Portfolios with Trading Constraints and Payout Restrictions

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General Problem

- (Very) long-term investor (example: university endowment)
- Payout from portfolio over time (want to keep payout from declining)
- Invest in various asset categories
- Decisions:
  - How much to payout (consume)?
  - How to invest in asset categories?
- Complication: restrictions on asset trades

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Outline

- Basic formulation
- General infinite horizon solution method
- Simplified problem and continuous time solution
- Results for restricted-trading portfolio
- Future issues

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Problem Formulation

- Notation:

  x – current state ($x \in X$)

  u (or u_x) – current action given x (u (or u_x) $\in U(x)$)

  δ – single period discount factor

  $P_{x,u}$ – probability measure on next period state y depending on x and u

  c(x,u) – objective value for current period given x and u

  V(x) – value function of optimal expected future rewards given current state x

- Problem: Find V such that

  $V(x) = \max_{u \in U(x)}\{c(x,u) + \delta\, E_{P_{x,u}}[V(y)]\}$
  for all $x \in X$.

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Approach

- Define an upper bound on the value function

$V^0(x) \geq V(x) \ \forall \ x \in X$

- Iteration k: upper bound V^k

Solve for some x^k

$TV^k(x^k) = \max_u c(x^k,u) + \delta \, E_{Pxk,u}[V^k(y)]$

Update to a better upper bound V^{k+1}

- Update uses an outer linear approximation on U^k

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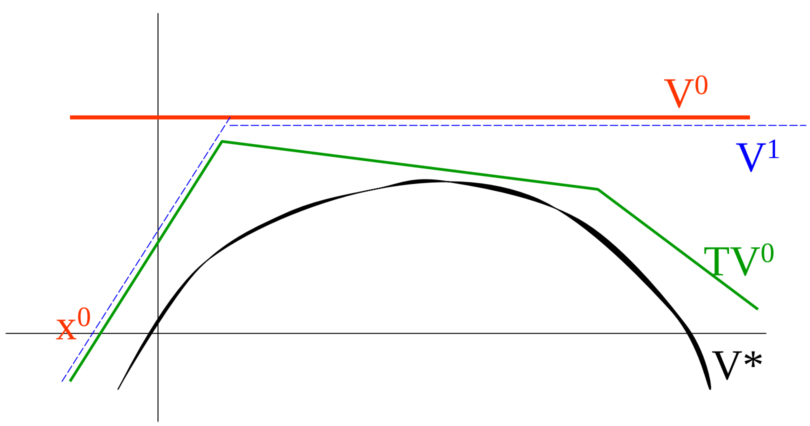
Successive Outer Approximation



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Properties of Approximation

- $V^* \leq TV^k \leq V^{k+1} \leq V^k$
- Contraction

  $\| TV^k - V^* \|_\infty \leq \delta \|V^k - V^*\|_\infty$
- Unique Fixed Point

  $TV^* = V^*$

$\Rightarrow$ if $TV^k \geq V^k$, then $V^k = V^*$.

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Convergence

- Value Iteration

  $T^k V^0 \to V^*$
- Distributed Value Iteration

  If you choose every $x \in X$ infinitely often,

  then $V^k \to V^*$.

  (Here, random choice of x, use concavity.)
- Deepest Cut

  Pick x^k to maximize $V^k(x)-TV^k(x)$

  DC problem to solve

  Convergence again with continuity (caution on boundary of domain of V^*)

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Details for Random Choice

- Consider any x
- Choose i and x^i s.t. $||x^i – x|| < \varepsilon^i$
- Suppose $|| \nabla V^i || \leq K \; \forall \; i$

$|| V^k(x) – V^*(x)|| \leq ||V^k(x)-V^k(x^k)+V^k(x^k)-V^*(x^k)+V^*(x^k)-V^*(x)||$

$\leq 2 \varepsilon^k K + \delta ||V^{k-1}(x^k)-V^*(x^k)||$

$\leq 2\sum_i \varepsilon^i K + \delta^k ||V^0(x^0) – V^*(x^0)||$

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Cutting Plane Algorithm

Initialization: Construct $V^0(x)=\max_u c^0(x,u) + \delta E_{Px,u}[V^0(y)]$, where $c^0 \geq c$ and c^0 concave.

V^0 is assumed piecewise linear and equivalent to

$V^0(x)=\max \{\theta \mid \theta \leq E^0 x + e^0\}$. k=0.

Iteration: Sample $x^k \in X$ (in any way such that the probability of $x^k \in A$ is positive for any $A \subset X$ of positive measure) and solve

$TV^k(x^k) = \max_u c(x^k,u) + \delta E_{Pxk,u}[V^k(y)]$ where

$V^k(y) = \max\{\theta \mid \theta \leq E^l y + e^l, l=0,\ldots,k.\}$

Find supporting hyperplanes defined by E^{k+1} and e^{k+1} such that $E^{k+1} x + e^{k+1} \geq TV^k(x)$. $k \leftarrow k+1$.

Repeat.

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Specifying Algorithm

Feasibility:

$Ax + Bu \leq b$

Transition:

$y=F_i u$ for some realization i with probability p_i

Iteration k Problem:

$TV^k(x^k) = \max_{u,\theta} c(x^k,u) + \delta \sum_i p_i \theta_i$

s.t. $A x^k + B u \leq b$, $- E^l(F^i u) - e^l + \theta^i \leq 0, \forall i,l.$

From duality:

$$TV^k(x^k) = \inf_{\mu,\lambda l,i} \max_{u,\theta} c(x^k,u) -\mu(Ax^k+Bu-b) + \delta \sum_i (p_i \theta_i+\sum_l \lambda^{i,l}(E^l(F^i u) + e^l - \theta^i))$$

$$\leq \max_{u,\theta} c(x^k,u) -\mu^k(Ax^k+Bu-b) + \delta \sum_i (p_i \theta_i+\sum_l \lambda^{i,l,k}(E^l(F^i u) +e^l - \theta^i)) \text{ for optimal } \mu^k, \lambda^{i,l,k} \text{ for } x^k$$

$$\leq c(x^k,u^k) + \nabla c(x^k,u^k)^T (x-x^k,-u^k) -\mu^k Ax +\mu^k b +\sum_i (\sum_l \lambda^{i,l,k}e^l)$$

Cuts:

$E^{k+1}= \nabla_x c(x^k,u^k)^T -\mu^k A$

e^{k+1} equal to the constant terms.

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Investment Problem

- Determine asset allocation and consumption policy to maximize the expected discounted utility of spending
  - State and Action

    x=(cons, risky, wealth) u=(cons_new,risky_new)
  - Two asset classes
    - Risky asset, with lognormal return distribution
    - Riskfree asset, with given return r_f
  - Power utility function

$$c(cons\_new) = \frac{cons\_new^{1-\gamma}}{1-\gamma}$$

  - Consumption rate constrained to be non-decreasing

    cons_new $\geq$ cons

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Existing Research

- Dybvig '95*
  - Continuous-time approach
  - Solution Analysis
    - Consumption rate remains constant until wealth reaches a new maximum
    - The risky asset allocation α is proportional to $w-c/r_f$, which is the excess of wealth over the perpetuity value of current consumption
    - α decreases as wealth decreases, approaching 0 as wealth approaches c/r_f (which is in absence of risky investment sufficient to maintain consumption indefinitely).
- Dybvig '01
  - Considered similar problem in which consumption rate can decrease but is penalized (soft constrained problem)

* "Duesenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment Given Intolerance for any Decline in Standard of Living" *Review of Economic Studies* 62, 1995, 287-313.

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Objectives

- Replicate Dybvig continuous time results using discrete time approach
- Evaluate the effect of trading restrictions for certain asset classes (e.g., private equity)
- Consider additional problem features
  - Transaction Costs
  - Multiple risky assets

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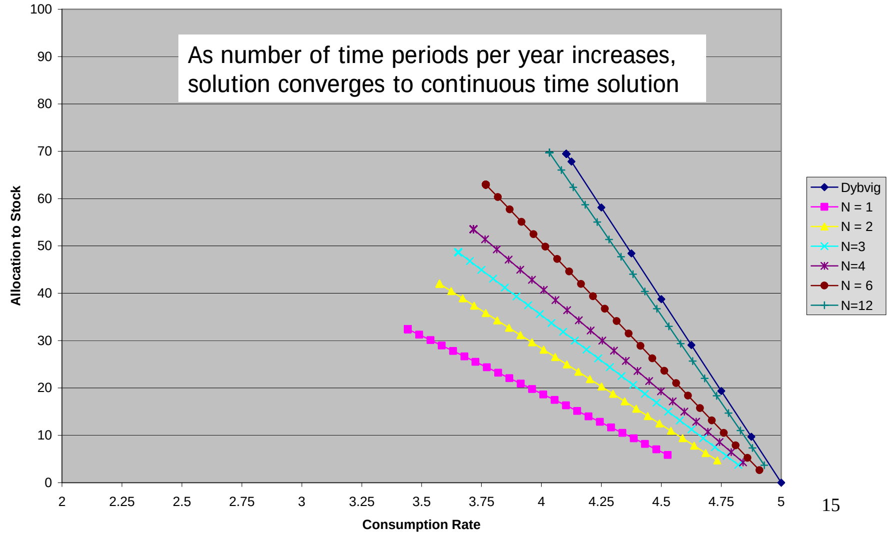
Results – Non-decreasing Consumption



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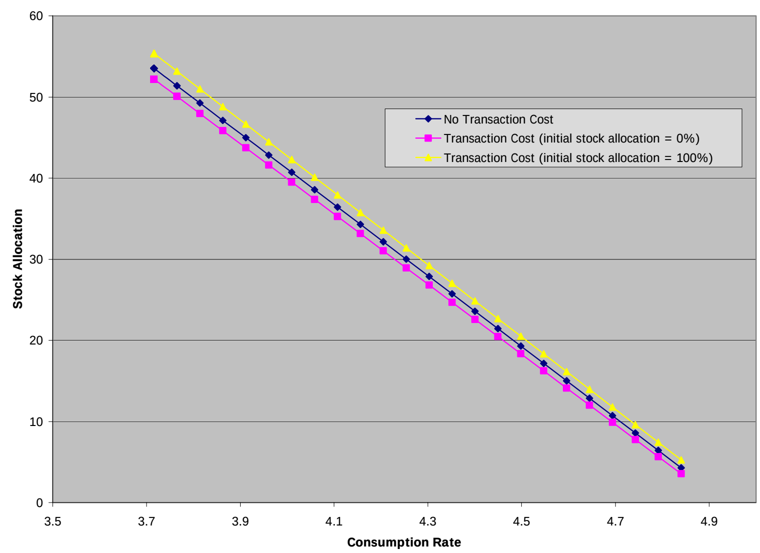
Results – Non-Decreasing Consumption with Transaction Costs



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Observations

- Effect of Trading Restrictions
  - Continuously traded risky asset: 70% of portfolio for 4.2% payout rate
  - Quarterly traded risky asset: 32% of portfolio for same payout rate
- Transaction Cost Effect
  - Small differences in overall portfolio allocations
  - Optimal mix depends on initial conditions

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Extensions

- Soft constraint on decreasing consumption
  - Allow some decreases with some penalty
- Lag on sales
  - Waiting period on sale of risky assets (e.g., 60-day period)
- Multiple assets
  - Allocation bounds

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Conclusions

- Can formulate infinite-horizon investment problem in stochastic programming framework
- Solution with cutting plane method
- Convergence with some conditions
- Results for trade-restricted assets significantly different from market assets with same risk characteristics

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Approach

- Application of typical stochastic programming approach complicated by infinite horizon

$$Q(\bar{x}) = \max \sum_i p_{\xi_i}\left(c_{\xi_i} x_{\xi_i} + e^{-\delta t} Q\left(x_{\xi_i}\right)\right)$$
$$s.t. \quad A x_{\xi_i} = b_{\xi_i} - T_{\xi_i}\bar{x}$$

- Initialization.
  - Define a valid constraint on *Q(x)*

$$Q(\bar{x}) \le -E^0 x_{\xi_i} + e^0$$

Requires problem knowledge. For optimal consumption problem, assume extremely high rate of consumption forever

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Approach (cont.)

- Iteration k

$$Find \quad \gamma^k = \min_x \left(U^k(x) - V^k(x)\right), \; where$$

$$V^k(x) = \min_{0 \le j \le k} \left\{-E^j x + e^j\right\}, \; and$$

$$U^k(x) = \max \quad \sum_i p_{\xi_i}\left(c_{\xi_i} x_{\xi_i} + e^{-\delta t}\Theta_{\xi_i}\right)$$

$$s.t. \quad A x_{\xi_i} = b_{\xi_i} - T_{\xi_i}\bar{x}$$

$$E^j x_{\xi_i} + \Theta_{\xi_i} \le e^j, \quad j = 0, \ldots, k-1$$

If $\gamma^k > -\varepsilon$, terminate.

Else, define a new cut

$$E^k = \sum_i - p_{\xi_i}\mu_{\xi_i} T_{\xi_i}$$

$$e^k = \sum_i p_{\xi_i}\left(\mu_{\xi_i} b_{\xi_i} + \rho_{\xi_i} e\right)$$

Expensive search over x, possible for the optimal consumption problem because of small number of variables