is bounded below. Let $\tilde{F}(x, v)$ be the piecewise affine interpolation of $F(x, v)$ where $x \in \mathbb{R}^m_+$ for given $v \in \mathbb{N}^n$. Suppose there exists an extension $\tilde{F}_e$ of $\tilde{F}$ from $\mathbb{R}^m_+ \times \mathbb{N}^n$ to $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ such that $\tilde{F}_e(x, v) = \tilde{F}(x, v)$ for $x \in \mathbb{R}^m_+$ and $v \in \mathbb{N}^n$ and $\tilde{F}_e(x, y)$ is jointly convex in $x \in \mathbb{R}^m_+$ and $y \in \mathbb{R}^n_+$. Then*

$$F^*(v) = \inf_{u \in \mathbb{N}^m} \{F(u, v)\} = \inf_{x \in \mathbb{R}^m_+} \{\tilde{F}_e(x, v)\}$$

is multimodular on $v \in \mathbb{N}^n$.

Proof: First notice that by Theorem A.1 $\tilde{F}(x, v)$ is a bounded-below polyhedral convex function on $x \in \mathbb{R}^m_+$ for a given $v \in \mathbb{N}^n$ and all extreme points of the graph $(x, \tilde{F}(x, v))$ lie in $\mathbb{N}^m \times \mathbb{R}$. Since the argument infimum of a bounded-below polyhedral convex function contains an extreme point,

$$\text{arginf}_{u \in \mathbb{N}^m} \{\tilde{F}(u, v)\} \subset \text{arginf}_{x \in \mathbb{R}^m_+} \{\tilde{F}(x, v)\}. \tag{A.1}$$

Define $F^*_e(y) = \inf_{x \in \mathbb{R}^m_+} \{\tilde{F}_e(x, y)\}$ for $y \in \mathbb{R}^n_+$. From relation A.1, it is immediate by definition that for all $v \in \mathbb{N}^n$

$$\begin{aligned}
F^*_e(v) &= \inf_{x \in \mathbb{R}^m_+} \{\tilde{F}_e(x, v)\} \\
&= \inf_{x \in \mathbb{R}^m_+} \{\tilde{F}(x, v)\} \\
&= \inf_{u \in \mathbb{N}^m} \{\tilde{F}(u, v)\} \text{ ,by relation A.1,} \\
&= \inf_{u \in \mathbb{N}^m} \{F(u, v)\} \\
&= F^*(v).
\end{aligned}$$

By Proposition B-4 from Heyman and Sobel (1984), $F_e^*(y)$ is convex in y. Next, for each atom $a \in \mathcal{A}_+^n$, we can determine a unique hyperplane $\hat{H}_a$ such that $\hat{H}_a(v) = F_e^*(v)$ for any extreme point v of atom a. Then, the piecewise affine approximation $\tilde{F}_e^*$ of F_e^* with respect to $\mathcal{A}_+^n$ becomes,

$$\tilde{F}_e^*(y) = \sup_{a \in \mathcal{A}_+^n} \{\hat{H}_a(y)\}.$$

Clearly, $\tilde{F}_e^*(y)$ is convex in y since it is the pointwise supremum of affine functions. Note that the piecewise affine approximation of F_e^* equals to the piecewise affine interpolation of F^*; therefore, by Theorem A.1, F^* on $\mathbb{N}^n$ is multimodular. □

Again, the immediate corollary of the Multimodularity Preservation theorem for anti-multimodular functions is stated as follows.

Corollary A.2 (Anti-Multimodularity Preservation) *Suppose $F(u, v)$ is anti-multimodular in $u \in \mathbb{N}^m$ for given $v \in \mathbb{N}^n$ and F is bounded above. Let $\tilde{F}(x, v)$ be the piecewise affine interpolation of $F(x, v)$ where $x \in \mathbb{R}_+^m$ for given $v \in \mathbb{N}^n$. Suppose there exists an extension $\tilde{F}_e$ of $\tilde{F}$ from $\mathbb{R}_+^m \times \mathbb{N}^n$ to $\mathbb{R}_+^m \times \mathbb{R}_+^n$ such that $\tilde{F}_e(x, v) = \tilde{F}(x, v)$ for $x \in \mathbb{R}_+^m$ and $v \in \mathbb{N}^n$ and $\tilde{F}_e(x, y)$ is jointly concave in $x \in \mathbb{R}_+^m$ and $y \in \mathbb{R}_+^n$. Then*

$$F^*(v) = \sup_{u \in \mathbb{N}^m} \{F(u, v)\} = \sup_{x \in \mathbb{R}_+^m} \{\tilde{F}_e(x, v)\}$$

is anti-multimodular on $v \in \mathbb{N}^n$.

We will use Corollary A.2, which is more appropriate for our profit maximization problem, along with Theorem 2 from Eberly and Van Mieghem (1997) to show that if the module sizes of the DMS and RMS are the same, the optimal policy is of ISD type.

Appendix B: Proof of Lemma 3.1

Proof: Since by definition $\hat{\gamma}_t(i, j; x) = \gamma_t(i, j; x)$ when the module sizes of DMS and RMS are identical ($\kappa_D = \kappa_R = \kappa$), we can rewrite Eqn. 2.1 as follows:

$$\hat{\gamma}_t(i, j; x) = \begin{cases} \pi_{D,t} x & , x \leq \kappa i \\ \pi_{D,t} \kappa i + \pi_{R,t}(x - \kappa i) & , \kappa i < x \leq \kappa(i + j) \\ \pi_{D,t} \kappa i + \pi_{R,t} \kappa j - \phi_t [x - \kappa(i + j)] & , x > \kappa(i + j) \end{cases} . \tag{B.1}$$

Note that for all x, $\hat{\gamma}(i, j; x)$ is increasing in i for a given j and increasing in j for a given i. We prove by considering all possible cases for items 1-3. In the interest of this space, we will only show the proof for item 1.

Case 1: $x \leq \kappa i$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i, j; x) = \hat{\gamma}_t(i + 1, j; x) = \hat{\gamma}_t(i, j + 1; x) = \hat{\gamma}_t(i + 1, j + 1; x) = \pi_{D,t} x.$$

Then,

$$\hat{\gamma}_t(i+1,j;x) - \hat{\gamma}_t(i,j;x) = 0 = \hat{\gamma}_t(i,j+1;x) - \hat{\gamma}_t(i+1,j+1;x).$$

Therefore, 1) is satisfied in Case 1.

Case 2: $\kappa i < x \leq \kappa(i+j)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i,j;x) = \hat{\gamma}_t(i,j+1;x) = \pi_{D,t}\kappa i + \pi_{R,t}(x - \kappa i).$$

Next, consider the following subcases:

Subcase 2.1: $\kappa(i+1) < x \leq \kappa(i+j)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1,j;x) = \hat{\gamma}_t(i+1,j+1;x) = \pi_{D,t}\kappa(i+1) + \pi_{R,t}(x - \kappa(i+1)).$$

Therefore,

$$\hat{\gamma}_t(i+1,j+1;x) - \hat{\gamma}_t(i+1,j;x) = 0 = \hat{\gamma}_t(i,j+1;x) - \hat{\gamma}_t(i,j;x).$$

Therefore, 1) is satisfied in Subcase 2.1.

Subcase 2.2: $\kappa i < x \leq \kappa(i+1)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1,j;x) = \hat{\gamma}_t(i+1,j+1;x) = \pi_{D,t}x.$$

Therefore,

$$\hat{\gamma}_t(i+1,j+1;x) - \hat{\gamma}_t(i+1,j;x) = 0 = \hat{\gamma}_t(i,j+1;x) - \hat{\gamma}_t(i,j;x).$$

Therefore, 1) is satisfied in Subcase 2.2.

Case 3: $x > \kappa(i+j)$. It follows from Eqn. B.1 that $\hat{\gamma}_t(i,j;x) = \pi_{D,t}\kappa i + \pi_{R,t}\kappa j - \phi_t[x - \kappa(i+j)]$. Next, consider the following subcases:

Subcase 3.1: $x > \kappa(i+j+1)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1,j;x) = \pi_{D,t}\kappa(i+1) + \pi_{R,t}\kappa j - \phi_t[x - \kappa(i+j+1)],$$

and

$$\hat{\gamma}_t(i,j+1;x) = \pi_{D,t}\kappa i + \pi_{R,t}\kappa(j+1) - \phi_t[x - \kappa(i+j+1)].$$

Consider the following subcases.

Subcase 3.1.1: $x > \kappa(i+j+2)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1,j+1;x) = \pi_{D,t}\kappa(i+1) + \pi_{R,t}\kappa(j+1) - \phi_t[x - \kappa(i+j+2)].$$

This implies that

$$\hat{\gamma}_t(i+1,j;x) - \hat{\gamma}_t(i,j;x) = (\pi_{D,t} + \phi_t)\kappa = \hat{\gamma}_t(i+1,j+1;x) - \hat{\gamma}_t(i,j+1;x).$$

Therefore, 1) is satisfied in Subcase 3.1.1.

Subcase 3.1.2: $\kappa(i+j+1) < x \le \kappa(i+j+2)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1, j+1; x) = \pi_{D,t}\kappa(i+1) + \pi_{R,t}(x - \kappa(i+1)).$$

This implies that

$$\begin{aligned}
& \hat{\gamma}_t(i+1, j+1; x) - \hat{\gamma}_t(i, j+1; x) \\
= \ & \pi_{D,t}\kappa(i+1) + \pi_{R,t}(x - \kappa(i+1)) \\
& \quad - \{\pi_{D,t}\kappa i + \pi_{R,t}\kappa(j+1) - \phi_t[x - \kappa(i+j+1)]\} \\
= \ & \pi_{D,t}\kappa + \pi_{R,t}(x - \kappa(i+j+2)) + \phi_t[x - \kappa(i+j+1)] - \phi_t\kappa + \phi_t\kappa \\
= \ & (\pi_{D,t} + \phi_t)\kappa + (\pi_{R,t} + \phi_t)[x - \kappa(i+j+2)] \\
\le \ & (\pi_{D,t} + \phi_t)\kappa, \text{ since } x \le \kappa(i+j+2) \text{ in this subcase and } \pi_{R,t}, \phi_t \ge 0, \\
= \ & \hat{\gamma}_t(i+1, j; x) - \hat{\gamma}_t(i, j; x).
\end{aligned}$$

Therefore, 1) is satisfied in Subcase 3.1.2.

Subcase 3.2: $\kappa(i+j) < x \le \kappa(i+j+1)$. There are 2 more subcases to consider:

Subcase 3.2.1: $\kappa(i+1) < x$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1, j; x) = \hat{\gamma}_t(i+1, j+1; x) = \pi_{D,t}\kappa(i+1) + \pi_{R,t}(x - \kappa(i+1)).$$

Therefore,

$$\hat{\gamma}_t(i+1, j+1; x) - \hat{\gamma}_t(i+1, j; x) = 0 \le \hat{\gamma}_t(i, j+1; x) - \hat{\gamma}_t(i, j; x),$$

since $\hat{\gamma}_t(i, j; x)$ is increasing in j for given i and x. Therefore, 1) is satisfied in Subcase 3.2.1

Subcase 3.2.2: $x \le \kappa(i+1)$. It follows from Eqn. B.1 that

$$\hat{\gamma}_t(i+1, j; x) = \hat{\gamma}_t(i+1, j+1; x) = \pi_{D,t}x.$$

Therefore,

$$\hat{\gamma}_t(i+1, j+1; x) - \hat{\gamma}_t(i+1, j; x) = 0 \le \hat{\gamma}_t(i, j+1; x) - \hat{\gamma}_t(i, j; x),$$

since $\hat{\gamma}_t(i, j; x)$ is increasing in j for a given i and x. Therefore, 1) is satisfied in Subcase 3.2.2.

Since i, j, x and t are arbitrary, 1) is true for all cases. □

Appendix C: Proof of Theorem 3.2

Proof: It is straightforward that one need only restrict attention to stationary policies in this case. Before proceeding, define

$$\begin{aligned}
\hat{\zeta}(k) &= \mathsf{E}\min(X, k\kappa), \Delta\hat{\zeta}(k) = \hat{\zeta}(k) - \hat{\zeta}(k-1), \\
\hat{\eta}(k) &= \mathsf{E}(X - k\kappa)^+, \Delta\hat{\eta}(k) = \hat{\eta}(k) - \hat{\eta}(k-1).
\end{aligned}$$

Assume that the firm already has $i-1$ modules and is considering weather to invest in the i-th module which would be used to manufacture the $[(i-1)\kappa+1]$-th to $[i\kappa]$-th units of demand. (Clearly, if demand is less than $(i-1)\kappa+1$ in any given period, we would not make use of this module in that period.) Denote $\Delta V_D(i)$ as the value of having a DMS module to satisfy the demand units $[(i-1)\kappa+1]$-th to $[i\kappa]$-th. This would involve investing in new DMS modules every time a new product generation is introduced. Then,

$$\Delta V_D(i) = \pi_D \Delta\hat{\zeta}(i) - \phi\Delta\hat{\eta}(i) - m_D + \delta[p(\Delta V_D(i) - (c_D - \tilde{s}_D)) + (1-p)\Delta V_D(i)],$$

where the first two terms of the left hand side of the above equation is the expected immediate profit from satisfying demand units $[(i-1)\kappa+1]$ to $[i\kappa]$. By rearranging the above equation, we have

$$(1-\delta)\Delta V_D(i) \quad = \quad \pi_D\Delta\hat{\zeta}(i) - \phi\Delta\hat{\eta}(i) - m_D - \delta p(c_D - \tilde{s}_D).$$

On the other hand, denote $\Delta V_R(i)$ as the value of having a RMS module to satisfy the demand units $[(i-1)\kappa+1]$-th to $[i\kappa]$-th. By the same analysis as in the previous case, it follows that

$$(1-\delta)\Delta V_R(i) = \pi_R\Delta\hat{\zeta}(i) - \phi\Delta\hat{\eta}(i) - m_R.$$

It is, therefore, optimal to invest in DMS module i to satisfy the demand units $[(i-1)\kappa+1]$-th to $[i\kappa]$-th unit, if $\Delta V_D(i) - c_D > \Delta V_R(i) - c_R$, which is equivalent to

$$\pi_D\Delta\hat{\zeta}(i) - m_D - \delta p(c_D - \tilde{s}_D) - (1-\delta)c_D > \pi_R\Delta\hat{\zeta}(i) - m_R - (1-\delta)c_R.$$

After some algebra, we get

$$p < \frac{(1-\delta)(c_R - c_D) + (m_R - m_D)}{\delta(c_D - \tilde{s}_D)} + \frac{(\pi_D - \pi_R)\Delta\hat{\zeta}(i)}{\delta(c_D - \tilde{s}_D)} = p_* + \frac{(\pi_D - \pi_R)\Delta\hat{\zeta}(i)}{\delta(c_D - \tilde{s}_D)}.$$

Note that $\hat{\zeta}(k)$ is an increasing concave function in k; therefore, $\Delta\hat{\zeta}(k)$ is a non-negative decreasing function in k. Along with the assumption that $\pi_D \geq \pi_R$ and $c_D > s_D \geq \tilde{s}_D$, it follows that $\frac{(\pi_D-\pi_R)\Delta\hat{\zeta}(k)}{\delta(c_D-\tilde{s}_D)}$ is non-negative and decreasing in k. Therefore, if $p < p_*$, it is clear that investing exclusively in DMS modules is optimal regardless of the distribution of X since p_* is independent of the distribution of X. Therefore, 1) follows.

By the same argument, it is optimal to invest in RMS module i to satisfy the demand units $[(i-1)\kappa+1]$-th to $[i\kappa]$-th unit, if $\Delta V_D(i) - c_D < \Delta V_R(i) - c_R$ which is equivalent to

$$p > p_* + \frac{(\pi_D - \pi_R)\Delta\hat{\zeta}(i)}{\delta(c_D - \tilde{s}_D)}.$$

Note that $\Delta\hat{\zeta}(1) = \mathsf{E}\min(X,\kappa)$ by the assumption that X is non-negative almost surely. Therefore, if $p > p^*$, this immediately implies that it is optimal to invest exclusively in RMS modules since $p^* \geq p_* + \frac{(\pi_D-\pi_R)\Delta\hat{\zeta}(i)}{\delta(c_D-\tilde{s}_D)}$ for all i. Consequently, 2) follows.

Finally, 3) immediately follows since if $p < p^*$, it is always better to invest in DMS modules to serve the demand units [1]-st to $[\kappa]$-th than invest in an RMS module to do so. $\square$

Appendix D: Proof of Proposition 4.2

Proof: Before proceeding, let $V_t(i,j;h_t^b;m_{D,\tau})$ and $\hat{W}_{t+1}(i,j;h_t^b;m_{D,\tau})$ denote the parametric versions of $V_t(i,j;h_t^b)$ and $\hat{W}_{t+1}(i,j;h_t^b)$, respectively, in the problem instance where the maintenance cost of a dedicated module in period τ is $m_{D,\tau}$ in Formulation F2. Similarly, for $\tau_1 \neq \tau_2$, let $V_t(i,j;h_t^b;m_{D,\tau_1},m_{D,\tau_2})$ and $\hat{W}_{t+1}(i,j;h_t^b;m_{D,\tau},m_{D,\tau_2})$ denote the parametric version of $V_t(i,j;h_t^b)$ and $\hat{W}_{t+1}(i,j;h_t^b)$, respectively, in the problem instance where the maintenance cost of a dedicated module in period τ_1 is m_{D,τ_1} and in period τ_2 is m_{D,τ_1} in Formulation F2.

We prove by induction on t. At $t=T$, the proposition is trivially true by the boundary condition. Assume that the proposition is also true at $t=t'+1$.

Consider the case $t=t'$. Since $V_{t'+1}(i,j;h_{t'+1}^b;\mathbf{m_D})$ is supermodular in $(i,j,\mathbf{m_D})$ on $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$ for a given $h_{t'+1}^b \in \mathscr{H}_{t'+1}^b$, by Theorem 2.6.1 from Topkis (1998), this implies that $V_{t'+1}$ has increasing differences on $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$, i.e.,

1. For all $0 \leq \tau \leq T, m_{D,\tau} \leq m'_{D,\tau}, i$ and j,

$$\begin{aligned} &V_{t'+1}(i,j+1;h_{t'+1}^b;m_{D,\tau}) - V_{t'+1}(i,j;h_{t'+1}^b;m_{D,\tau}) \\ &\quad \leq \quad V_{t'+1}(i,j+1;h_{t'+1}^b;m'_{D,\tau}) - V_{t'+1}(i,j;h_{t'+1}^b;m'_{D,\tau}). \end{aligned}$$

2. For all $0 \leq \tau \leq T, m_{D,\tau} \leq m'_{D,\tau}, i$ and j,

$$\begin{aligned} &V_{t'+1}(i+1,j;h_{t'+1}^b;m_{D,\tau}) - V_{t'+1}(i,j;h_{t'+1}^b;m_{D,\tau}) \\ &\quad \geq \quad V_{t'+1}(i+1,j;h_{t'+1}^b;m'_{D,\tau}) - V_{t'+1}(i,j;h_{t'+1}^b;m'_{D,\tau}). \end{aligned}$$

3. For all $0 \leq \tau_1,\tau_2 \leq T, \tau_t \neq \tau_2, m_{D,\tau_1} \leq m'_{D,\tau_1}, m_{D,\tau_2} \leq m'_{D,\tau_2}, i$ and j

$$\begin{aligned} &V_{t'+1}(i,j;h_{t'+1}^b;m_{D,\tau_1},m'_{D,\tau_2}) - V_{t'+1}(i,j;h_{t'+1}^b;m_{D,\tau_1},m_{D,\tau_2}) \\ &\quad \leq \quad V_{t'+1}(i,j;h_{t'+1}^b;m'_{D,\tau_1},m'_{D,\tau_2}) - V_{t'+1}(i,j;h_{t'+1}^b;m'_{D,\tau_1},m_{D,\tau_2}). \end{aligned}$$

4. For all i and j,

$$\begin{aligned} &V_{t'+1}(i+1,j;h_{t'+1}^b;\mathbf{m_D}) - V_{t'+1}(i,j;h_{t'+1}^b;\mathbf{m_D}) \\ &\quad \geq \quad V_{t'+1}(i+1,j+1;h_{t'+1}^b;\mathbf{m_D}) - V_{t'+1}(i,j+1;h_{t'+1}^b;\mathbf{m_D}). \end{aligned}$$

Next, we show that (i) :

$$\begin{aligned}
&\hat{W}_{t'+1}(i,j+1;h^b_{t'};m_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m_{D,\tau}) \\
&\quad \leq \quad \hat{W}_{t'+1}(i,j+1;h^b_{t'};m'_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m'_{D,\tau}),
\end{aligned}$$

(ii) :

$$\begin{aligned}
&\hat{W}_{t'+1}(i+1,j;h^b_{t'};m_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m_{D,\tau}) \\
&\quad \geq \quad \hat{W}_{t'+1}(i+1,j;h^b_{t'};m'_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m'_{D,\tau}).
\end{aligned}$$

(iii) :

$$\begin{aligned}
&\hat{W}_{t'+1}(i,j;h^b_{t'};m_{D,\tau_1},m'_{D,\tau_2}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m_{D,\tau_1},m_{D,\tau_2}) \\
&\quad \leq \quad \hat{W}_{t'+1}(i,j;h^b_{t'};m'_{D,\tau_1},m'_{D,\tau_2}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m'_{D,\tau_1},m_{D,\tau_2}).
\end{aligned}$$

and (iv) :

$$\begin{aligned}
&\hat{W}_{t'+1}(i+1,j;h^b_{t'};\mathbf{m_D}) - \hat{W}_{t'+1}(i,j;h^b_{t'};\mathbf{m_D}) \\
&\quad \geq \quad \hat{W}_{t'+1}(i+1,j+1;h^b_{t'};\mathbf{m_D}) - \hat{W}_{t'+1}(i,j+1;h^b_{t'};\mathbf{m_D}).
\end{aligned}$$

(i) follows since

$$\begin{aligned}
&\hat{W}_{t'+1}(i,j+1;h^b_{t'};m_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m_{D,\tau}) \\
&= \quad \hat{\alpha}_{t'}(i,j+1;h^b_{t'}) - \hat{\alpha}_{t'}(i,j;h^b_{t'}) - m_{R,t'} \\
&\quad +\delta\mathsf{E}[p(H^a_{t'})(V_{t'+1}(0,j+1;H^b_{t'+1,N};m_{D,\tau}) - V_{t'+1}(0,j;H^b_{t'+1,N};m_{D,\tau})) \\
&\quad + (1-p(H^a_{t'}))(V_{t'+1}(i,j+1;H^b_{t'+1,C};m_{D,\tau}) - V_{t'+1}(i,j;H^b_{t'+1,C};m_{D,\tau}))|h^b_{t'}] \\
&\leq \quad \hat{\alpha}_{t'}(i,j+1;h^b_{t'}) - \hat{\alpha}_{t'}(i,j;h^b_{t'}) - m_{R,t'} \\
&\quad +\delta\mathsf{E}[p(H^a_{t'})(V_{t'+1}(0,j+1;H^b_{t'+1,N};m'_{D,\tau}) - V_{t'+1}(0,j;H^b_{t'+1,N};m'_{D,\tau})) \\
&\quad + (1-p(H^a_{t'}))(V_{t'+1}(i,j+1;H^b_{t'+1,C};m'_{D,\tau}) - V_{t'+1}(i,j;H^b_{t'+1,C};m'_{D,\tau}))|h^b_{t'}], \\
&= \quad \hat{W}_{t'+1}(i,j+1;h^b_{t'};m'_{D,\tau}) - \hat{W}_{t'+1}(i,j;h^b_{t'};m'_{D,\tau}).
\end{aligned}$$

and the cases for $(ii), (iii), and (iv)$ are similar.

In all cases, the inequality follows from the fact that $V_{t'+1}$ has increasing differences on each pair of its arguments. These results altogether imply that $\hat{W}_{t'+1}(k,l;h^b_{t'};\mathbf{m_D})$ has increasing differences in $(k,l,\mathbf{m_D})$ on $\mathcal{S}_E \times \mathbb{R}^{T+1}_+$. Also, $-(\beta_{D,t'}(k-i) + \beta_{R,t'}(l-j))$ has jointly increasing differences in (i,j) and (k,l) on $\mathcal{S}_E \times \mathcal{S}_E$ by Lemma 4.1. Define

$$F_{t'}(k,l;i,j,h^b_{t'},\mathbf{m_D}) = -(\beta_{D,t'}(k-i) + \beta_{R,t'}(l-j)) + \hat{W}_{t'+1}(k,l;h^b_{t'};\mathbf{m_D}).$$

Therefore, it immediately follows that $F_{t'}(k,l;i,j,h^b_{t'},\mathbf{m_D})$ has jointly increasing differences in (i,j), (k,l), and $\mathbf{m_D}$ on $\mathcal{S}_E \times \mathcal{S}_E \times \mathbb{R}^{T+1}_+$. The fact that $F_{t'}(k,l;i,j,h^b_{t'},\mathbf{m_D})$ is supermodular in $(k,l;i,j,m_D)$ on

$\mathcal{S}_E \times \mathcal{S}_E \times \mathbb{R}_+^{T+1}$ then follows from Corollary 2.6.1 of Topkis (1998). Finally, by Theorem 2.7.6 from Topkis (1998),

$$V_{t'}(i, j; h_{t'}^b; \mathbf{m_D}) = \max_{(k,l) \in \mathcal{S}_E} F_{t'}(k, l; i, j, h_{t'}^b, \mathbf{m_D})$$

is supermodular in $(i, j, \mathbf{m_D})$ on $\mathcal{S}_E \times \mathbb{R}_+^{T+1}$. Therefore, the proposition is true for all t. □

Appendix E: Counterexample for Section 4

In this appendix, we provide a counterexample to show that if we compare two probability functions one of which stochastically dominates the other, the optimal policy switching curves do not move in a monotonic fashion as discussed in Section 4. Consider a situation where maximum product life of each product generation is fixed at 5 periods. The conditional probability of new product introduction $p = (p_1, p_2, \ldots, p_5 = 1)$ where p_t is the (conditional) probability that the new product generation will be introduced in period t given the current product has been in the market for $t-1$ periods. In this example, let $p = (0.005, 0.3, 0.5, 0.65, 1)$. We assume that demand level is constant at 50 parts per period independent of product generation or product life. The remaining problem parameters of this counterexample are described as follows: For all t, $\pi_{D,t} = \pi_{R,t} = \pi = 1, \phi_t = 0, \delta = 0.7, c_{D,t} = 3.6, c_{R,t} = 7.0, s_{D,t} = s_{R,t} = 0, m_{D,t} = m_{R,t} = 0, \kappa = 3$ over infinite horizon. The optimal decision regions of this problem instance is displayed on the left hand side of Figure E.1.

The left hand side of Figure E.1 represents the optimal decision regions as we vary how long the current product generation has been on the market and $p = (0.005, 0.3, 0.5, 0.65, 1)$. For example, when the product has been on the market for one period only, if we currently have one reconfigurable, and zero dedicated modules, it is optimal to buy 15 more dedicated modules. Similarly, the right-hand side of Figure E.1 represents the optimal decisions as we vary how long the current product generation has been on the market and $p = (0.005, 0.3, 0.5, 0.95, 1)$. In this case, when the product has been on the market for two periods or fewer, the decisions are identical. However, when the product has been on the market for three periods, when $p = (0.005, 0.3, 0.5, 0.65, 1)$, it is optimal to buy more reconfigurable modules in a very large number states but for $p = (0.005, 0.3, 0.5, 0.95, 1)$, it is in fact optimal to do nothing. Therefore, as the probability of the next generation arrival gets larger, in this case the optimal decision is to wait out for the arrival of the next generation so as to invest in the new dedicated modules after the next generation is introduced. That is, once it becomes nearly certain that the next generation product will arrive, it may in fact make reconfigurable systems less desirable especially if as in our example, the price difference between reconfigurable and dedicated is significant.

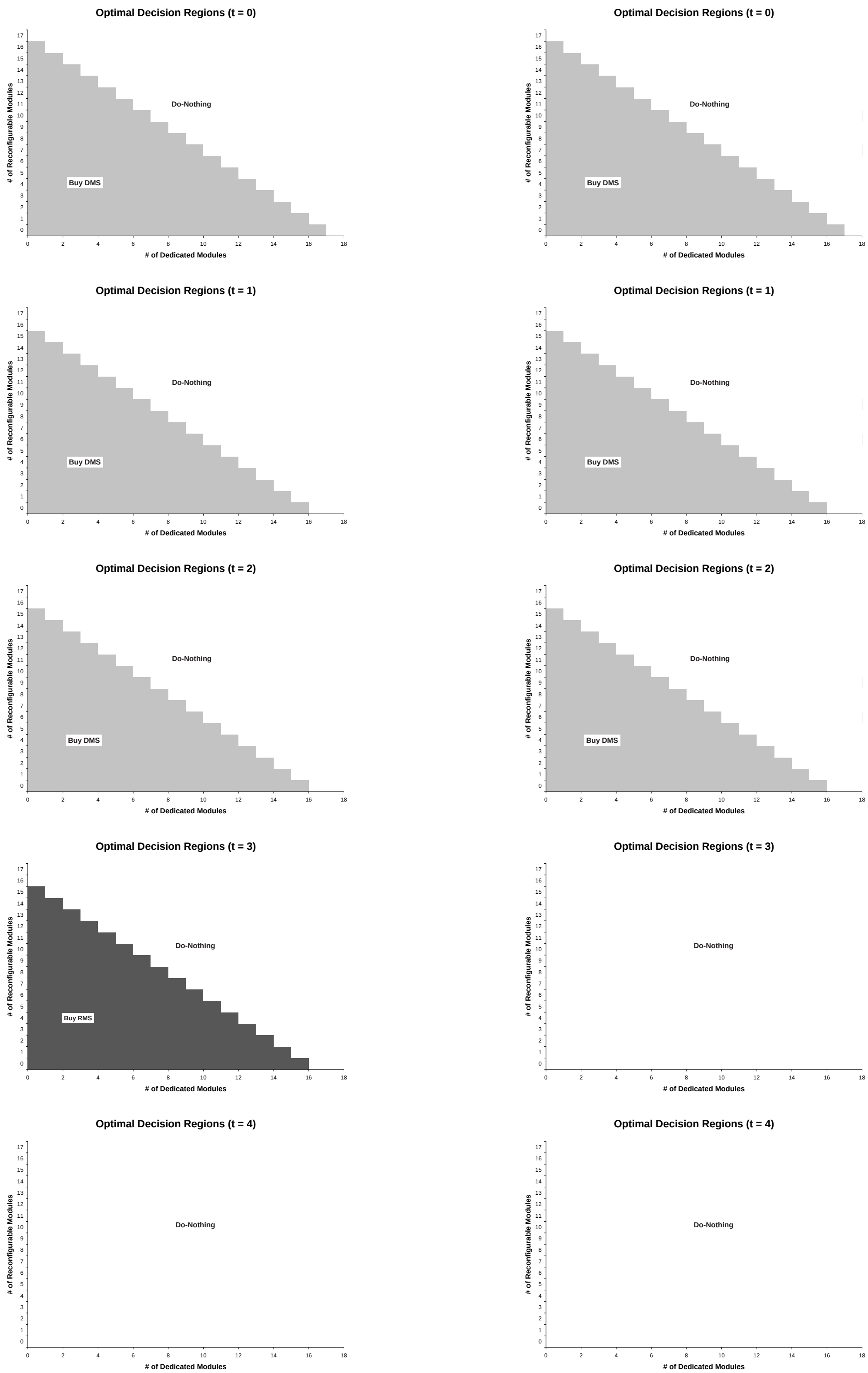

Figure E.1: Optimal Decision Regions of a Counterexample when p increases but Investing Less in RMS Modules. ($p_4 = 0.65$ on the left panel and $p_4 = 0.95$ on the right panel.)

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