An Upper Bound on the Network Recourse Function[1]

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Abstract: This paper considers network flow problems in which the link upper capacities are independently distributed random variables. The optimal objective value is then a non-increasing, convex function of these random capacities. This function is known as the network recourse function for two-stage stochastic programs in which the second stage is a pure network flow problem. This paper exploits the network structure to show that this function also has a property called convex marginal return functions with respect to upper capacities of links with a common initial or terminal node. This results in the ability to establish an upper bound on the expected value of the optimal objective value which only considers high and low extreme values for upper capacities of links with common initial or terminal nodes. This often significantly decreases the function evaluations needed, and still provides an effective bound. Experimental results are given.

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1 Introduction

The classic two-stage stochastic program with recourse (see Dantzig [7], Walkup and Wets [15]) is the problem of finding a first-stage decision which is heavily influenced by uncertainty in the second stage. Second-stage decisions, or recourse actions, are taken after the uncertain second-stage problem parameters are known with certainty. The goal is to choose a first stage decision which minimizes the cost incurred in the first stage plus the expected costs incurred in the second period, given the first stage decision. Solving for the expected costs in the second period, also known as the expected value of the recourse function, can be difficult.

Decomposition methods are effective for stochastic programs since large stochastic programs can be partitioned into much smaller individual scenarios or subproblems (see Van Slyke and Wets [14], Birge [2]). Further, decomposition methods allow straightforward use of parallel computing which can yield significant computational benefits (see Birge et. al [3]).

Exploiting specific structure of second-stage problems is another means for overcoming the difficulty with finding the expected value of the recourse function. In stochastic programs in which the second stage problem is a pure network flow problem, the second stage problem is called the network recourse function. Wallace [16] considered such stochastic programs and developed computationally effective methods for solving large numbers of networks which differ only by node supplies and demands.

The problem of solving for the expected value of the recourse function, however, becomes overwhelming when the number of possible realizations of the random variables is large. For general stochastic programs, effective approximation methods exist which allow further simplification (see Birge and Wets [5]). Further, specific bounds for the network recourse function have been studied.

Wallace [17] developed an upper bound for the network recourse function which is found by solving three separate minimum cost network flow problems, regardless of the number of random elements. Frantzeskakis and Powell [9] and Cheung and Powell [6] developed lower bounds for multistage stochastic networks which are based upon linear approximations and convex approximations, respectively. Both bounds use backward recursion and were successfully applied to dynamic vehicle allocation problems in which some links have stochastic upper bounds.

In this paper, the network recourse function is considered for stochastic programs in which each of the second stage problems is a pure minimum cost network flow problem differing only by the link upper capacities. We consider second-stage problems in which the number of random elements is large and the random elements are independent, in some cases giving a practically infinite number of scenarios. One stage of the dynamic vehicle allocation problem in which a fleet carrier must route vehicles according to the demands of many independent customers is an example of such a problem. This problem, as well as approximation methods, are discussed extensively in [6] and [9].

In this paper, an new effective and efficient upper bound is developed for the expected value of the network recourse function. Section 2 gives a mathematical formulation of a stochastic minimum cost network flow problem. We then show that the network recourse function, when viewed as a function of the random capacities, is a non-increasing convex function with a property called convex marginal return functions with respect to upper capacities of links having either a common initial node or a common terminal node. These properties are then used to find an effective upper bound on the expected value of the network recourse function which does not require prohibitive amounts of computation. In the final section, these bounds are applied to several network problems to demonstrate their effectiveness.

2 Formulation of a Stochastic Minimum Cost Network Flow Problem

Consider a directed network G, consisting of a set of nodes $\mathcal{N} = 1, 2, \ldots, m$ and a set of directed links $\mathcal{A}$ joining various pairs of nodes in $\mathcal{N}$. For any link $(i, j) \in \mathcal{A}$, node i is the *initial node* of (i, j) and node j is the *terminal node* of (i, j). The goal of the program is to route a given supply of a commodity from specified sources, denoted $\breve{S} \subset \mathcal{N}$, to specified sinks, denoted $\breve{T} \subset \mathcal{N}$, at minimum cost. Each node i in $\breve{S}$ has an amount $b_i \geq 0$ of the commodity available at that source. Each node i in $\breve{T}$ has an amount $b_i \leq 0$ of the commodity that must be routed to that sink i. The cost of routing each unit along link (i, j) in $\mathcal{A}$ is c_{ij}. Finally, each directed link (i, j) in $\mathcal{A}$ has lower and upper capacity constraints, l_{ij} and u_{ij} respectively, on the amount of flow that can be sent along that link. Throughout this paper, the upper capacities u_{ij} are assumed to be stochastic, governed by the independent, non-negative, discrete random variables ϕ_{ij}.

Let x_{ij} denote the flow along link (i, j). Then, the problem can be viewed as a function of the stochastic upper capacities, ϕ_{ij}, as follows:

$$
\begin{array}{llcll}
f(\phi) = & \min & \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij} & & \\
& \text{subject to:} & \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} & = & \left\{ \begin{array}{ll} b_i & \text{if } i \in \breve{S} \\ 0 & \text{if } i \in \mathcal{N} \setminus (\breve{S} \cup \breve{T}) \\ -b_i & \text{if } i \in \breve{T} \end{array} \right. \\
& & x_{ij} & \leq & u_{ij} = \phi_{ij} \qquad \forall\ (i,j) \in \mathcal{A} \\
& & x_{ij} & \geq & l_{ij} \qquad \forall\ (i,j) \in \mathcal{A}
\end{array}
$$

The problem can easily be transformed to a circulatory network flow problem by creating a super-source node $\mathbf{s}$, a super-sink node $\mathbf{t}$, links $(\mathbf{s}, i)$ for each $i \in \breve{S}$ with upper and lower capacity b_i, links $(j, \mathbf{t})$ for each $j \in \breve{T}$ with upper and lower capacity b_j, and a link $(\mathbf{t}, \mathbf{s})$ with upper and lower capacity $\sum_{i\in\breve{S}} b_i = \sum_{j\in\breve{T}} b_j$. The cost per unit flow on all these links is zero. Let $\mathcal{N}^* = \mathcal{N} \bigcup \{\mathbf{s}, \mathbf{t}\}$ and $\mathcal{A}^* = \mathcal{A} \bigcup \{(\mathbf{s}, i) \forall i \in \breve{S}, (j, \mathbf{t}) \forall j \in \breve{T}, (\mathbf{t}, \mathbf{s})\}$. Then the problem can be rewritten as follows:

$$
\begin{array}{llclll}
f(\phi) = & \text{Min} & \sum_{(i,j)\in\mathcal{A}} c_{ij} x_{ij} & & & \\
& \text{subject to:} & \sum_{(i,j)\in\mathcal{A}^*} x_{ij} - \sum_{(j,i)\in\mathcal{A}^*} x_{ji} & = & 0 & \forall i \in \mathcal{N}^* \\
& & x_{ij} & \leq & u_{ij} = \phi_{ij} & \\
& & x_{ij} & \geq & l_{ij} &
\end{array}
\tag{1}
$$

The applications of minimum cost flow problems are vast (see, e.g., Bazaara, Jarvis and Sherali [1], Murty [12]). Any such problem which is governed by demand for commodity movement may be viewed as a stochastic minimum cost network flow problem. For applications of this type of problem, see [13], [16].

3 Properties of the Network Recourse Function

The function $f(\phi)$ has several properties which can be exploited to make approximations easier and more effective. First, the problem is a stochastic linear program. Thus, $f(\phi)$ is a piecewise linear, convex function in ϕ (see, e.g., Birge and Louveaux [4]). Further, since any routing which is possible for ϕ^1 is also possible for $\phi^2 \geq \phi^1$, it is readily seen that $f(\phi)$ is a non-increasing function of ϕ.

The next property to show needs to be defined first.

Definition: A function $f(d_1, \ldots, d_n)$ is said to have *convex marginal return functions with respect to d_i and d_j*, $1 \leq i < j \leq n$, if, for all $\lambda > 0, \epsilon > 0$,

$$\begin{aligned} &f(d_1, \ldots, d_i + \lambda, \ldots, d_j + \epsilon, \ldots, d_n) - f(d_1, \ldots, d_i, \ldots, d_j + \epsilon, \ldots, d_n) \\ &\quad \geq f(d_1, \ldots, d_i + \lambda, \ldots, d_j, \ldots, d_n) - f(d_1, \ldots, d_i, \ldots, d_j, \ldots, d_n). \end{aligned}$$

The remainder of this section shows that $f(\phi)$ has convex marginal return functions with respect to upper capacities of links having either commom initial nodes or common terminal nodes. Briefly, the explanation for convex marginal return functions is as follows.

Consider a network flow problem in which Link 1 and Link 2 have the same initial node. Suppose that the network flow problem is solved for particular values of ϕ_1 and ϕ_2, the upper capacities for Link 1 and Link 2, respectively. Now increase the value of ϕ_1 by one, while leaving all other capacities unchanged. Suppose that the new problem is solved by a network algorithm such as the out-of-kilter method which recursively deletes negative cost cycles, and the solution to the previous problem is used as the starting solution. The only possible negative cost cycle would have to include an increase of flow on Link 1, since the previous solution was optimal and the only changed aspect of the problem is the upper capacity of Link 1. But that implies that the solution to the updated problem must either decrease or leave unchanged the flow along Link 2. In either event, if ϕ_2 is now increased by one, the effect of that increase is going to be less than if ϕ_2 had been increased before ϕ_1. A similar argument can be used if Link 1 and Link 2 have the same terminal node.

Now in order to prove this property rigorously, the optimality conditions and the basic ideas behind the *Out-of-Kilter* Algorithm need to be reviewed.

Let the dual variables from program (1) be defined as:

$$\begin{aligned} \mu_i &\equiv \text{dual variable corresponding to conservation of flow constraint (1) at node } i, i \in \mathcal{N}^*, \\ \lambda_{ij} &\equiv \text{dual variable corresponding to lower capacity constraint (3) on link } (i,j), \forall (i,j) \in \mathcal{A}^*, \\ \delta_{ij} &\equiv \text{dual variable corresponding to upper capacity constraint (2) on link } (i,j), \forall (i,j) \in \mathcal{A}^*. \end{aligned}$$

Then, examining the complimentary slackness conditions for this problem gives the following result for all minimum cost network flow problems (see, e.g., Bazaara, Jarvis and Sherali [1]).

Theorem 3.1 *Let* $\mathbf{x}$ *be any conserving flow, and let* μ *be any integral vector. Then* $\mathbf{x}$ *and* μ *are respectively primal and dual optimal solutions to program (1) if and only if for all* (i,j) *in* $\mathcal{A}^*$,

$$\begin{array}{lll} c_{ij} - \mu_i + \mu_j > 0 & \textit{implies} & x_{ij} = l_{ij} \\ c_{ij} - \mu_i + \mu_j < 0 & \textit{implies} & x_{ij} = u_{ij} \\ c_{ij} - \mu_i + \mu_j = 0 & \textit{implies} & l_{ij} \leq x_{ij} \leq u_{ij}. \end{array}$$

Proof: See Fulkerson [10].

Fulkerson [10] used this theorem to develop the *Out-of-Kilter* Method for solving minimum cost network flow problems. A link (i,j) which satisfies the conditions of Theorem 3.1 is said to be *in-kilter*, while those which do not satisfy those conditions are *out-of-kilter*. Each link is given a kilter number, which corresponds to the minimal change of flow required to bring that link into kilter; clearly, *in-kilter* links have a kilter number of zero.

The goal of the algorithm is to have the sum of all kilter numbers equal zero. The method accomplishes this by successively canceling negative cost residual cycles. A link (i,j) is selected from among those links which are currently *out-of-kilter*. If flow needs to be increased along (i,j), then node j is labeled. If flow needs to be decreased, node i is labeled. The method then proceeds to look from labeled nodes to see if flow along links to adjacent nodes can be increased or decreased, as appropriate. The labels generally contain information about the direction of flow change through that node and the maximum change possible through that node.

While all *out-of-kilter* links are allowed to have flow changes (which will bring them closer to being *in-kilter*), only *in-kilter* links for which $\mu_i - \mu_j - c_{ij} = 0$ are allowed to have flow change. The labeling continues until either a circuit is found which includes link (i,j) or no more nodes can be labeled. In the latter case, a dual variable change is made in which the dual variables, μ_k, for all labeled nodes are increased by an equal amount. This step allows for more labeling to be done in the next labeling step, or to determine that the problem is infeasible. In the former case, a flow change is made in which the flow along (i,j) is changed. In either case, the kilter number of each link either remains the same or strictly decreases. The kilter numbers never increase. In any labeling step, let $\mathcal{X}$ denote the set of labeled nodes, and let $\bar{\mathcal{X}}$ denote the set of unlabeled nodes.

For a full description of the method, see Fulkerson [10], Bazaara, Jarvis & Sherali [1], or Murty [12].

One insightful point to make about network flows in general and, in particular, the *Out-of-Kilter* Algorithm, is the following restatement of the objective function, which will be used later. If the matrix E is used to denote the node-arc incidence matrix, then any conserving flow satisfies $Ex = 0$, and so $\mu Ex = 0$. This implies that for a conserving flow which satisfies the lower and upper capacity constraints, the objective function can be written:

$$\begin{aligned} f(x) &= \sum_{(i,j)\in\mathcal{A}^*} c_{ij}x_{ij} \\ &= \sum_{(i,j)\in\mathcal{A}^*} (c_{ij} - \mu_i + \mu_j)x_{ij}. \end{aligned} \tag{2}$$

The $(c_{ij} - \mu_i + \mu_j)$ used in Equation (2) is exactly the equation used in Theorem 3.1 to determine if link (i,j) is *in-kilter* or not.

Another point to note is the inability to increase flow along certain links during certain primal flow changes. As mentioned, the *Out-of-Kilter* Algorithm works by cancelling all negative cost residual cycles. Suppose a particular network is solved, then some Link k has its upper capacity increased by one. Since the previous problem had been solved, all kilter numbers were zero exactly. Now the only possible nonzero kilter number is that of Link k. Its only possible nonzero value is one. Since all primal flow changes decrease the kilter number of at least one of the links involved in the cycle by one, the new network is

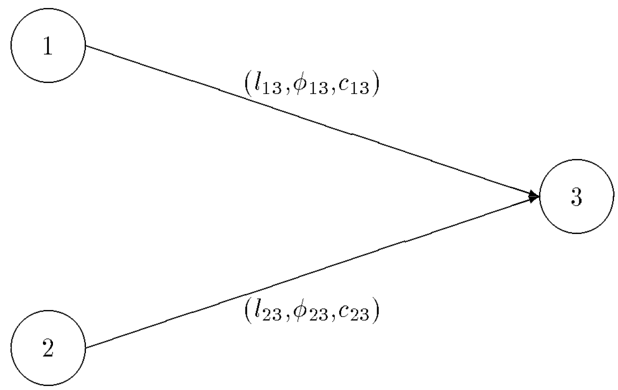

Figure 1: Link 1 and Link 2 with a common terminal node.

at most one cycle change away from optimality. That cycle change (if necessary) must involve increasing flow on Link k by one unit.

This result implies that a cycle cannot include increasing flow along any other link which has the same terminal node as Link k. Otherwise, the result would not be a cycle. Similarly, the cycle cannot include increasing flow along any link which has the same initial node as link k. These two observations will be needed later in the proofs of convex marginal return functions for links having either a common initial or terminal node.

To develop the idea of convex marginal return functions, consider a minimum cost network flow problem in which Link 1 flows from Node 1 to Node 3 and Link 2 flows from Node 2 to Node 3. For a particular realization of ϕ, the upper capacity of Link 1 is $u_{13} = \phi_{13}$ and the upper capacity of Link 2 is $u_{23} = \phi_{23}$. For simplification, assume that only these two links have stochastic capacities, so that program (1) can be viewed as $f(\phi_{13}, \phi_{23})$. Figure 1 gives a partial diagram of a possible network depiction of this situation.

Claim 3.1 $f(\phi_{13}+1, \phi_{23}) - f(\phi_{13}, \phi_{23}) \leq f(\phi_{13}+1, \phi_{23}+1) - f(\phi_{13}, \phi_{23}+1)$.

Proof: Let the all components of the optimal solution to $f(\phi_{13}, \phi_{23})$ have an accent bar ($\bar{x}$), all components of the optimal solution to $f(\phi_{13}+1, \phi_{23})$ have an accent breve ($\breve{x}$), all components of the optimal solution to $f(\phi_{13}, \phi_{23}+1)$ have an accent hat ($\hat{x}$), and all components of the optimal solution to $f(\phi_{13}+1, \phi_{23}+1)$ have an accent tilde ($\tilde{x}$).

Suppose $\breve{x}_1 \leq \phi_{13}$. This implies that $f(\phi_{13}, \phi_{23}) = f(\phi_{13}+1, \phi_{23})$. Then, when u_{23} is raised to $\phi_{23}+1$ in $f(\phi_{13}+1, \phi_{23}+1)$, by the comments above, $\tilde{x}_1$ remains less than or equal to ϕ_{13}. This implies that $f(\phi_{13}, \phi_{23}+1) = f(\phi_{13}+1, \phi_{23}+1)$. This shows that the claim is valid when $\breve{x}_1 \leq \phi_{13}$ or $\hat{x}_2 \leq \phi_{23}$ (since the same argument can be repeated).

Assume $\breve{x}_1 = \phi_{13} + 1$ and $\hat{x}_2 = \phi_{23} + 1$. Suppose that $\tilde{x}_1 \leq \phi_{13}$. This implies that $f(\phi_{13}, \phi_{23} + 1) = f(\phi_{13}+1, \phi_{23}+1)$. Since $f(\phi)$ is a non-increasing function of ϕ, we know that $f(\phi_{13}+1, \phi_{23}) \leq f(\phi_{13}, \phi_{23})$. Hence, the claim remains valid. Again, the same argument can be repeated for $\tilde{x}_2$.

Finally, assume $\breve{x}_1 = \phi_{13} + 1$, $\hat{x}_2 = \phi_{23} + 1$, $\tilde{x}_1 = \phi_{13} + 1$, and $\tilde{x}_2 = \phi_{23} + 1$. Examine what happens in each problem.

1. $\underline{f(\phi_{13}, \phi_{23})}$ $\Rightarrow$ Optimal flow $\bar{x}$ has value $f(\phi_{13}, \phi_{23})$.
$\bar{x}_1 = \phi_{13}, \bar{\mu}_1 - \bar{\mu}_3 > c_{13}$, otherwise $\breve{x}_1 \leq \phi_{13}$.
$\bar{x}_2 = \phi_{23}, \bar{\mu}_2 - \bar{\mu}_3 > c_{23}$, otherwise $\breve{x}_2 \leq \phi_{23}$.

2. $\underline{f(\phi_{13} + 1, \phi_{23})}$ $\Rightarrow$ Optimal flow $\breve{x}$ has value $f(\phi_{13} + 1, \phi_{23})$.
$\breve{x}_1 = \phi_{13} + 1$, by assumption.
$\breve{\mu}_1 - \breve{\mu}_3 = \bar{\mu}_1 - \breve{\mu}_3 > c_{13}$, ($\breve{\mu}_1 = \bar{\mu}_1$ since Node 1 $\in \bar{\mathcal{X}}$ in any dual variable change; otherwise a circuit would have been found).
$\breve{x}_2 = \phi_{23}$ and $\breve{\mu}_2 - \breve{\mu}_3 > c_{23}$. Otherwise $\tilde{x}_2 \leq \phi_{23}$.
$\breve{\mu}_3 \geq \bar{\mu}_3$, since any dual variable change strictly increases μ for all nodes in $\mathcal{X}$, and Node 3 is always in $\mathcal{X}$ (since link 1 is the only *out-of-kilter* link).

Then by Equation (2), since the only difference between flow $\bar{x}$ and $\breve{x}$ is the circuit including link 1 and the only possible links which could be included in that circuit other than Link 1 according to the *Out-of-Kilter* method are those links (k, l) for which $\mu_k - \mu_l - c_{kl} = 0$, the following is established:

$$\begin{aligned} f(\phi_{13} + 1, \phi_{23}) &= f(\phi_{13}, \phi_{23}) - (c_{13} - \bar{\mu}_1 + \breve{\mu}_3)\phi_{13} + (c_{13} - \bar{\mu}_1 + \breve{\mu}_3)(\phi_{13} + 1) \\ &= f(\phi_{13}, \phi_{23}) + (c_{13} - \bar{\mu}_1 + \breve{\mu}_3). \end{aligned} \tag{3}$$

Since the return to in-kilter was by circuit,

$$c_{13} - \bar{\mu}_1 + \breve{\mu}_3 < 0.$$

3. $\underline{f(\phi_{13}, \phi_{23} + 1)}$ $\Rightarrow$ Optimal flow $\hat{x}$ has value $f(\phi_{13}, \phi_{23} + 1)$.
$\hat{x}_1 = \phi_{13}$, and $\hat{\mu}_1 - \hat{\mu}_3 > c_{13}$. Otherwise $\tilde{x}_1 \leq \phi_{13}$.
$\hat{x}_2 = \phi_{23} + 1$, by assumption.
$\hat{\mu}_2 - \hat{\mu}_3 = \bar{\mu}_2 - \hat{\mu}_3 > c_{23}$ and $\hat{\mu}_3 \geq \bar{\mu}_3$.
Again, by Equation (2),

$$\begin{aligned} f(\phi_{13}, \phi_{23} + 1) &= f(\phi_{13}, \phi_{23}) - (c_{23} - \bar{\mu}_2 + \hat{\mu}_3)\phi_{23} + (c_{23} - \bar{\mu}_2 + \hat{\mu}_3)(\phi_{23} + 1) \\ &= f(\phi_{13}, \phi_{23}) + (c_{23} - \bar{\mu}_2 + \hat{\mu}_3). \end{aligned} \tag{4}$$

By assumption, the return to in-kilter status is by circuit. Hence,

$$c_{23} - \bar{\mu}_2 + \hat{\mu}_3 < 0.$$

4. $\underline{f(\phi_{13} + 1, \phi_{23} + 1)}$ (starting with feasible flow $\breve{x}$) $\Rightarrow$ Optimal flow $\tilde{x}$ has value $f(\phi_{13} + 1, \phi_{23} + 1)$.

$\tilde{x}_1 = \phi_{13}+1, \tilde{x}_2 = \phi_{23}+1$, by assumption.
$\tilde{\mu}_2 - \tilde{\mu}_3 = \breve{\mu}_2 - \tilde{\mu}_3 > c_{23}$.
$\tilde{\mu}_1 - \tilde{\mu}_3 \geq c_{13}$ and $\tilde{\mu}_3 \geq \breve{\mu}_3$.
By Equation (2),

$$\begin{aligned} f(\phi_{13}+1, \phi_{23}+1) &= f(\phi_{13}+1, \phi_{23}) - (c_{23} - \breve{\mu}_2 + \tilde{\mu}_3)\phi_{23} + (c_{23} - \breve{\mu}_2 + \tilde{\mu}_3)(\phi_{23}+1) \\ &= f(\phi_{13}+1, \phi_{23}) + (c_{23} - \breve{\mu}_2 + \tilde{\mu}_3). \end{aligned} \tag{5}$$

Similarly, starting with feasible flow $\hat{x}$, Equation (2) gives:

$$\begin{aligned} f(\phi_{13}+1, \phi_{23}+1) &= f(\phi_{13}, \phi_{23}+1) - (c_{13} - \hat{\mu}_1 + \tilde{\mu}_3)\phi_{13} - (c_{13} - \hat{\mu}_1 + \tilde{\mu}_3)(\phi_{13}+1) \\ &= f(\phi_{13}, \phi_{23}+1) + (c_{13} - \hat{\mu}_1 + \tilde{\mu}_3). \end{aligned} \tag{6}$$

Now, from $f(\phi_{13}, \phi_{23})$, consider the possibility of increasing both u_{13} and u_{23} by 1. Regardless of which link is chosen first, Node 3 must be labeled $(+\bullet, 1)$. Thus, if a circuit to either Node 1 or Node 2 is sought starting from labeled Node 3, one node must be reached before the other, and, until one is reached, neither μ_1 or μ_2 is increased in any dual variable change. Thus, either $\breve{\mu}_2 = \bar{\mu}_2$ or $\hat{\mu}_1 = \bar{\mu}_1$. Without loss of generality, assume $\hat{\mu}_1 = \bar{\mu}_1$. Then, using equations (3) and (6),

$$\begin{aligned} & f(\phi_{13}+1, \phi_{23}) - f(\phi_{13}, \phi_{23}) - f(\phi_{13}+1, \phi_{23}+1) + f(\phi_{13}, \phi_{23}+1) \\ &\qquad = (c_{13} - \bar{\mu}_1 + \breve{\mu}_3) - (c_{13} - \hat{\mu}_1 + \tilde{\mu}_3) \\ &\qquad = \breve{\mu}_3 - \tilde{\mu}_3 \\ &\qquad \leq 0. \end{aligned}$$

Thus, the claim is valid. $\square$

Next, the same property will be shown for links having a common initial node. Here, consider a network flow problem in which Link 1 flows from Node 1 to Node 3 and Link 2 flows from Node 1 to Node 3. For a particular realization of ϕ, the upper capacity of Link 1 is $u_{12} = \phi_{12}$ and the upper capacity of Link 2 is $u_{13} = \phi_{13}$. For simplification, assume that only these two links have stochastic capacities, so that program (1) can be viewed as $f(\phi_{13}, \phi_{12})$. Figure 2 gives a partial diagram of a possible network depiction of this situation.

Claim 3.2 $f(\phi_{12}+1, \phi_{13}) - f(\phi_{12}, \phi_{13}) \leq f(\phi_{12}+1, \phi_{13}+1) - f(\phi_{12}, \phi_{13}+1)$.

Proof: This proof closely matches that of Claim 3.1. Let the all components of the optimal solution to $f(\phi_{13}, \phi_{12})$ have an accent bar ($\bar{x}$), all components of the optimal solution to $f(\phi_{13}+1, \phi_{12})$ have an accent breve ($\breve{x}$), all components of the optimal solution to $f(\phi_{13}, \phi_{12}+1)$ have an accent hat ($\hat{x}$), and all components of the optimal solution to $f(\phi_{13}+1, \phi_{12}+1)$ have an accent tilde ($\tilde{x}$).

Suppose $\breve{x}_1 \leq \phi_{13}$. This implies that $f(\phi_{13}, \phi_{12}) = f(\phi_{13}+1, \phi_{12})$. Then, when u_{12} is raised to $\phi_{12}+1$ from $f(\phi_{13}+1, \phi_{12}+1)$, by the comments above, we know that $\tilde{x}_1$ remains less than or equal to ϕ_{13}. This implies that $f(\phi_{13}, \phi_{12}+1) = f(\phi_{13}+1, \phi_{12}+1)$. This shows that the claim is valid when $\breve{x}_1 \leq \phi_{13}$

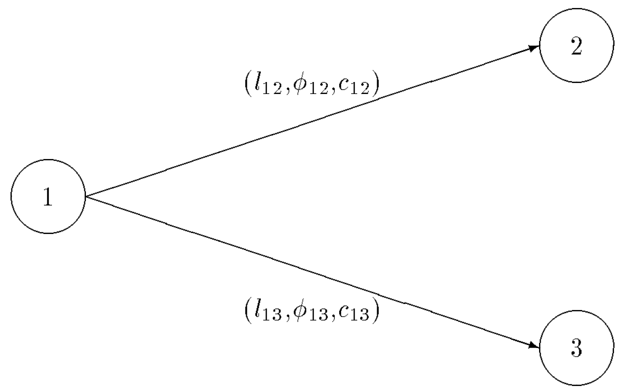

Figure 2: Link 1 and Link 2 with a common initial node.

or $\hat{x}_2 \leq \phi_{12}$ (since the same argument can be repeated).

Assume $\breve{x}_1 = \phi_{13} + 1$ and $\hat{x}_2 = \phi_{12} + 1$. Suppose that $\tilde{x}_1 \leq \phi_{13}$. This implies that $f(\phi_{13}, \phi_{12} + 1) = f(\phi_{13} + 1, \phi_{12} + 1)$. But since $f(\phi)$ is a non-increasing function of ϕ, we know that $f(\phi_{13} + 1, \phi_{12}) \leq f(\phi_{13}, \phi_{12})$. Thus, the claim remains valid. The same argument can be repeated for $\tilde{x}_2$.

Finally, assume $\breve{x}_1 = \phi_{13} + 1$, $\hat{x}_2 = \phi_{12} + 1$, $\tilde{x}_1 = \phi_{13} + 1$, and $\tilde{x}_2 = \phi_{12} + 1$. Examine what happens in each problem.

1. $\underline{f(\phi_{13}, \phi_{12})} \Rightarrow$ Optimal flow $\bar{x}$ has value $f(\phi_{13}, \phi_{12})$.
$\bar{x}_1 = \phi_{13}, \bar{\mu}_1 - \bar{\mu}_3 > c_{13}$. Otherwise, $\breve{x}_1 \leq \phi_{13}$.
$\bar{x}_2 = \phi_{12}, \bar{\mu}_1 - \bar{\mu}_2 > c_{12}$. Otherwise, $\breve{x}_2 \leq \phi_{12}$.

2. $\underline{f(\phi_{13} + 1, \phi_{12})} \Rightarrow$ Optimal flow $\breve{x}$ has value $f(\phi_{13} + 1, \phi_{12})$.
$\breve{x}_1 = \phi_{13} + 1$, by assumption.
$\breve{\mu}_1 - \breve{\mu}_3 = \bar{\mu}_1 - \breve{\mu}_3 > c_{13}$, ($\breve{\mu}_1 = \bar{\mu}_1$ since Node $1 \in \bar{\mathcal{X}}$ in any dual variable change; otherwise a circuit would have been found).
$\breve{x}_2 = \phi_{12}$ and $\breve{\mu}_1 - \breve{\mu}_2 > c_{12}$. Otherwise, $\tilde{x}_2 \leq \phi_{12}$.
$\breve{\mu}_3 \geq \bar{\mu}_3$, since any dual variable change strictly increases μ for all nodes $\in \mathcal{X}$, and Node 3 is always in $\mathcal{X}$ (since Link 1 is the only *out-of-kilter* link).
Then by Equation (2), since the only difference between the flow $\bar{x}$ and $\breve{x}$ is the circuit including Link 1 and the only possible links which could be included in that circuit other than Link 1 according to the *Out-of-Kilter* method are those links (k, l) for which $\mu_k - \mu_l - c_{kl} = 0$,

the following is established:

$$\begin{aligned} f(\phi_{13}+1,\phi_{12}) &= f(\phi_{13},\phi_{12}) - (c_{13}-\bar{\mu}_1+\breve{\mu}_3)\phi_{13} + (c_{13}-\bar{\mu}_1+\breve{\mu}_3)(\phi_{13}+1) \\ &= f(\phi_{13},\phi_{12}) + (c_{13}-\bar{\mu}_1+\breve{\mu}_3). \end{aligned} \tag{7}$$

Since the return to in-kilter was by circuit,

$$c_{13}-\bar{\mu}_1+\breve{\mu}_3 < 0.$$

3. $\underline{f(\phi_{13},\phi_{12}+1)}$ $\Rightarrow$ Optimal flow $\hat{x}$ has value $f(\phi_{13},\phi_{12}+1)$.
$\hat{x}_1 = \phi_{13}$, and $\hat{\mu}_1 - \hat{\mu}_3 > c_{13}$. Otherwise, $\tilde{x}_1 \leq \phi_{13}$.
$\hat{x}_2 = \phi_{12}+1$, by assumption.
$\hat{\mu}_1 - \hat{\mu}_2 = \bar{\mu}_1 - \hat{\mu}_2 > c_{12}$, and $\hat{\mu}_1 \geq \bar{\mu}_1$.
Again, by Equation (2),

$$\begin{aligned} f(\phi_{13},\phi_{12}+1) &= f(\phi_{13},\phi_{12}) - (c_{12}-\bar{\mu}_1+\hat{\mu}_2)\phi_{12} + (c_{12}-\bar{\mu}_1+\hat{\mu}_2)(\phi_{12}+1) \\ &= f(\phi_{13},\phi_{12}) + (c_{12}-\bar{\mu}_1+\hat{\mu}_2). \end{aligned} \tag{8}$$

Since, by assumption, the return to in-kilter status is by circuit,

$$c_{12}-\bar{\mu}_1+\hat{\mu}_2 < 0.$$

4. $\underline{f(\phi_{13}+1,\phi_{12}+1)}$ (starting with feasible flow $\breve{x}$) $\Rightarrow$ Optimal flow $\tilde{x}$ has value $f(\phi_{13}+1,\phi_{12}+1)$.
$\tilde{x}_1 = \phi_{13}+1, \tilde{x}_2 = \phi_{12}+1$, by assumption.
$\tilde{\mu}_1 - \tilde{\mu}_2 = \breve{\mu}_1 - \tilde{\mu}_2 > c_{12}$.
$\tilde{\mu}_1 - \tilde{\mu}_3 \geq c_{13}$ and $\tilde{\mu}_3 \geq \breve{\mu}_3$.
By Equation (2),

$$\begin{aligned} f(\phi_{13}+1,\phi_{12}+1) &= f(\phi_{13}+1,\phi_{12}) - (c_{12}-\breve{\mu}_1+\tilde{\mu}_2)\phi_{12} + (c_{12}-\breve{\mu}_1+\tilde{\mu}_2)(\phi_{12}+1) \\ &= f(\phi_{13}+1,\phi_{12}) + (c_{12}-\breve{\mu}_1+\tilde{\mu}_2). \end{aligned} \tag{9}$$

Similarly, starting with feasible flow $\hat{x}$, Equation (2) gives:

$$\begin{aligned} f(\phi_{13}+1,\phi_{12}+1) &= f(\phi_{13},\phi_{12}+1) - (c_{13}-\hat{\mu}_1+\tilde{\mu}_3)\phi_{13} - (c_{13}-\hat{\mu}_1+\tilde{\mu}_3)(\phi_{13}+1) \\ &= f(\phi_{13},\phi_{12}+1) + (c_{13}-\hat{\mu}_1+\tilde{\mu}_3). \end{aligned} \tag{10}$$

Now, from $f(\phi_{13},\phi_{12})$, consider the possibility of increasing both u_{13} and u_{12} by 1. Regardless of which link is chosen, a cycle is not found resolving either *out-of-kilter* status until Node 1 becomes labeled. Thus, until Node 1 is reached, μ_1 is not increased in any dual variable change. Thus, either $\breve{\mu}_2 = \bar{\mu}_2$ or $\hat{\mu}_1 = \bar{\mu}_1$. Without loss of generality, assume $\hat{\mu}_1 = \bar{\mu}_1$. Hence, using equations (7) and (10),

$$\begin{aligned} f(\phi_{13}+1,\phi_{12}) - f(\phi_{13},\phi_{12}) - f(\phi_{13}+1,\phi_{12}+1) + f(\phi_{13},\phi_{12}+1) & \\ = (c_{13}-\bar{\mu}_1+\breve{\mu}_3) - (c_{13}-\hat{\mu}_1+\tilde{\mu}_3) & \\ = \breve{\mu}_3 - \tilde{\mu}_3 & \\ \leq 0. & \end{aligned}$$

Thus, the claim is valid. □

4 Upper Bound for a Non-increasing Convex Function with Convex Marginal Return Functions

Let Φ be an n-dimensional random vector, where each Φ_i is independently distributed with a bounded support, finite mean and cumulative distribution function F_i. Further, let ϕ_i^L, ϕ_i^H be the realizations of Φ_i defined as:

$$\phi_i^L = \sup\{\phi_i : F_i(\phi_i) \leq 0\}, \tag{11}$$

$$\phi_i^H = \inf\{\phi_i : F_i(\phi_i) \leq 1\}, \tag{12}$$

so that $\phi_i^L < \phi_i^H$ for all i. Let $p_i^L \equiv prob\{\Phi_i = \phi_i^L\}$ and $p_i^H \equiv prob\{\Phi_i = \phi_i^H\}$ for all i be such that

$$p_i^L \phi_i^L + p_i^H \phi_i^H = E[\Phi_i], \quad p_i^L + p_i^H = 1, \quad p_i^L, p_i^H \geq 0.$$

Let $f(\Phi)$ be a non-increasing convex function of Φ with convex marginal return functions with respect to any pair $(\Phi_i, \Phi_j), i = 1, \ldots, n,\ j = 1 \ldots, n$. Then the following proposition, proven in Donohue and Birge [8], shows that an upper bound on $E[f(\Phi)]$ can be established with only 2 function evaluations.

Proposition 4.1 *Let Φ and $f(\Phi)$ be as defined above. Let $p^L \equiv \max\{p_1^L, p_2^L, \ldots, p_n^L\}$ and $p^H \equiv 1 - p^L$. Then*

$$E[f(\Phi)] \leq p^L f(\phi_1^L, \phi_2^L, \ldots, \phi_n^L) + p^H f(\phi_1^H, \phi_2^H, \ldots, \phi_n^H).$$

Having shown that the minimum cost network flow problem with stochastic upper link capacities is a non-increasing convex function of the stochastic upper capacities, and that this function also has convex marginal return functions with respect to capacities of links having either a common initial or common terminal node, Proposition 1 can be applied to solve for an upper bound on the expected value of this stochastic minimum cost network program which involves solving 2^{k_1} or 2^{k_2} realizations, where k_1 is the number of nodes having outbound links with stochastic upper capacities and k_2 is the number of nodes having inbound links with stochastic upper capacities. In many cases, this is a significant reduction to the 2^L realizations that are needed to be solved for the Edmundson-Madansky bound, where L is the number of links having stochastic upper capacities.

Consider a network in which link i has stochastic upper capacity ϕ_i, $i = 1, \ldots, k$. Then by the Edmunson-Madansky [11] bound,

$$E[f(\Phi)] \leq \sum_{i^1 \in \{L,H\}} \cdots \sum_{i^k \in \{L,H\}} \prod_{j=1}^{k} (p_j^{i^j}) f(\phi_1^{i^1}, \ldots, \phi_k^{i^k}). \tag{13}$$

Let $n_1, \ldots, n_h$ be the set of terminal nodes of links $1, \ldots, k$. Let t_i denote the terminal node of link i, $i = 1, \ldots, k$, and let $p_{n_j}^L = \max\{p_i^L : t_i = j\}, p_{n_j}^H = 1 - p_{n_j}^L$. Since $f(\Phi)$ is a non-increasing, convex function with convex marginal return functions, Proposition 4.1 can be applied over each set of links with a common terminal node. Hence,

$$E[f(\Phi)] \leq \sum_{j^1 \in \{L,H\}} \cdots \sum_{j^h \in \{L,H\}} \prod_{l=1}^{h} (p_{n_l}^{j^l}) f(\phi_1^{j^{t_1}}, \ldots, \phi_k^{j^{t_k}}). \tag{14}$$

Dist. Number, i	L_i	H_i	$\Pr(X_i = L_i)$	$\Pr(X_i = L_i + 1)$	$\Pr(X_i = L_i + 2)$	$\Pr(X_i = L_i + 3)$
1	1	4	0.35	0.30	0.25	0.10
2	2	5	0.40	0.20	0.20	0.20
3	3	6	0.30	0.25	0.25	0.20
4	4	6	0.50	0.30	0.20	-
5	5	8	0.40	0.30	0.15	0.15
6	6	8	0.45	0.35	0.20	-
7	7	9	0.50	0.25	0.25	-
8	8	11	0.35	0.30	0.20	0.15
9	9	11	0.45	0.40	0.15	-
10	10	12	0.40	0.35	0.25	-

Table 1: Possible Distributions for Links with Stochastic Upper Capacity for Transportation Problem.

Similarly, by letting t_i denote the initial node of link i, the same inequality results from applying Proposition 4.1 over each set of links with a common initial node. Refinements of the bound in Proposition 4.1 are found in [8] which can be applied here to improve the effectiveness of this bound.

5 Experimental Result

To test the effectiveness of this upper bound, two test problems were created and solved. The first problem is a transportation problem with 15 source nodes, 15 sink nodes and 105 links connecting the sources and sinks. Three different versions of this problem were considered. The second problem is a vehicle allocation problem with uncertain demands. This type of problem determines the expected value of a specific initial vehicle distribution, given uncertainty about the demand for vehicle routings along several or all links. This problem has 20 nodes and 92 links.

Three versions of the transportation problem were considered. First, all 105 links were assumed to be stochastic, making finding an upper bound using Edmundson-Madansky computationally impossible. Next, 30 of the links were randomly selected to have stochastic upper capacities, and each of the other links were given capacity near the mean of its respective distribution. These 30 links had 9 common terminal nodes, allowing an upper bound to be found with $2^9 = 512$ function evaluations. Finally, all of the links out of node 8 were considered stochastic, giving a total of 9 stochastic links. Here, an upper bound is established with just two function evaluations. Further, a refinement of the upper bound by partitioning was calculated for comparison. Table 2 lists the cost per unit flow between all sources and sinks, where flow between the corresponding nodes is possible. Table 3 lists the supply and demand requirements at each of the source and sink nodes. Each link has a lower capacity bound of zero. The distribution of the random variable corresponding to the upper capacity of each link was one of ten possibilities. Let X_i denote the random variable with distribution i, and let L_i (H_i) denote the lowest (highest) possible value of distribution i. The ten possible distributions are listed in Table 1. Table 4 gives the distribution that was assigned to each link.

The second problem is a vehicle allocation problem, one stage of a dynamic vehicle allocation problem with uncertain demands (see [13]). Figure 5.1 depicts the network over which the fleet of vehicles must be routed, given the initial distribution of vehicles at the source nodes 1 through 5. The problem only

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
s_1	109	323	-	96	107	230	380	-	254	352	-	-	-	-	385
s_2	-	105	-	99	-	405	-	-	-	-	-	-	428	249	158
s_3	-	-	-	315	-	-	217	66	-	139	73	157	-	-	-
s_4	202	-	175	277	-	-	402	290	-	-	-	233	-	-	-
s_5	-	123	-	-	-	-	-	-	167	94	169	296	154	447	-
s_6	-	-	-	143	-	-	216	382	-	112	229	149	-	-	-
s_7	291	-	127	-	-	-	279	-	226	436	-	-	-	-	-
s_8	-	-	388	137	280	239	-	-	-	81	318	201	245	-	97
s_9	384	-	-	-	-	334	339	-	137	317	-	-	-	65	-
s_{10}	197	-	-	-	-	357	-	415	362	272	246	223	-	393	-
s_{11}	-	442	-	257	242	307	420	-	119	-	384	-	149	-	209
s_{12}	172	-	136	-	-	186	-	111	-	322	-	106	-	-	312
s_{13}	159	267	-	-	67	76	-	362	427	401	-	-	174	-	-
s_{14}	-	-	244	-	110	291	-	211	-	-	292	228	-	205	-
s_{15}	352	-	-	-	-	-	426	350	-	-	-	-	97	94	355

Table 2: Cost per unit flow from source s_i to sink t_j in Transportation Problem.

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_i	55	46	35	26	44	38	24	58	38	44	32	46	50	34	30
t_i	47	32	23	43	13	59	50	41	44	72	38	47	30	30	31

Table 3: Source and sink node supply and demand requirements in Transportation Problem.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
s_1	8	5	-	9	1	9	5	-	9	10	-	-	-	-	3
s_2	-	10	-	7	-	10	-	-	-	-	-	-	1	10	10
s_3	-	-	-	2	-	-	7	8	-	5	6	9	-	-	-
s_4	8	-	3	6	-	-	3	8	-	-	-	1	-	-	-
s_5	-	7	-	-	-	-	-	-	7	6	9	7	8	2	-
s_6	-	-	-	9	-	-	4	3	-	6	10	8	-	-	-
s_7	6	-	6	-	-	-	6	-	7	2	-	-	-	-	-
s_8	-	-	7	6	2	8	-	-	-	10	7	8	10	-	3
s_9	1	-	-	-	-	4	8	-	7	10	-	-	-	9	-
s_{10}	10	-	-	-	-	2	-	1	10	9	7	8	-	1	-
s_{11}	-	2	-	9	1	4	9	-	3	-	2	-	2	-	4
s_{12}	6	-	1	-	-	9	-	7	-	8	-	7	-	-	10
s_{13}	10	9	-	-	5	9	-	6	2	9	-	-	3	-	-
s_{14}	-	-	7	-	8	7	-	3	-	-	2	2	-	8	-
s_{15}	2	-	-	-	-	-	10	7	-	-	-	-	8	6	3

Table 4: Distribution for upper capacity of link from source s_i to sink t_j in Transportation Problem.

D_i	L_i	H_i	$\Pr(X_i = L_i)$	$\Pr(X_i = L_i + 1)$	$\Pr(X_i = L_i + 2)$	$\Pr(X_i = L_i + 3)$	$\Pr(X_i = L_i + 4)$
1	0	2	0.40	0.30	0.30	-	-
2	0	3	0.40	0.25	0.20	0.15	-
3	0	3	0.35	0.25	0.20	0.20	-
4	0	4	0.30	0.25	0.20	0.15	0.10
5	0	4	0.35	0.20	0.15	0.15	0.15
6	1	3	0.45	0.35	0.20	-	-
7	1	3	0.40	0.35	0.25	-	-
8	1	4	0.45	0.20	0.20	0.15	-
9	1	4	0.40	0.25	0.20	0.15	-
10	2	4	0.40	0.30	0.30	-	-

Table 5: Possible Distributions for Stochastic Links in Vehicle Allocation Problem.

requires that flow is conserved throughout the network, given the initial distribution of vehicles. Thus, Node 20 is actually an artificial sink node and the links into Node 20 are artificial as well. These links are uncapacitated and add no value to the problem. All other links shown in the diagram handle the flow of loaded vehicles between the adjacent nodes.

Since the need to move loads is governed by demand, the upper capacities of these links are stochastic. Flow along these links generate positive revenue. Running parallel to each of these links, but not in the diagram, is another link which handles the flow of empty vehicles. These links are uncapacitated. Flow along these links generate positive cost. The goal is to minimize negative profits (cost - revenue). The given initial distribution of vehicles is (10,12,12,10,7). The links in this problem were numbered by considering each node in numerical order (as seen in the diagram) and numbering consecutively from top to bottom. Table 6 gives the cost per unit of loaded flow along each link (r_i), the cost of per unit of empty flow along each link (c_i), and the distribution given to the random variable corresponding to that link's upper capacity(D_i). As before, the distributions were chosen from among 10 different possibilities, which are shown in Table 5.

Again, three versions of this problem were considered. In the first version, all of the links handling the flow of loaded vehicles were assumed to have stochastic upper capacities. In the second version, 15 links were chosen at random to have stochastic capacities. These 15 links have 8 distinct initial nodes and 11 distinct terminal nodes, so that the upper bound from links with a common terminal node differs only slightly from the Edmundson-Madansky bound, both in value and computation. The upper bound from links with common initial nodes, though, is small enough that that probability space can be partitioned and the initial upper bound can be refined. Finally, all links into and out of Node 8 are assumed to have stochastic upper capacities. Thus, a total of eight links have stochastic capacities. Since the network is acyclic, none of the links into Node 8 are affected when the bound for links with a common initial node is used over the links out of Node 8. Hence, the bound for links with a common terminal node can be used on all the links into Node 8. Thus, only four function evaluations are necessary.

A program was written in C which recursively updated the right-hand side of each network problem as needed, then passed the new program to be solved by IBM's Optimization Subroutine Library (OSL). The program was run on an IBM RS\6000 workstation.

The results are shown in the table below. The table includes information about the number of stochastic links in each version, and the number of distinct problems that had to be solved. If two values are

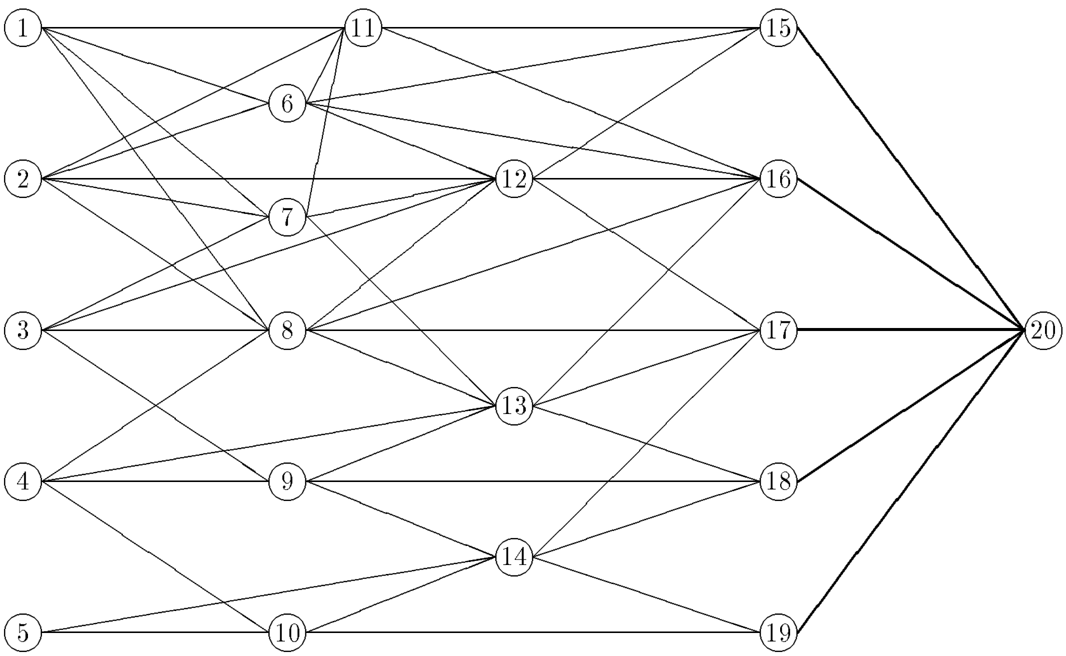

Figure 3: Vehilcle Allocation Problem with Uncertain Demands

i	r_i	c_i	D_i	i	r_i	c_i	D_i	i	r_i	c_i	D_i	i	r_i	c_i	D_i
1	327	81	4	13	255	73	7	25	177	63	9	36	263	93	4
2	211	67	10	14	265	76	3	26	348	91	10	37	285	98	7
3	243	70	1	15	360	105	9	27	148	51	10	38	281	99	7
4	268	70	7	16	161	60	10	28	312	76	8	39	242	87	1
5	301	75	10	17	264	78	8	29	332	83	9	40	301	108	6
6	204	73	3	18	348	86	10	30	284	75	3	41	277	93	4
7	356	95	8	19	267	78	8	31	317	76	6	42	168	77	5
8	190	63	5	20	146	54	7	32	367	103	6	43	175	78	7
9	225	69	1	21	330	86	4	33	329	81	2	44	273	94	8
10	198	66	1	22	352	93	2	34	214	65	9	45	252	87	3
11	343	85	2	23	188	61	6	35	326	78	1	46	252	88	2
12	261	80	3	24	225	74	3								

Table 6: Revenue for loaded flow, cost of empty flow, and distribution of capacity for links in Vehicle Allocation Problem.

shown for the number of problems solved, the first is for the upper bound for links with a common terminal node, the second for links with a common initial node. If only one value is shown, the values are the same. Also shown is the optimal objective value when all stochastic components are at their lowest value (Greatest Objective Value), and the optimal objective value when all stochastic components are at their highest value (Least Objective Value). This was done to show the range of possible values. Finally, the table shows the lower bound found using Jensen's Inequality, the upper bound using links with a common terminal node, and the upper bound using links with a common initial node. The problem Trans.3.a shows the effect of the refining the bound in problem Trans.3. Problem Veh.Allo.3 shows the bound obtained by clustering both the links into Node 8 and links out of Node 8.

Problem	Number of Stochastic Links	Number of Problems Solved	Greatest Objective Value	Least Objective Value	Lower Bound (Jensen)	Upper Bound (Terminal Node)	Upper Bound (Initial Node)
Trans.1.	105	2^{15}	132,095	114,190	124,154.90	126,838.00	126,671.59
Trans.2.	30	2^{9}	132,095	126,007	129,519.45	130,064.32	130,098.17
Trans.3.	9	2	130,303	127,165	128,766.40	-	129,126.25
Trans.3.a	9	3	130,303	127,165	128,766.40	-	129,054.28
Veh.Allo.1	46	2^{14}	-2058	-33270	-17,835.50	-16,082.53	-14,831.20
Veh.Allo.2	15	2^{11},2^{8}	-8038	-20,258	-13,179.10	-12,912.32	-12,558.45
Veh.Allo.3	8	4	-6861	-16952	-11086.65	-10787.42	

The bounds for the Transportation problem are remarkably tight, as the upper bound is within 2% of the lower bound in all cases, and as close as .224% in problem Trans.3.a. This would suggest that the function is almost linear in the stochastic capacities ϕ, and that Jensen's inequality is very close to the actual expected value. It is reasonable to assume that more error exists from the upper bound here than the lower bound, as is typically the case with convex function approximations. Note, however, that the effectiveness of Jensen's inequality in this problem would be more difficult to prove without an effective upper bound. The best upper bounds for problems Veh.Allo.1, Veh.Allo.2, and Veh.Allo.3 are within 9.8%, 2.0%, and 2.7%, respectively, of the given lower bound.

6 Conclusion

As stochastic programming reaches new heights in the number of realizations of the second-stage problem parameters being considered, new methods must be developed for approximating the expected value of recourse functions of even more realizations. To know the value of any approximation method, tight bounds, both upper and lower, must be available on the expected value of the recourse function. Finding effective approximation methods for stochastic programs often requires that problem structures be exploited so as to maximize computational efficiency. In this paper, we exploited the structure of minimum cost network flow problems in such a way as to allow an effective upper bound to be established with a reduced number of function evaluations.

Further, the ideas of this paper may possibly be extended for special cases where the flow through certain nodes can be shown to be independent of one another. Also, the results of this paper also suggest ideas for better partitioning methods used for refining lower bounds. Given the result about convex marginal return functions, it seems likely that the best partitioning order is going to be one which partitions on links having as few common initial and terminal nodes as possible.

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