Portfolios with Trading Constraints and Payout Restrictions

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General Problem

- (Very) long-term investor (example: university endowment)
- Payout from portfolio over time (want to keep payout from declining)
- Invest in various asset categories
- Decisions:
  - How much to payout (consume)?
  - How to invest in asset categories?
- Complication: restrictions on asset trades
Outline

• Basic formulation
• General infinite horizon solution method
• Simplified problem and continuous time solution
• Results for restricted-trading portfolio
• Future issues

Problem Formulation

• Notation:
  x – current state (x ∈ X)
  u (or u_x) – current action given x (u (or u_x) ∈ U(x))
  δ – single period discount factor
  P_{x,u} – probability measure on next period state y depending on x and u
  c(x,u) – objective value for current period given x and u
  V(x) – value function of optimal expected future rewards given current state x

• Problem: Find V such that
  V(x) = \max_{u \in U(x)} \{c(x,u) + \delta E_{P_{x,u}}[V(y)]\}
  for all x ∈ X.
Approach

• Define an upper bound on the value function
  \[ V^0(x) \geq V(x) \ \forall \ x \in X \]

• Iteration k: upper bound \( V^k \)
  
  Solve for some \( x^k \)
  
  \[ TV^k(x^k) = \max_u c(x^k,u) + \delta E_{P_i,a}[V^k(y)] \]

  Update to a better upper bound \( V^{k+1} \)

• Update uses an outer linear approximation on \( U^k \)

Successive Outer Approximation
Properties of Approximation

- $V^* \leq TV^k \leq V^{k+1} \leq V^k$
- Contraction
  \[ \|TV^k - V^*\|_\infty \leq \delta \|V^k - V^*\|_\infty \]
- Unique Fixed Point
  \[ TV^* = V^* \]
  \[ \Rightarrow \text{if } TV^k \geq V^k, \text{ then } V^k = V^*. \]

Convergence

- Value Iteration
  \[ T^k V^0 \rightarrow V^* \]
- Distributed Value Iteration
  If you choose every $x \in X$ infinitely often, then $V^k \rightarrow V^*$.
  (Here, random choice of $x$, use concavity.)
- Deepest Cut
  Pick $x^k$ to maximize $V^k(x) - TV^k(x)$
  DC problem to solve
  Convergence again with continuity (caution on boundary of domain of $V^*$)
Details for Random Choice

- Consider any $x$
- Choose $i$ and $x_i$ s.t. $||x_i - x|| < \epsilon_i$
- Suppose $|| \nabla V^i || \leq K \forall i$
  \[
  || V^k(x) - V^*(x) || \leq || V^k(x) - V^k(x^k) + V^k(x^k) - \nabla V^k(x) ||
  \leq 2 \epsilon_k K + \delta || V^{k-1}(x^k) - V^*(x^k) ||
  \leq 2 \sum_i \epsilon_i K + \delta^k || V^0(x^0) - V^*(x^0) ||
  \]

Cutting Plane Algorithm

**Initialization:** Construct $V^0(x) = \max_u c^0(x,u) + \delta \epsilon_{\text{Prd}}[V^0(y)]$, where $c^0 \geq c$ and $c^0$ concave.

$V^0$ is assumed piecewise linear and equivalent to

$V^0(x) = \max \{ \theta | \theta \leq E^0 x + e^0 \}$. $k=0$.

**Iteration:** Sample $x^k \in X$ (in any way such that the probability of $x^k \in A$ is positive for any $A \subset X$ of positive measure) and solve

$TV^k(x^k) = \max_u c(x^k,u) + \delta \epsilon_{\text{Prd}}[V^k(y)]$ where

$V^k(y) = \max \{ \theta | \theta \leq E^l y + e^l, l=0,\ldots,k \}$

Find supporting hyperplanes defined by $E^{k+1}$ and $e^{k+1}$ such that $E^{k+1} x + e^{k+1} \geq TV^k(x^k)$. $k \leftarrow k+1$.

Repeat.
Specifying Algorithm

Feasibility:
\[ Ax + Bu \leq b \]

Transition:
\[ y = F_i u \] for some realization \( i \) with probability \( p_i \)

Iteration k Problem:
\[
\begin{align*}
TV^k(x^k) &= \max_{u, \theta} c(x^k, u) + \delta \sum_i p_i \theta_i \\
& \text{s.t. } A x^k + B u \leq b, \quad -E(F^i u) - e^i + \theta^i \leq 0, \quad \forall i, l.
\end{align*}
\]

From duality:
\[
TV^k(x^k) = \inf_{\mu, \lambda} \max_{u, \theta} c(x^k, u) - \mu(A x^k + B u - b) \\
+ \delta \sum_i (p_i \theta_i + \sum l \lambda_{i,l}(E(F^i u) + e^i - \theta^i)) \\
\leq \max_{a, b} c(x^k, u^b) - \mu(A x^k + B u^b) + \delta \sum_i (p_i \theta_i + \sum l \lambda_{i,l}(E(F^i u) + e^i - \theta^i)) \text{ for optimal } \mu^k, \lambda_{i,l,k}^k \text{ for } x^k
\]

Cuts:
\[
E^{k+1} = \nabla c(x^k, u^b)^T - \mu^k A \\
c^{k+1} \text{ equal to the constant terms.}
\]

Investment Problem

- Determine asset allocation and consumption policy to maximize the expected discounted utility of spending
  - State and Action
    \[ x = (\text{cons, risky, wealth}) \quad u = (\text{cons\_new, risky\_new}) \]
  - Two asset classes
    - Risky asset, with lognormal return distribution
    - Riskfree asset, with given return \( r_f \)
  - Power utility function
    \[
c(\text{cons\_new}) = \frac{\text{cons\_new}^{1-\gamma}}{1-\gamma}
\]
  - Consumption rate constrained to be non-decreasing
    \[ \text{cons\_new} \geq \text{cons} \]
Existing Research

- Dybvig ‘95*
  - Continuous-time approach
  - Solution Analysis
    - Consumption rate remains constant until wealth reaches a new maximum
    - The risky asset allocation $\alpha$ is proportional to $w - c/r_f$, which is the excess of wealth over the perpetuity value of current consumption
    - $\alpha$ decreases as wealth decreases, approaching 0 as wealth approaches $c/r_f$ (which is in absence of risky investment sufficient to maintain consumption indefinitely).

- Dybvig ’01
  - Considered similar problem in which consumption rate can decrease but is penalized (soft constrained problem)


Objectives

- Replicate Dybvig continuous time results using discrete time approach
- Evaluate the effect of trading restrictions for certain asset classes (e.g., private equity)
- Consider additional problem features
  - Transaction Costs
  - Multiple risky assets
Results – Non-decreasing Consumption

As number of time periods per year increases, solution converges to continuous time solution

Results – Non-Decreasing Consumption with Transaction Costs
Observations

• Effect of Trading Restrictions
  • Continuously traded risky asset: 70% of portfolio for 4.2% payout rate
  • Quarterly traded risky asset: 32% of portfolio for same payout rate

• Transaction Cost Effect
  • Small differences in overall portfolio allocations
  • Optimal mix depends on initial conditions

Extensions

• Soft constraint on decreasing consumption
  • Allow some decreases with some penalty

• Lag on sales
  • Waiting period on sale of risky assets (e.g., 60-day period)

• Multiple assets
  • Allocation bounds
Conclusions

• Can formulate infinite-horizon investment problem in stochastic programming framework
• Solution with cutting plane method
• Convergence with some conditions
• Results for trade-restricted assets significantly different from market assets with same risk characteristics

Approach

• Application of typical stochastic programming approach complicated by infinite horizon

\[ Q(\bar{x}) = \max \sum_{i} p_{\bar{e}_i} \left( c_{\bar{e}_i} x_{\bar{e}_i} + e^{-\delta Q(x_{\bar{e}_i})} \right) \]

\[ s.t. \quad Ax_{\bar{e}_i} = b_{\bar{e}_i} - T_{\bar{e}_i} \bar{x} \]

• Initialization.
  • Define a valid constraint on \( Q(x) \)

\[ Q(\bar{x}) \leq -E^0 x_{\bar{e}_i} + e^0 \]

Requires problem knowledge. For optimal consumption problem, assume extremely high rate of consumption forever
Approach (cont.)

• Iteration k

Find \( \gamma^k = \min_x \left( U^k(x) - V^k(x) \right) \), where

\[
V^k(x) = \min_{0 \leq j \leq k} \left\{ -E^j x + e^j \right\}, \text{ and }
\]

\[
U^k(x) = \max \sum_i p_{zi} \left( c_{zi} x_{zi} + e^{-k} \Theta_{zi} \right)
\]

s.t. \( Ax_{zi} = b_{zi} - T_{zi} \bar{x} \)

\[
E^j x_{zi} + \Theta_{zi} \leq e^j, \quad j = 0, \ldots, k - 1
\]

If \( \gamma^k > -\varepsilon \), terminate.

Else, define a new cut

\[
E^k = \sum_i -p_{zi} \mu_{zi} T_{zi}
\]

\[
e^k = \sum_i p_{zi} \left( \mu_{zi} b_{zi} + \rho_{zi} e \right)
\]