# Comparing Process Flexibility with Inventory Flexibility via Product Substitution

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**Abstract**

This paper studies the optimal control of, interaction between, and relative value of two types of flexibility under Markov models of demand and production: *process flexibility* and *inventory flexibility*. In our model, process flexibility is generated by a multi-functional production facility that can produce two types of products, and inventory flexibility is manifested in firm-driven one-way product substitution. Both process flexibility and inventory flexibility are important drivers of supply chain performance and are strategic design considerations. To compare the value generated by these two types of flexibility, we model a dynamically controlled two-product, make-to-stock system with stochastic processing times and stochastic demand. We characterize the complex joint optimal production and post-production policy for a special case and numerically show that a simply structured multi-threshold policy is a near-optimal heuristic policy for the general case. We gain further insight into the impact of system parameters on the value of process flexibility and inventory flexibility via a comprehensive numerical study. We find that: (*i*) the model exhibits much greater benefit from process flexibility than inventory flexibility for nearly all of the cases, and (*ii*) for a wide range of capacity and cost parameters, process flexibility and inventory flexibility complement each other, so pursuing both forms of flexibility is effective.

**Keywords**: Production and inventory systems, product substitution, inventory flexibility, process flexibility, Markov Decision Process, multi-threshold heuristic algorithm.

# 1 Introduction

Flexibility is a general concept that is often viewed as a firm's ability to match production to demand in the face of uncertainty and variability. Incorporating flexibility in a firm's supply chain has received considerable attention due to the uncertainty stemming from the characteristics of the contemporary markets such as product proliferation, increased customization, short product life cycles, changes in technology, and most importantly demand uncertainty. *Process flexibility*, whereby a multi-functional production facility can produce multiple products (and switch between them without cost or lost time), provides an effective way of coping with uncertainty. In terms of when the flexibility is utilized, process flexibility is a form of *production flexibility* as it is used during the production phases of a product. Thus, it is limited by the fact that this decision must be made in anticipation of future demand. Examples of flexibility that seek to mitigate the ill effects of anticipating demand include component commonality and delayed differentiation (postponement).

We use the term post-production flexibility to refer to a mechanism applied after production has completed. We highlight a form of post-production flexibility known as product *substitution*, which refers to the use of one product (component) to satisfy the demand for a different product (component). In this paper, we model product substitution as a decision made by the firm to steer demand based on current inventory levels at the moment a demand arrives.

In our model the firm, not the customer, makes the decision to substitute. For cases in which one product is superior to the other, the firm has the option to meet a demand for the lower-quality product by making a dynamic upgrade offer to customer to substitute an in-stock item of the higher-quality product. This type of substitution, where only one of the products substitutes for the other is known as "one-way substitution" or "downward substitution". One-way substitution occurs in many settings such as semiconductor chips where a faster processor can be substituted for a slower processor (Hsu and Bassok, 1999) and usage of cadillac boxes as a method of customization at IBM (Rao et al., 2004). Product substitution is a source of flexibility because it makes the stocks of some products "multi-functional", in the sense that inventory of the higher-quality product can be used to meet either its own demand or for the demand of lower-quality products. This paper refers to the flexibility provided through product substitution as *inventory flexibility.* In terms of when the flexibility is utilized, inventory flexibility is a form of *post-production flexibility.* Other examples of post-production flexibility are customer-driven product substitution and dynamic pricing.

Surveying the salient process flexibility literature, we find that Fine and Freund (1990), Gupta et al. (1992), Van Mieghem (1998), and Bish and Wang (2004) examine investments in dedicated production facilities versus flexible production facilities. Chod and Rudi (2005) analyze the effect of demand variance and correlation on flexible capacity and expected profits. They quantify the value of process flexibility by comparing a firm with flexible facility to a firm with dedicated facilities when the investment costs are the same. Goyal and Netessine (2007) study the impact of competition and demand uncertainty on capacity decisions and adoption of process flexibility. Jordan and Graves (1995) look at partial process flexibility, where a production facility can produce a subset of products considered, so that the structure of process flexibility takes the form of a chain. They show that, in a single-stage supply chain with multiple products and plants, adding partial process flexibility in the right way can achieve nearly all the benefits of total process flexibility. Graves and Tomlin (2003) extend this work to multi-stage supply chains. All the papers mentioned so far focus on settings with stochastic demands and deterministic production processes. Iravani et al. (2005) analyze the effect of the partial process flexibility structure on the performance in single-stage supply chains. They propose an index that effectively determines the performance ranking of different structures.

In the literature on substitution, Bassok et al. (1999), Smith and Agrawal (2000), and Rao et al. (2004) focus on single-period, multiproduct inventory models with stochastic demand and one-way substitution. Smith and Agrawal (2000) provides a review of literature on the product substitution in various contexts such as retailing, yield management, and resource allocation. Bassok et al. (1999) study a single-period, multiproduct inventory model with one-way substitution and zero production lead time. Their model is designed as a two-stage decision problem with an ordering stage and an allocation stage. They suggest a greedy algorithm that solves the allocation stage optimally and prove the optimality of a base-stock policy. Hsu et al. (2005) consider a finite-horizon, multiproduct inventory problem with deterministic demand and one-way substitution. They provide a dynamic programming algorithm and a heuristic to solve the problem. Hsu and Bassok (1999) consider a single-period, multiproduct production/inventory system with random production yield and one-way substitution. They present methods to solve the model for the optimal allocation and the optimal production input quantities. Rao et al. (2004) analyze a single-period, multiproduct production/inventory system with one-way substitution and setup costs. In both Hsu and Bassok (1999) and Rao et al. (2004), it is assumed that production capacity is infinite and products always become available before demand is realized.

In this paper, we compare flexibility types utilized at different stages of the supply chain by focusing on a commonly practiced form of production flexibility as well as post-production flexibility. For this purpose, we analyze a dynamically controlled model of a two-product, make-to-stock system utilizing both process flexibility and inventory flexibility to satisfy stochastic demands. Process flexibility is incorporated via a multi-functional facility capable of producing both of the products with limited capacity (and with stochastic processing times). Inventory flexibility is manifested in the form of one-way substitution, which allows the firm to meet demand for lower-quality product using stock of higher-quality product. The model we treat could also arise in a manufacturing environment involving assembly of end products from substitutable components produced within the firm. Our managerial insights and analysis provide several important contributions to the literature. After partially characterizing the structural properties of the joint optimal production and substitution policy, we present a much simpler multi-threshold policy that can be used as an alternative to the optimal policy. Through a very extensive test suite with more than 75,000 cases, we verify that the multi-threshold heuristic we propose performs very well for a wide range of model parameters. We also present managerial insights in the form of observations related to the interaction between, and comparison of flexibility types utilized at different stages of the supply chain by focusing on the process and the inventory flexibilities.

To summarize, some of the interesting questions we answer with our managerial insights and analysis are as follows: (1) What are the structural properties of the joint optimal production and substitution policy, and is there a much simpler and cost effective alternative to the optimal policy? (2) What is the impact of the system parameters such as production capacities, demand rates, and different costs on the value of inventory flexibility or process flexibility? (3) Is the value of inventory (process) flexibility higher for systems with or without process (inventory) flexibility? (4) Under what circumstances does process flexibility have more value than inventory flexibility, or vice versa?

In Section 2, we present the details of the model and construct the corresponding Markov Decision Process (MDP) formulation. Section 3 explores the structure of an optimal policy, Section 4 characterizes an important special case, and Section 5 uses extensive numerical analysis to gain interesting and practical insights into the above issues.

# 2 Model Formulation

The firm produces products of type $i \in \{1, 2\}$ in its multi-functional production facility at an average rate $\mu_i$. Without loss of generality, we let product 1 and product 2 denote higher-quality and lower-quality products, respectively. The firm must decide which type of product to make (or whether to stay idle) at any instant of time in anticipation of future demand. This is the proactive form of flexibility in our model. The firm keeps the finished products in inventory in order to satisfy demands quickly and to manage limited capacity and production/demand variability (i.e., it is a make-to-stock system). We assume that the products have independent demand arrival processes with arrival rate $\lambda_i$ for type $i$.

The firm is allowed to use one-way substitution to meet demand for product 2 using stock of product 1, the higher quality product of two similar offerings. This is the reactive form of flexibility in our model in the sense of being applied after observing demand. In one-way substitution, product 1 may be substituted for product 2 in all cases (and the consumer pays the lower price for product 2), but the market will not accept product 2 as a substitute for product 1. The inventory flexibility is augmented by the fact that our model allows substitution even when product 2 is not out of stock. In addition, we allow more inventory flexibility by letting the firm decide either to substitute or not to substitute an order for product 2 with product 1. These substitution options above may be attractive in some states due to the stochastic processing times, stochastic demands, and cost differentials of the products.

The firm incurs a substitution cost of $C \geq 0$ per substitution to account for the net lost profit which may include any necessary conversion costs. The firm incurs an inventory holding/carrying cost of $h_i$, $i = 1, 2$, per item per unit time for product $i$ kept in inventory.

Whenever a unit demand for product 1 finds the stock empty, a lost sales cost of $P_1$ is incurred to account for lost profit as well as loss of goodwill. An alternate interpretation of the lost sales cost is that it is a proxy for the firm's cost for expediting (e.g., overtime, subcontracting) to supplement its regular production at a net cost of $P_i$ per product $i = 1, 2$. In the face of a demand for product 2, the firm makes the choice of whether or not to substitute a product 1 (provided it is available). Observe that under stockout of product 2, a demand for product 2 will result in an immediate penalty of $P_2$ if product 1 is not substituted and $C$ if it is.

We model the demand for product $i$ as a Poisson process with rate $\lambda_i$ and production times for product $i$ follow an exponential distribution with rate $\mu_i$. This allows us to utilize the MDP model to study the structure of the joint optimal production and substitution policy. Let $n_i(t) \in \mathbf{Z}^+$, where $\mathbf{Z}^+$ is the set of non-negative integers $\{0, 1, 2, \ldots\}$, represent the inventory level of product $i$ at time $t$. Then the state of the inventory process at time $t$ is given by the row vector $\mathbf{n}(t) = (n_1(t), n_2(t))$ in the state space $\mathcal{S} = \mathbf{Z}^+ \times \mathbf{Z}^+$.

A control policy $\pi$ states the action taken at any time given the current state of the system $\mathbf{n}(t)$. At any time, the control action set for state $\mathbf{n}(t)$ is a union of allowable production and substitution actions. For any state $\mathbf{n}(t)$, allowable production actions are idling, producing item 1, and producing item 2. Upon arrival of a customer requesting product 2, allowable upgrading actions are: $(i)$ offer an upgrade (if item 1 is available), and $(ii)$ do not offer an upgrade.

Let $\alpha \in (0, 1)$ denote the discount rate. Also, let $h(\mathbf{n}(t)) = \sum_{i=1}^{2} h_i n_i(t)$ be the inventory holding cost rate function, where $h_i > 0$ is the unit holding cost rate of product $i$. We assume that $h_1 \geq h_2$ due to the increased quality and value of product 1. Let $l_i(t)$ be the accumulated lost sales of product $i$ up to time $t$, and $s(t)$ be the accumulated number of substitutions up to time $t$. We require $P_1 > P_2$ for consistency with our modeling of asymmetric settings in which one item has strictly better quality and thus one-way substitution. We seek to find the optimal control policy $\pi^*$ that minimizes the firm's total *discounted* cost over an infinite horizon given as follows:

$$\inf_{\pi} E^{\pi}_{n_0} \left[ \int_0^{\infty} e^{-\alpha t} h(\mathbf{n}(t)) dt + C \int_0^{\infty} e^{-\alpha t} ds(t) + \sum_{i=1}^{2} P_i \int_0^{\infty} e^{-\alpha t} dl_i(t) \right], \tag{1}$$

where $E^{\pi}_{n_0}$ denotes the expectation over demand, for a given policy $\pi$ and initial inventory levels $n_0 = (n_1(0), n_2(0))$. We are also interested in the firm's total *average* cost given as:

$$\inf_{\pi} \lim_{T\to\infty} \frac{1}{T}\left( E^{\pi}_{n_0}\left[ \int_0^T h(\mathbf{n}(t))dt + C\int_0^T ds(t) + \sum_{i=1}^{2} P_i \int_0^T dl_i(t)\right]\right). \tag{2}$$

Without loss of generality, we redefine the time scale such that $\nu + \alpha = 1$, where $\nu = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$ is the uniformization rate. Let the unit row vectors be defined as $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Also, let $\mathbf{I}_{(R)}$ be the indicator function of event $R$. By the uniformization device of Lippman (1975), the optimality equation of the MDP for (1) can be compactly written as:

$$V_\alpha(\mathbf{n}) = h(\mathbf{n}) + \lambda_1\left[V_\alpha(\mathbf{n} - \mathbf{e}_1\mathbf{I}_{(n_1>0)}) + P_1\mathbf{I}_{(n_1=0)}\right] + \lambda_2\mathbf{T}_s V_\alpha(\mathbf{n}) + \mathbf{T}_p V_\alpha(\mathbf{n}) \tag{3}$$

where the operators $\mathbf{T}_s$ and $\mathbf{T}_p$ are defined as:

$$\mathbf{T}_s V_\alpha(\mathbf{n}) = \min \begin{cases} V_\alpha(\mathbf{n} - \mathbf{e}_2\mathbf{I}_{(n_2>0)}) + P_2\mathbf{I}_{(n_2=0)} & : \text{ No upgrading; or if } n_1 = 0, \\ V_\alpha(\mathbf{n} - \mathbf{e}_1) + C & : \text{ Upgrading.} \end{cases} \tag{4}$$

and

$$\mathbf{T}_p V_\alpha(\mathbf{n}) = \min \begin{cases} (\mu_1 + \mu_2)V_\alpha(\mathbf{n}) & : \text{ Idling,} \\ \mu_1 V_\alpha(\mathbf{n} + \mathbf{e}_1) + \mu_2 V_\alpha(\mathbf{n}) & : \text{ Produce item 1,} \\ \mu_1 V_\alpha(\mathbf{n}) + \mu_2 V_\alpha(\mathbf{n} + \mathbf{e}_2) & : \text{ Produce item 2.} \end{cases} \tag{5}$$

In (3), $V_\alpha(\mathbf{n})$ is the optimal total discounted cost over an infinite horizon when the initial inventory levels are $\mathbf{n} = (n_1, n_2)$. The first term in (3) is the holding cost incurred at state $\mathbf{n}$. The terms with $\lambda_1$ (a product 1 demand arrival) either denote the lost sales cost incurred in the case of a stockout of product 1 or denote the new state of the system when a demand for the product 1 is met from inventory. The third term in (3), $\mathbf{T}_s V_\alpha(\mathbf{n})$, corresponds to the choice between upgrading or not upgrading. Provided that there is a product 1 in the inventory, the firm might choose to upgrade a demand for the lower quality item, product 2, and satisfy it with product 1 by incurring a substitution cost of $C$. If the firm chooses not to upgrade the demand for product 2, it either meets the demand from the inventory or incurs a lost sales cost in case of a stockout for product 2. The last term in (3), $\mathbf{T}_p V_\alpha(\mathbf{n})$, denotes the choice between idling, producing a product 1, and producing a product 2.

As it is never optimal to keep infinitely many items in inventory, we can set an upper bound on $\mathbf{n}$. As a result, the long-run average cost problem can be obtained as the limit of discounted cost problem as the discounting rate $\alpha$ goes to zero (see Weber and Stidham

(1987) and Véricourt et al. (2002)). The structural properties of the discounted cost case are retained for the average cost case. In the average cost case, the optimality equation of the MDP is:

$$V(\mathbf{n}) + g = h(\mathbf{n}) + \lambda_1 \left[ V(\mathbf{n} - \mathbf{e}_1 \mathbf{I}_{(n_1>0)}) + P_1 \mathbf{I}_{(n_1=0)} \right] + \lambda_2 \mathbf{T}_s V(\mathbf{n}) + \mathbf{T}_p V(\mathbf{n}) \qquad (6)$$

where $g$ is the optimal cost per unit time, $V$ is the optimal relative value function, and the operators $\mathbf{T}_s$ and $\mathbf{T}_p$ are as defined before.

Having presented the model in detail, we can now describe accurately some related research. While the heuristics and insights we obtain are dramatically different than those of Hu et al. (2008) and Zhao et al. (2008), the MDP model we formulated independently of them does share some similar features, even though they model transshipment.

Hu et al. (2008) address the optimal control of inventory and transhipment for a firm that produces a *single* product in two locations and faces capacity uncertainty in those production facilities. They employ a periodic setting with lost sales, focus on the structure of the optimal policy, and provide results of the sensitivity of the optimal policy with respect to model parameters. In their periodic review setting, at the beginning of each period, the firm determines the desired production quantity at each location. After the production and demand uncertainties are revealed, firm decides how much inventory to transship from one location to the other and unsatisfied demand is lost. Their model allows for heterogenous parameters at each location, so their setting can be considered as a model with two products (as in our paper), each product corresponding to a location. In that case, transshipment would be a "two-way substitution" and would generate *inventory flexibility*. However, in that case, as each production facility is dedicated to a product as opposed to flexible facilities that could produce both products, their model would not have *process flexibility* generated via multi-functional facilities. Furthermore, our model is designed as a dynamically controlled queueing system in continuous time, as opposed to a periodic control model.

Zhao et al. (2008) also focus on characterizing the optimal control of inventory and transhipment for a firm that produces a *single* product in two locations, but they model each location as a single-server make-to-stock queueing system, where a customer demand is backordered if it is not immediately satisfied. They propose three heuristics for the problem.

In their setting, each facility can either produce for its default location or produce and transship to the other location with a transshipment cost. The firm also has the option to transship on hand inventory to meet demand after incurring the same transshipment cost. Even though the models are similar, the differences between them are very significant. Our model focuses on a single facility model with two products and lost sales, whereas Zhao et al. (2008) utilizes a two facility model with one product and backordering. Similar to Hu et al. (2008), Zhao et al. (2008) can also be interpreted as a two-product model by considering each location as a product. But in that case, it can be observed that the process flexibility mechanics of their model is not well-suited for a two-product setting with substitution. Each facility produces one of the products without any additional cost, but production of the other product incurs a transshipment cost. The same transshipment cost is also used as the inventory substitution cost. In Section 6 as a part of our numerical study, in addition to our single-facility model, we also use a two-facility model. That model is closer to, but still different than the one used in Zhao et al. (2008) due to the cost structure and the backordering assumption in that paper.

Both Hu et al. (2008) and Zhao et al. (2008) utilize inventory flexibility generated by "two-way substitution." Even though two-way product substitution is suitable for transshipment models with a single product, its applicability is questionable for a two-product setting with firm-driven product substitution, which requires the customer to accept the offer. One-way substitution successfully represents the phenomenon where a higher-quality product is offered as an alternative to the lower-quality product. In this sense, a model of two-way substitution would only be realistic in rare cases where the actual or perceived quality/price difference of the products is negligible. One-way substitution is much more prevalent as a practical business practice.

# 3 Complexity of the Structure of the Joint Optimal Policy

In this section, we study the behavior of the joint optimal production and substitution policy. We first show the non-monotonic behavior of the optimal policy, and then illustrate the impact of the optimal substitution policy on production decisions.

### 3.1 Complex Non-monotonic Behavior of the Joint Optimal Control Policy

In order to understand the structural properties of the joint optimal production and substitution policy, we performed an extensive numerical study. It revealed that the joint optimal policy is state-dependent with a very complex structure. Furthermore, the structure of the production policy, and to a lesser extent, substitution policy, is sensitive to the model parameters. We illustrate this with the examples in Figures 1 to 3, which show the optimal production and substitution policies for a system based on the parameters $\lambda_1 = 4$, $\lambda_2 = 5$, $h_1 = 1.5$, $h_2 = 1$, $P_1 = 15$, $P_2 = 10$, and with different production rate combinations.

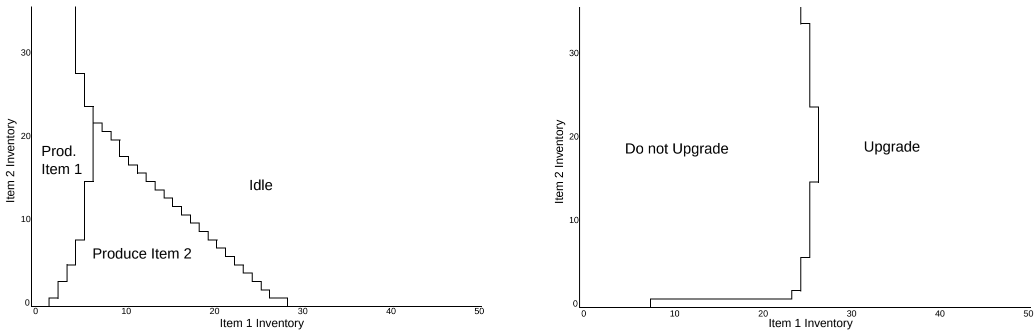


Figure 1: The optimal policy for the case with $\lambda_1 = 4$, $\lambda_2 = 5$, $\mu_1 = 6$, $\mu_2 = 7$, $h_1 = 1.5$, $h_2 = 1$, $P_1 = 15$, $P_2 = 10$, and $C = 2$. *Left:* Production policy; *Right:* Upgrading policy.

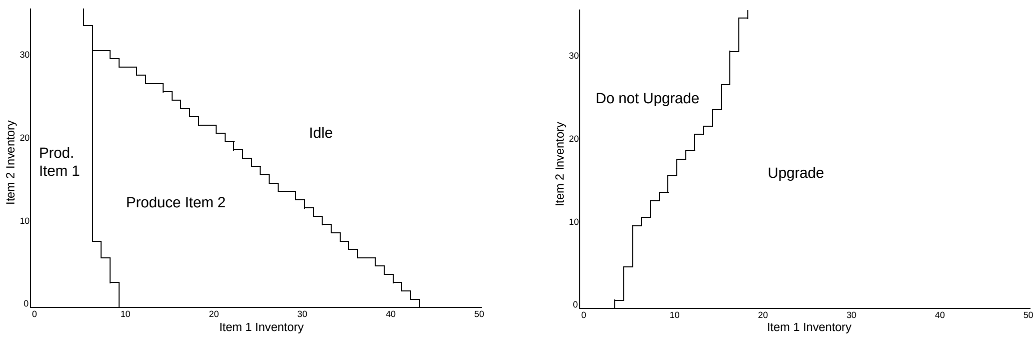


Figure 2: The optimal policy for the example in Figure 1, when $\mu_2$ is reduced from 7 to 4. *Left:* Production policy; *Right:* Upgrading policy.

Figure 2 shows the optimal production and substitution policy when the production rate of product 2, $\mu_2$, in the example in Figure 1 is reduced from 7 to 4. In this case, low production capacity for product 2 (i.e., $\lambda_2 = 5 > \mu_2 = 4$) and ample production capacity for product 1 ($\lambda_1 = 4 < \mu_1 = 6$) substantially increases the use of upgrading (i.e., Figure 2.*Right* has more states with upgrading than Figure 1.*Right*). Furthermore, it is

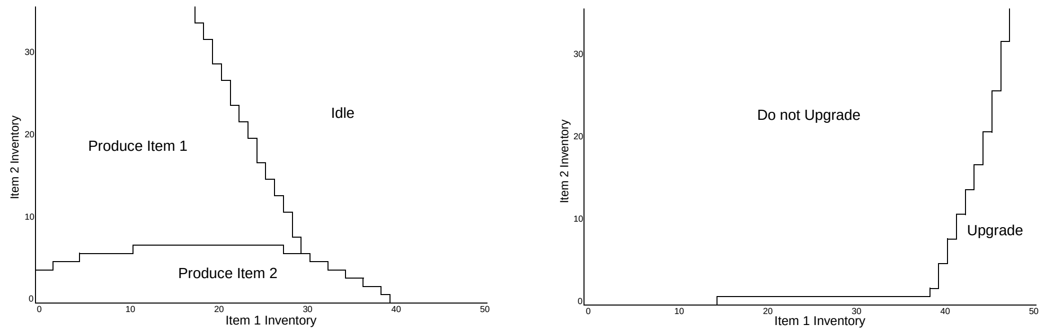


Figure 3: The optimal policy for the example in Figure 1, when $\mu_1$ is reduced from 6 to 3. *Left:* Production policy; *Right:* Upgrading policy.

more attractive to produce product 1 when inventory of both products are low, due to the inadequate capacity for product 2 combined with excess capacity for product 1. That is to say, production of product 1 combined with upgrading is a better strategy than producing product 2 (which still incurs lost sale penalty costs, $P_2$, due to inadequate capacity). Finally, the reduction in the production rate for product 2, $\mu_2$, shifts the threshold separating product 2 production and idling towards higher product 1 inventory values (i.e., build higher inventory levels). In summary, Figure 2 illustrates how a capacity shortage for product 2 allows the manufacturer to use upgrading as a device to convert product 1 capacity to product 2 capacity.

In Figure 3, the example in Figure 1 is changed in only one respect: the production rate of product 1, $\mu_1$, is reduced from 6 to 3. In this case, low production capacity for product 1 ($\lambda_1 = 4 > \mu_1 = 3$) and high production capacity for product 2 ($\lambda_2 = 5 < \mu_2 = 7$) substantially decreases the use of upgrading. Hence, Figure 3.*Right* has less states with upgrading than Figure 1.*Right*. Even if there is a product 2 stockout, upgrading is used sparingly – requiring at least 14 items of product 1 in inventory for upgrading to become beneficial. The reduction in the production capacity of product 1 forces the multi-functional facility to adjust its production policy. When both items are near stockout, the firm no longer prefers to produce product 1, because production times of product 1 are too long. Indeed, a loss rate of 1 product 1 job and 5 product 2 jobs per unit time is ensured if the server were devoted exclusively to product 1 production.

The nonmonotonic behavior of the optimal production and substitution policy, as shown in the above figures, adds to the complexity of the structure of the optimal policy and further emphasizes on the need for simple and cost-effective heuristic policies.

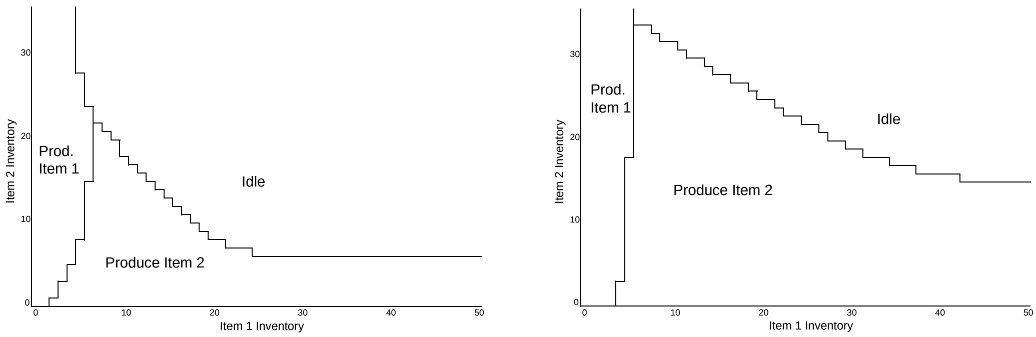


Figure 4: The optimal production policy when the upgrading option is removed. *Left:* For the case in Fig. 1; *Right:* For the case in Fig. 2.

## 3.2 The Impact of Substitution on an Optimal Production Policy

It is helpful to illustrate the role of product substitution, which can be identified simply by removing the upgrading option from the model. The resulting contrast between the optimal production policy with product substitution and that without can be seen in Figure 1.*Left* and Figure 4.*Left*, respectively. The main qualitative difference is the modification of the production decisions when the inventory of product 1 (product 2) is high (is low). Without substitution, when product 2 inventory is zero, the system will always produce either product 1 or product 2. With substitution, it is optimal not to produce product 2 at all when product 1 inventory is high enough, even when product 2 inventory is empty. The reason can be observed from Figure 1.*Right*. For each product 2 inventory level, there is a product 1 inventory level above which we use product 1 to satisfy product 2 demands. Therefore, even if product 2 inventory is extremely low, or even zero, if product 1 inventory is sufficiently high we will upgrade product 2 demands and do not need to produce product 2. In this case, Figure 1.*Right* shows that if product 2 inventory is empty, it is optimal to upgrade when there are at least 8 of type 1 in inventory; however, we require at least 24 product 1 in inventory to start upgrading if there is a product 2 in inventory.

As mentioned before, due to the capacity shortage for product 2, upgrading is more important for the case given in Figure 2 than the case given in Figure 1. Therefore, the change in the production policy in Figure 2.*Left* is more significant when the upgrading option is removed, as seen in Figure 4.*Right*.

Above all, our intention in this section is to convey how the number of parameters and the complexity of the joint production and substitution policy (e.g., non-monotonic behavior) makes the characterization of the optimal policy very difficult in general. Nevertheless, for cases in which production rates are equal (i.e., $\mu_1 = \mu_2$), we are able to characterize the structure of the optimal production and substitution policy in the next section.

# 4 Characterization of an Optimal Policy for Equal Production Rates

We provide a detailed analysis of the joint optimal production and substitution policy when the production rates of the two products are equal, i.e., $\mu_1 = \mu_2 = \mu$. Taking computer assembly as an example, the time it takes to assemble the hard disks or CPU's in different desktop models may be exactly the same, regardless of the hard disk storage capacity or CPU speed. Modular design of components is a strategic decision made to increase flexibility, and therefore the case of homogeneous service times more likely to occur in practice than one might otherwise suspect.

For the equal production rate case, we define the optimality equation of the MDP given in (3) using the operator $\mathbf{T}_s$ as given in (4) and defining the operators $\mathbf{T}$ and $\mathbf{T}_p$ as follows:

$$V_\alpha(\mathbf{n}) = \mathbf{T}\,V_\alpha(\mathbf{n}) = h(\mathbf{n}) + \lambda_1\big[V_\alpha(\mathbf{n} - \mathbf{e}_1\mathbf{I}_{(n_1>0)}) + P_1\mathbf{I}_{(n_1=0)}\big] + \lambda_2\mathbf{T}_s\,V_\alpha(\mathbf{n}) + \mu\mathbf{T}_p\,V_\alpha(\mathbf{n}) \quad (7)$$

$$\mathbf{T}_p\,V_\alpha(\mathbf{n}) = \min\Big\{V_\alpha(\mathbf{n}), V_\alpha(\mathbf{n}+\mathbf{e}_1), V_\alpha(\mathbf{n}+\mathbf{e}_2)\Big\}. \quad (8)$$

For our analysis, it is convenient to define the first difference operator, $\mathbf{D}_i$, as well as the second difference operator, $\mathbf{D}_{ij}$, on real-valued functions $v$ defined over the state space $\mathcal{S}$ for $i, j \in \{1, 2\}$ as:

$$\mathbf{D}_i v(\mathbf{n}) = v(\mathbf{n}+\mathbf{e}_i) - v(\mathbf{n})$$
$$\mathbf{D}_{ij} v(\mathbf{n}) = \mathbf{D}_i(\mathbf{D}_j v(\mathbf{n})).$$

The following Proposition is key to our main result regarding the optimal policy. The proofs of Proposition 1 and Theorem 1 can be found in online Appendix A.

**Proposition 1** *Let $\mathcal{U}$ be the set of functions on the state space $\mathcal{S}$ which satisfy the properties* **P1** *to* **P4** *given below. If a function $v \in \mathcal{U}$, then $\mathbf{T}_s v \in \mathcal{U}$, $\mathbf{T}_p v \in \mathcal{U}$, and $\mathbf{T}v \in \mathcal{U}$.*

- **P1** *Supermodularity:* $\mathbf{D}_{12}v(\mathbf{n}) \geq 0$
- **P2** *Diagonal dominance:* $\mathbf{D}_{11}v(\mathbf{n}) \geq \mathbf{D}_{12}v(\mathbf{n})$, $\mathbf{D}_{22}v(\mathbf{n}) \geq \mathbf{D}_{12}v(\mathbf{n})$
- **P3** *Convexity:* $\mathbf{D}_{11}v(\mathbf{n}) \geq 0$, $\mathbf{D}_{22}v(\mathbf{n}) \geq 0$
- **P4** *Lower bound:* $\mathbf{D}_1 v(\mathbf{n}) \geq -P_1$, $\mathbf{D}_2 v(\mathbf{n}) \geq -P_2$.

*Furthermore, the optimal cost function* $V_\alpha \in \mathcal{U}$*.*

Theorem 1 presents our main results regarding the structure of the optimal production and substitution policy when the production rates are equal.

**Theorem 1** *The joint optimal production and substitution policy has the following properties:*

*(i) If idling is optimal at state* $\mathbf{n} \in \mathcal{S}$*, then idling is also optimal at states* $\mathbf{n} + \mathbf{e}_1$ *and* $\mathbf{n} + \mathbf{e}_2$*.*

*(ii) If producing the higher quality product (product 1) is optimal at state* $\mathbf{n} \in \mathcal{S}$ *with* $n_1 \geq 1$*, then producing product 1 is also optimal at states* $\mathbf{n} - \mathbf{e}_1$ *and* $\mathbf{n} - \mathbf{e}_1 + \mathbf{e}_2$*.*

*(iii) If producing the lower quality product (product 2) is optimal at state* $\mathbf{n} \in \mathcal{S}$ *with* $n_2 \geq 1$*, then producing product 2 is also optimal at states* $\mathbf{n} - \mathbf{e}_2$ *and* $\mathbf{n} - \mathbf{e}_2 + \mathbf{e}_1$*.*

*(iv) If upgrading the demand for product 2 is optimal at state* $\mathbf{n} \in \mathcal{S}$*, then it is also optimal to upgrade at states* $\mathbf{n} + \mathbf{e}_1$*, and if* $n_2 \geq 1$*, at state* $\mathbf{n} - \mathbf{e}_2$*.*

Some of the properties listed in Theorem 1 for systems with equal production rates do not hold for the general case when $\mu_1 \neq \mu_2$. For example, Figure 3.*Left* shows that, if producing product 1 is optimal at state $\mathbf{n} \in \mathcal{S}$ with $n_1 \geq 1$, then producing product 1 is not necessarily optimal at state $\mathbf{n} - \mathbf{e}_1$ when $\mu_1 \neq \mu_2$. Similarly, Figure 2.*Left* shows that, if producing product 2 is optimal at state $\mathbf{n} \in \mathcal{S}$ with $n_2 \geq 1$, then producing product 2 is not necessarily optimal at state $\mathbf{n} - \mathbf{e}_2$ when $\mu_1 \neq \mu_2$. Lastly, Figure 1.*Right* shows that, if upgrading is optimal at state $\mathbf{n} \in \mathcal{S}$ with $n_2 \geq 1$, it is not necessarily optimal to upgrade at state $\mathbf{n} - \mathbf{e}_2$ when $\mu_1 \neq \mu_2$.

As shown in Figures 1, 2, and 3, and supported by the examples provided above, the structure of the joint optimal production and substitution policy is complex. This motivates the need for cost efficient and easy-to-implement policies. In the next section, we present such a policy and show that our multi-threshold policy, while considerably simpler than the optimal, has a cost close to that of the optimal policy.

# 5 Multi-threshold Policy

We see that the joint optimal production and substitution policy is not easy to implement in practice. This motivates the need for a near optimal heuristic policy with a simple structure that can be used instead to approximate the joint optimal production and substitution policy. We construct a multi-threshold policy to capture some features of the optimal production and substitution policy combinations presented in Figures 1 to 3 based upon the insights and knowledge obtained from our extensive numerical study and Theorem 1. As illustrated in Figures 5 and 6, for different parameter combinations, the proposed policy consists of two different policies with a total of five non-negative parameters which are interdependent:

- *Production policy*: Three parameters of the multi-threshold policy, $A_1, A_2$, and $A_3$ (with $A_1 \geq A_2$), constitute the production side of the multi-threshold policy.
- *Substitution policy*: Two parameters, $B_1$ and $B_2$, determine the substitution decisions of the multi-threshold policy.

Let $\mathcal{S}_i$ denote the set of states at which the firm produces product $i = 1, 2$, and let $\mathcal{S}_u$ denote the set of states at which it upgrades the demand. Then the proposed multi-threshold policy can be expressed as follows:

$$\begin{aligned}
\text{Produce product 1:} \quad & \text{If } \mathbf{n} \in \mathcal{S}_1 = \{\mathbf{n} : n_1 < A_2 \text{ or } A_2 + A_3 < n_1 < A_1,\ A_3 < n_2\} \\
& (\text{or equivalently } \mathcal{S}_1 = \{\mathbf{n} : n_1 < A_1,\ \mathbf{n} \notin \mathcal{S}_2\}) \\
\text{Produce product 2:} \quad & \text{If } \mathbf{n} \in \mathcal{S}_2 = \{\mathbf{n} : A_2 \leq n_1 \leq A_2 + A_3,\ n_2 \leq A_3\} \\
\text{Idle:} \quad & \text{If not producing product 1 or product 2, i.e., } \mathbf{n} \notin \mathcal{S}_1 \text{ and } \mathbf{n} \notin \mathcal{S}_2 \\
\text{Upgrade:} \quad & \text{If } \mathbf{n} \in \mathcal{S}_u = \{\mathbf{n} : B_1 < n_1,\ n_2 \leq B_2\}
\end{aligned}$$

Figure 6.*Right* shows the general structure of the substitution policy for the proposed multi-threshold policy. Figures 5 and 6.*Left* show the production decisions of the multi-threshold policy for different combinations of parameters $A_1$, $A_2$, and $A_3$. In Figure 5.*Left*, $A_1 > A_2 + A_3$, in Figure 5.*Right*, $A_2 + A_3 \geq A_1 > A_2$, and finally in Figure 6.*Left*, $A_2 + A_3 \geq A_1 = A_2$ holds.

The multi-threshold policy has several cases formed by different combinations of production (substitution) policy related parameters $A_1$, $A_2$, and $A_3$ ($B_1$ and $B_2$). In the next section, we analyze a representative case of the proposed multi-threshold policy.

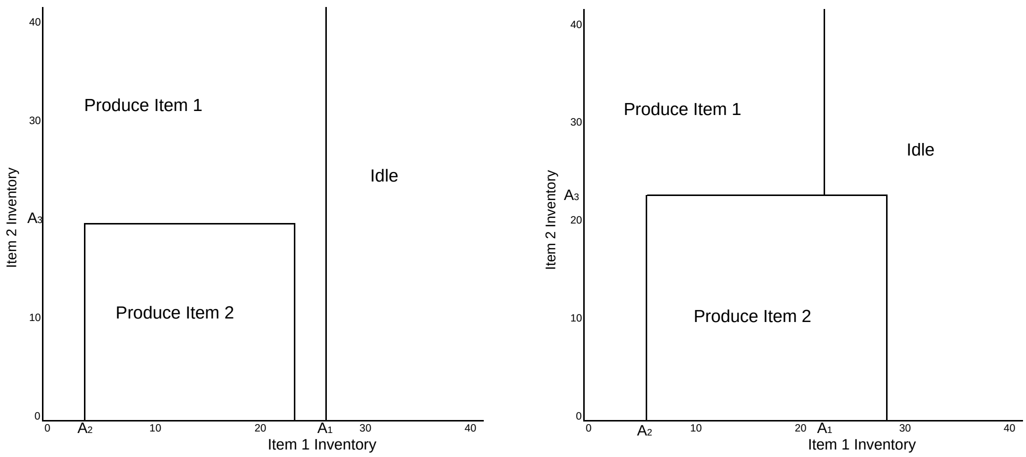


Figure 5: Multi-threshold policy examples. *Left:* Production policy for the case with $A_1 = 27$, $A_2 = 4$, and $A_3 = 20$, *Right:* Production policy for the case with $A_1 = 22$, $A_2 = 6$, and $A_3 = 22$.

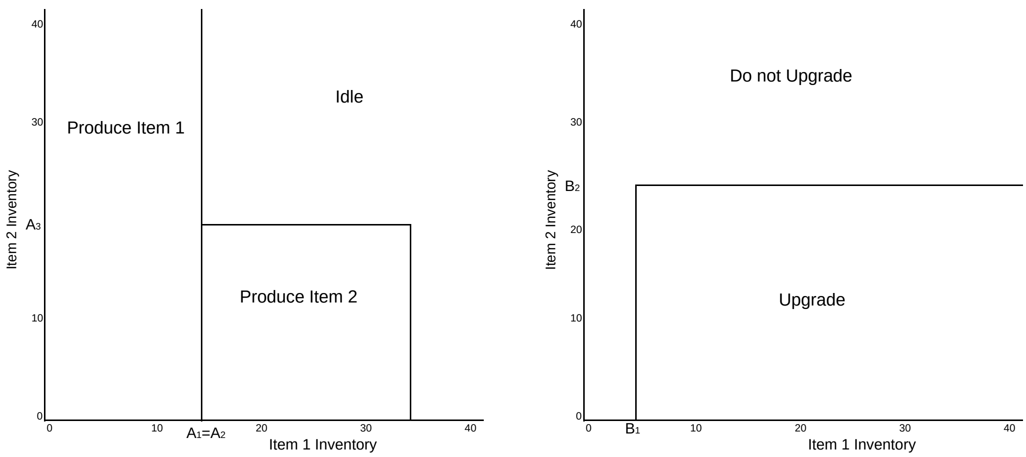


Figure 6: Multi-threshold policy examples. *Left:* Production policy for the case with $A_1 = A_2 = 15$, and $A_3 = 20$, *Right:* Upgrade policy for the case with $B_1 = 5$ and $B_2 = 24$.

### 5.1 Exact Analysis of the Multi-threshold Policy When $0 < B_1 < A_2 \leq A_1 \leq A_2 + A_3$ and $0 \leq A_3 \leq B_2$

We now develop a queueing model to analyze a representative case of the multi-threshold policy. Analyses of the other cases are similar and therefore omitted. We formulate and solve an exact model of the case where $0 < B_1 < A_2 \leq A_1 \leq A_2 + A_3$ and $0 \leq A_3 \leq B_2$ (cases in both Figure 5.*Right* and Figure 6 are examples) to obtain the system's performance measures such as average total cost per unit time and average number of substitutions per unit time.

Upon fixing the control policy used, the MDP model reduces to a continuous-time Markov chain that can be analyzed to obtain performance measures such as average inventory, average number of lost sales for product 1 and 2, and respective costs. For this purpose, we define $\pi_{i,j}$ to represent the steady-state probability that the number of products 1 and 2 in the firms' inventory are $i$ and $j$, respectively. Due to the idle action region, the state space of the resulting Markov chain is positive recurrent on only a single, finite class of states given as $\Omega = \{\pi_{i,j} : 0 \leq i \leq A_2, 0 \leq j \leq A_3 + 1\}$. It can be shown that the Markov chain defined on $\Omega$ is ergodic, the steady-state probabilities exist and they are unique (Kulkarni, 1995).

The balance equations of the Markov chain are presented in online Appendix B. Let the generating function for the steady-state probabilities be defined as follows:

$$\mathbf{\Pi}(v, w) = \sum_{i=0}^{A_2} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j}.$$

We have the following Proposition:

**Proposition 2** *The generating function* $\mathbf{\Pi}(v, w)$ *for the steady state probabilities* $\pi_{i,j}$ *of the model is:*

$$\begin{aligned} \mathbf{\Pi}(v, w) &= \frac{1}{H(v, w)} \Big\{ \lambda_1 w(v-1)\mathcal{N}_1(v, w) + \lambda_2 \big( v(w-1)\mathcal{N}_2(v, w) + (w-v)\mathcal{N}_3(v, w) \big) \\ &+ \mu_1 vw(1-v)\big(\mathcal{N}_4(v, w) + v^{A_2} w^{A_3+1} \pi_{A_2, A_3+1}\big) + \mu_2 vw(w-1)\mathcal{N}_4(v, w) \big) \Big\} \quad (9) \end{aligned}$$

*where*

$$H(v, w) = \lambda_1 w(v-1) + \lambda_2 v(w-1) + \mu_1 vw(1-v) \quad (10)$$

$$\mathcal{N}_k(v, w) = \begin{cases} \sum_{j=0}^{A_3+1} w^j \pi_{0,j} & : k = 1 \\ \sum_{i=0}^{B_1} v^i \pi_{i,0} & : k = 2 \\ \sum_{i=B_1+1}^{A_2} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j} & : k = 3 \\ \sum_{j=0}^{A_3} v^{A_2} w^j \pi_{A_2,j} & : k = 4 \end{cases} \quad (11)$$

The balance equations of the Markov chain could, in principle, be solved directly. However, this approach limits the size of the problem that can be solved with finite computing power. Thus, it is useful to exploit the structural properties of the balance equations to obtain efficient algorithms. We present two approaches to overcome this problem.

Our first approach incorporates generating function analysis to reduce the dimension of the system of equations to be solved. Notice that the generating function $\mathbf{\Pi}(v, w)$ in (9) has $(A_2 - B_1 + 1)(A_3 + 2) + B_1$ unknown probabilities. The algorithm presented in online Appendix B shows how to obtain the steady-state probabilities in (9) by solving $(A_2 - B_1 + 1)(A_3 + 2) + B_1$ equations instead of solving all of the $(A_2 + 1)(A_3 + 2)$ balance equations. An $(A_3 + 1)B_1$ reduction in balance equations could be significant. For example, if $A_2 = A_3 = 50$ and $B_1 = 49$, then obtaining the probabilities by using all of the balance equations involves solving 2652 equations, whereas only 153 equations are required through the algorithm provided in online Appendix B.

Our second approach avoids solving a system of equations to get the unknown probabilities. This approach, detailed in online Appendix C, exploits the Markov chain's structure in such a way that, by setting $\pi_{0,A_3+1} = 1$, we obtain the un-normalized values of the state probabilities via a recursion on the state space. Afterwards, the actual value of the probability for each state can be recovered by simply dividing the un-normalized value by the sum of the un-normalized values for all states.

Let $E(I_1)$ and $E(I_2)$ be the average inventories for the products 1 and 2, respectively. $E(I_1)$ and $E(I_2)$ can be obtained as follows:

$$E(I_1) = \sum_{i=1}^{A_2} h_1 \left( \sum_{j=1}^{A_3+1} \pi_{i,j} \right) = \left. \frac{\partial \Pi(v,1)}{\partial v} \right|_{v=1} \quad \text{and} \quad E(I_2) = \sum_{j=1}^{A_3+1} h_2 \left( \sum_{i=1}^{A_2} \pi_{i,j} \right) = \left. \frac{\partial \Pi(1,w)}{\partial w} \right|_{w=1}$$

Similarly, let $E(L_1)$ and $E(L_2)$ be the average lost sales per unit time for products 1 and 2, respectively. $E(L_1)$ and $E(L_2)$ are given as follows:

$$E(L_1) = \lambda_1 \sum_{j=0}^{A_3+1} \pi_{0,j} = \lambda_1 \mathcal{N}_1(1,1) \quad \text{and} \quad E(L_2) = \lambda_2 \sum_{i=0}^{B_1} \pi_{i,0} = \lambda_2 \mathcal{N}_2(1,1).$$

where $\mathcal{N}_1(1,1)$ and $\mathcal{N}_2(1,1)$ represent the proportion of time a product 1 and a product 2 lost sale happens, respectively. Note that, $\mathcal{N}_2(1,1) = \sum_{i=0}^{B_1} \pi_{i,0}$ is the proportion of time that a lost sale for product 2 happens, as the firm is upgrading product 2 demands to product 1 whenever $n_2 = 0$ and $n_1 > B_1$ under the multi-threshold policy.

Lastly, the average number of substitution per unit time, denoted by $E(S)$, is given as follows:

$$E(S) = \lambda_2 \sum_{i=B_1+1}^{A_2} \sum_{j=0}^{A_3+1} \pi_{i,j} = \lambda_2 \mathcal{N}_3(1,1).$$

Therefore, the total average cost per unit time, $E[TC]$, is:

$$E[TAC] = \sum_{i=1}^{2}(h_i E(I_i) + P_i E(L_i)) + CE(S) \quad (12)$$

$$= h_1 \frac{\partial \Pi(v,1)}{\partial v}\bigg|_{v=1} + h_2 \frac{\partial \Pi(1,w)}{\partial w}\bigg|_{w=1} + \lambda_1 P_1 \mathcal{N}_1(1,1) + \lambda_2(P_2 \mathcal{N}_2(1,1) + C\mathcal{N}_3(1,1)). \quad (13)$$

As Proposition 2 shows, the average total cost per unit time given in (12) is a function of $A_2, A_3,$ and $B_1$. Optimal values can be found by searching for the $A_2^*, A_3^*,$ and $B_1^*$ values that minimize (12) subject to the constraints $0 < B_1 < A_2 \leq A_1 \leq A_2 + A_3$ and $0 \leq A_3 \leq B_2$.

## 5.2 Cost Effectiveness of the Multi-Threshold Policy

In this section, we present the results of the numerical study conducted to investigate whether the performance of the multi-threshold policy is close to that of the optimal joint production and substitution policy.

The relative cost difference (i.e., the error percentage) between the optimal integrated production and substitution policy and the multi-threshold policy is defined as follows:

$$\Delta = \frac{TAC(\mathbf{A}, \mathbf{B}) - TC^*}{TC^*} \times 100\%$$

where $TC^*$ is the total average cost per unit time under the joint optimal production and substitution policy and $TC(\mathbf{A}, \mathbf{B})$ is the total average cost per unit time under the optimal multi-threshold policy computed via (12) with parameter vectors $\mathbf{A} = \{A_1^*, A_2^*, A_3^*\}$ and $\mathbf{B} = \{B_1^*, B_2^*\}$.

Our numerical study, which is based on the test suite introduced in the Appendix, shows that the proposed multi-threshold policy performs extremely well. The average error percentage for over 75,000 cases is only 0.54%, and in approximately 95.1% of the test cases the error percentage is less than or equal to 3%. The maximum error percentage for the test suite is found as 6.78%. Our numerical study indicates that the multi-threshold policy is a good candidate to be used instead of the optimal joint production and substitution policy.

# 6 Production Flexibility versus Post-Production Flexibility

In this section, we investigate the following questions: (1) What is the impact of the system parameters such as production capacities, substitution costs, lost sales costs, and holding costs on the value of inventory flexibility or process flexibility? (2) Is the value of adding inventory (or process) flexibility higher in systems which already have one types of flexibility (inventory or process) compared to those that have neither types of flexibility? (3) Under what circumstances does process flexibility have more value than inventory flexibility, or vice versa?

These are important questions of strategic managerial interest that give seek to gain insight into the value of proactive process flexibility and/or reactive inventory flexibility. Our work above indicated our finding that optimal policies and performance for this class of problems are complex and cannot be obtained in closed form. To address these questions, the best way is via computation of optimal performance over a wide range of settings. We have done just that, and our test suite consists of more than 23,400 cases. The test suite, which is described in more detail in the Appendix, is designed to cover a wide range for the ratio of lost sales cost to holding cost, ($P_i/h_i$ from 1 to 100 for $i = 1, 2$), and the ratio of lost sales cost to substitution cost, ($P_i/C$ ranges from 0.5 to 100). The product arrival rate and production rate combinations tested include cases with $0.5 \leq \lambda_i/\mu_i \leq 2$, ranging from low utilization through high utilization, including cases in which nearly half of the arrivals overflow and are lost in the absence of process flexibility or inventory flexibility.

## 6.1 Alternative System Settings

Some of the questions analyzed in the following sections involve the interaction between, and comparison of, inventory flexibility and process flexibility. Therefore, besides the multi-functional model in Section 2, we also need to develop a model to find the firm's optimal cost when it utilizes dedicated (non-flexible) production facilities.

We now present a model to find the firm's optimal cost when it utilizes two production facilities, each dedicated to produce one of the products, instead of using a multi-functional facility to produce both. Similar to Section 2, we formulate the system as an MDP and unless stated otherwise, we use the definitions introduced in that section.

Let $\mu_i^D$ be the production rate of the facility dedicated to produce product $i = 1, 2$. Using $\mathbf{T}_s$ as given in (4) and the operator $\mathbf{T}_p$ defined below, the average cost optimality equation for the MDP is:

$$V(\mathbf{n}) + g = h(\mathbf{n}) + \lambda_1\big[V(\mathbf{n} - \mathbf{e}_1\mathbf{I}_{(n_1>0)}) + P_1\mathbf{I}_{(n_1=0)}\big] + \lambda_2\mathbf{T}_s V(\mathbf{n}) + \mathbf{T}_p V(\mathbf{n}) \tag{14}$$

$$\mathbf{T}_p V(\mathbf{n}) = \min \begin{cases} (\mu_1^D + \mu_2^D)V(\mathbf{n}) & : \text{ Idling,} \\ \mu_1^D V(\mathbf{n} + \mathbf{e}_1) + \mu_2^D V(\mathbf{n}) & : \text{ Produce item 1,} \\ \mu_1^D V(\mathbf{n}) + \mu_2^D V(\mathbf{n} + \mathbf{e}_2) & : \text{ Produce item 2,} \\ \mu_1^D V(\mathbf{n} + \mathbf{e}_1) + \mu_2^D V(\mathbf{n} + \mathbf{e}_2) & : \text{ Produce item 1 and 2.} \end{cases} \tag{15}$$

The differences between this model and (6) are the terms related to production decisions, given by equations (5) and (15). In the dedicated facility case, we have four possible actions: (*i*) both of the facilities are idling, (*ii*) the facility for product 2 is idling while the other facility is producing product 1, and (*iii*) the facility for product 1 is idling while the other facility is producing product 2, and lastly (*iv*) both facilities are producing.

Note that, $\mu_i$ is the capacity of our multi-functional facility (i.e., number of product $i$ produced per unit time), if the multi-functional facility is devoted to product $i$. Since the multi-functional facility does not always produce one type of product in our base model, we consider $\mu_i^D$, the capacity of the dedicated system for product $i$, to be a fraction of the capacity $\mu_i$. The fraction is proportional to the demand for product $i$, so:

$$\mu_i^D = \Big(\frac{\lambda_i}{\lambda_1 + \lambda_2}\Big)\mu_i \quad \Longrightarrow \quad \mu_i = \Big(\frac{\lambda_1 + \lambda_2}{\lambda_i}\Big)\mu_i^D \qquad \text{for } i = 1, 2. \tag{16}$$

While referring to the alternative system settings, we use the subscript $i$ (subscript $ni$) for systems with inventory flexibility (no inventory flexibility) and subscript $p$ (subscript $np$) for process flexibility (no process flexibility). Alternative system settings considered are presented in Figure 7.

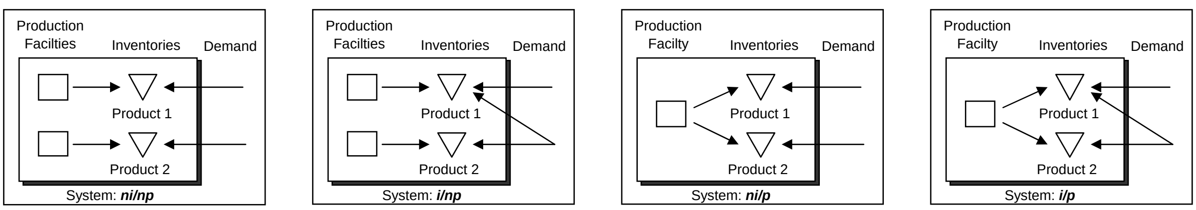


Figure 7: Alternative system settings with varying flexibility levels.

Note that the system presented in Section 2 corresponds to the system $i/p$ and the systems introduced in this section correspond to the system $i/np$ in Figure 7. The MDP model for the system $ni/np$ (system $ni/p$) can simply be obtained from the MDP model for the system $i/np$ (system $i/p$), by removing the "upgrading" action from the operator $\mathbf{T}_s$ given in (4).

## 6.2 Value of Inventory Flexibility

We would like to know if, and by how much, inventory flexibility can reduce the total average cost. Furthermore, we would also like to analyze the interaction between process flexibility and inventory flexibility. Before we introduce the measures that we use to analyze the value of inventory flexibility, we need to define $TC_{i/p}$ as the firm's total average cost per unit time in systems with inventory flexibility and process flexibility. Similarly, $TC_{ni/p}$, $TC_{i/np}$, and $TC_{ni/np}$ denote the firm's total average cost per unit time in alternative system settings discussed in Section 6.1. In order to measure the value of inventory flexibility, we consider the following measures:

$$I_P = \frac{TC_{ni/p} - TC_{i/p}}{TC_{ni/np}} \times 100\% \; ; \qquad I_{NP} = \frac{TC_{ni/np} - TC_{i/np}}{TC_{ni/np}} \times 100\%.$$

where $I_P$ ($I_{NP}$, respectively) represent the firm's relative cost saving due to adding inventory flexibility to a system which already has (does not have) process flexibility. These measures can be compared because both use the system with no flexibility (i.e., the $ni/np$ system) as a benchmark (in the denominator).

We obtained the metrics $I_P$ and $I_{NP}$ for the broad test suite defined in the Appendix, where for each parameter combination we parameterized our base system (i.e., system $ni/np$) using (16). Table 1 provides summary information on $I_P$ and $I_{NP}$, as the numerical study is too large to present in detail. Note that $I_P - I_{NP}$ represents how much the benefit of inventory flexibility is higher in systems that already have process flexibility compared to systems without process flexibility.

Table 1: Performance measure statistics.

| | $I_P$ | $I_{NP}$ | $I_P - I_{NP}$ |
|---|---|---|---|
| Average Value | 4.58% | 1.65% | 2.92% |
| Maximum Value | 79.32% | 25.49% | 72.05% |

### 6.2.1 Impact of Process Flexibility on the Value of Inventory Flexibility

As Table 1 shows, the value of inventory flexibility can be very significant, i.e., up to 25.49% in systems with no production flexibility, and up to 79.32% in systems with production flexibility. Furthermore, the value of inventory flexibility is larger in systems that already have production flexibility, i.e., $I_P - I_{NP}$ is nonnegative. We observed this in 78.2% of our cases. This led us to the following observation. In the rest of the paper we refer to $P_1 - P_2$ as the "difference between lost sales costs," and we refer to $h_1 - h_2$ as the "difference between inventory holding costs."

**Observation 1** *In most cases, the benefit of inventory flexibility is higher in systems that already have production flexibility compared to systems without production flexibility. However, as the production capacity of the lower-quality product decreases, systems without production flexibility might benefit more from inventory flexibility than systems with production flexibility. In such cases, $I_P - I_{NP}$ is reduced further when the difference between lost sales costs decreases.*

One would expect that the $I_{NP}$ values dominate the $I_P$ values in most cases, since $I_P$ represents the value of the inventory flexibility added as a second option on top of the process flexibility, whereas $I_{NP}$ represents the value of the inventory flexibility added as the first option to a system with no flexibility at all. However, as stated in Observation 1, it is perhaps surprising that the existence of process flexibility increases the value of inventory flexibility in majority of the cases by providing better capacity allocation through a multi-functional facility. Furthermore, a multi-functional facility also helps mitigate the increased risk of product 1 stockouts when inventory flexibility is used, further complementing the effectiveness of inventory flexibility.

On the other hand, as the capacity for product 2 in the base system with no flexibility decreases, as noted in Observation 1, after some point the benefit of capacity pooling of the multi-functional facility starts reducing the value of inventory flexibility.

The maximum value of both $I_P$ and $I_{NP}$ is obtained when the substitution cost, $C$, is zero. If we restrict attention to the case $C = 0$, the average $I_P - I_{NP}$ difference increases to 4.57%, further supporting Observation 1.

As we mentioned above, the capacity pooling of the multi-functional facility might reduce the value of inventory flexibility in a system that already has process flexibility (i.e., decreasing $I_P$). To observe the impact of the *degree of process flexibility* on value of inventory flexibility, we replaced the fully flexible production models for $ni/p$ and $i/p$ that are rightmost in Figure 7 with models that have only limited (server 2 can produce both types 1 and 2 now) as shown in Figure 8.

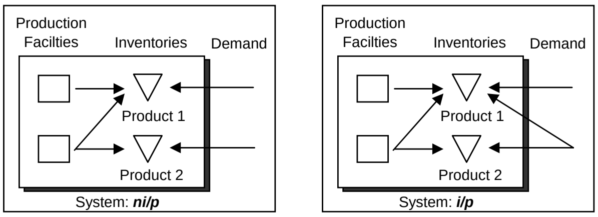


Figure 8: Alternative system settings with one-way process flexibility.

The two systems in Figure 8 have process flexibility, but only a lower level process flexibility, which we call "one-way" flexibility. Note that facility 2 in the figure can produce both types of products, while facility 1 can only produce product of type 1. As product 1 has higher lost sales cost and inventory holding cost than product 2 in our model, the value of one-way process flexibility that can shift capacity to product 1 would typically be higher than the one that can shift capacity to product 2. One-way process flexibility does not introduce full capacity pooling and it cannot shift capacity to product 2. We repeated the numerical study. As expected after Observation 1 (however, still an interesting result in itself), *in general, the benefit of inventory flexibility is higher in systems that already have one-way production flexibility compared to systems without one-way production flexibility.* In this case, $I_P - I_{NP}$ is negative in only less than 0.1% of the cases, and the maximum and the average $I_P - I_{NP}$ values increase to 81.9% and 7%, respectively. This indicates that process and inventory flexibility continue to be complementary to each other.

### 6.2.2 Impact of System Parameters on the Value of Inventory Flexibility

We have also investigated the numerical study to identify the impact of systems parameters on the value of inventory flexibility, which lead us to the following observation, which we will then explain.

**Observation 2** *The value of inventory flexibility $I_{NP}$ ($I_P$) increases as any of the following occurs: (1) substitution cost $C$ decreases, (2) the production capacity for the higher-quality product increases, (3) production capacity for the lower-quality product decreases, (4) the* *difference between lost sales costs or (5) the difference between inventory holding costs decrease.*

The intuition behind this can be understood as follows. When the production capacity for product 1 is large, it is easier for the system to build up product 1 inventory, therefore increasing the chances that a higher-quality product will be available for substitution, if needed. A reduction in the production capacity for product 2 also increases the value of inventory flexibility, as without substitution the system will suffer more frequent product 2 stockouts. An increase in the lost sales cost of product 2 increases the value of inventory flexibility, as the alternative to substitution is higher lost sales of product 2 per unit time. Similarly, higher inventory holding cost for product 2 also increases the value of inventory flexibility, because substitution becomes more attractive and reduces the inventory requirement for product 2. The effects of the lost sales cost and inventory holding cost of product 1 can be explained similarly.

## 6.3 Value of Process Flexibility

We have also evaluated the value of process flexibility through the following two metrics:

$$P_I = \frac{TC_{i/np} - TC_{i/p}}{TC_{ni/np}} \times 100\% \; ; \qquad P_{NI} = \frac{TC_{ni/np} - TC_{ni/p}}{TC_{ni/np}} \times 100\%.$$

Table 2 provides summary information on $P_I$ and $P_{NI}$. We observed that the value of process flexibility in systems without inventory flexibility can be up to 80.9%, with an average of 29.9%. These numbers are 80.9% and 32.8% for systems that already have inventory flexibility.

Table 2: Performance measure statistics.

| | $P_I$ | $P_{NI}$ | $P_I - P_{NI}$ |
|---|---|---|---|
| Average Value | 32.82% | 29.90% | 2.92% |
| Maximum Value | 80.92% | 80.90% | 72.05% |

### 6.3.1 Impact of Inventory Flexibility on the Value of Process Flexibility

Note that $P_I - P_{NI}$ represents how much the benefit of process flexibility is higher in systems that already have inventory flexibility compared to systems without inventory flexibility. It is easy to show that, $P_I - P_{NI} = I_P - I_{NP}$, which results in the same observation as that in Observation 1, rephrased for process flexibility. Specifically, we will have:

**Observation 3** *In most cases, the benefit of process flexibility is higher in systems that already have inventory flexibility compared to systems without inventory flexibility. However, as the production capacity of the lower-quality product decreases, systems without inventory flexibility might benefit more from process flexibility than systems with inventory flexibility. In such cases,* $P_I - P_{NI}$ *reduces further when the difference between lost sales costs decreases.*

### 6.3.2 Impact of System Parameters on the Value of Process Flexibility

The following closely related observations, 4 and 5, present the impact of systems parameters on the value of process flexibility in systems with inventory flexibility as well as without.

**Observation 4** *In systems without flexibility, when the production capacity of the higher-quality product becomes sufficiently small, the value of adding process flexibility* $P_I$ $(P_{NI})$ *increases as any of the following occurs: (1) the production capacity for the lower-quality product increases, (2) the difference between lost sales costs increases, or (3) the difference between inventory holding costs decreases.*

**Observation 5** *In systems without flexibility, when the production capacity of the lower-quality product becomes sufficiently small, the value of adding process flexibility* $P_I$ $(P_{NI})$ *increases as any of the following occurs: (1) the production capacity for the higher-quality product increases, (2) the difference between lost sales costs decreases, or (3) the difference between inventory holding costs increases.*

In scenarios where process flexibility provides value mainly by shifting capacity to a product type, say $i$, the value of the process flexibility increases when the differences between the product cost parameters emphasize the importance of product type $i$. However, because product 1 is assumed to have higher holding and lost sales costs due to its quality, shifting capacity to product 2 usually generates less value than vice versa.

## 6.4 Process Flexibility versus Inventory Flexibility

We now compare the value of process flexibility only ($P_{NI}$) to the value of inventory flexibility only ($I_{NP}$) in Table 3. Note that $P_{NI} - I_{NP}$ would be positive if the value of process flexibility is larger than the value of inventory flexibility. Based on the results of our numerical study, we can present the following observation:

Table 3: Performance measure statistics.

| | $P_{NI}$ | $I_{NP}$ | $P_{NI} - I_{NP}$ |
|---|---|---|---|
| Average Value | 29.90% | 1.65% | 28.25% |
| Maximum Value | 80.90% | 25.49% | 80.90% |

**Observation 6** *In general, process flexibility results in more cost savings than inventory flexibility. The benefit of process flexibility over inventory flexibility increases as the production capacity for the lower-quality product increases, or the difference between lost sales costs increases.*

We observed that in 99.05% of the cases the value of process flexibility is at least as large as the value of inventory flexibility. On average, the value of process flexibility is 28.2% more than the value of inventory flexibility for the test suite, with a maximum advantage ($P_{NI} - I_{NP}$) of 80.9%. Only 0.95% of the cases showed $I_{NP} - P_{NI}$ to be positive, the maximum being 5.4%. These cases corresponds to the situation where the value of inventory flexibility is very high and the value of process flexibility is low at the same time. This occurs when the production capacity for the higher-quality product is high, production capacity for the lower-quality product is low, the substitution cost $C$ is low, and the difference in inventory holding costs and the lost sales costs of both products are small. All these conditions favor producing product 1 all the time and use it as a substitution for product 2.

One might think that the substitution cost, $C$, is the main reason for the performance difference between process flexibility and inventory flexibility stated in Observation 6. Specifically, one might consider substitution cost $C$ as a per unit cost of utilizing inventory flexibility, while utilizing process flexibility (switching from producing one type of product to the other) is costless. To investigate this, we eliminated the substitution cost (i.e., we set $C = 0$), which maximizes the effectiveness of inventory flexibility, but the average value of process flexibility is still higher than the value of inventory flexibility by 24.1%, and the maximum difference is still 80.9%. Furthermore, in approximately 97.7% of the cases the value of process flexibility is at least as large as the value of inventory flexibility. Therefore, the substitution cost is not the main reason why the value of process flexibility is typically higher.

The other reason that one might consider as the driver of the superior performance of process flexibility over inventory flexibility is the fact that we compare one-way inventory flexibility to "two-way" process flexibility. To check this claim, we repeated our numerical study and compared inventory flexibility (i.e., systems *ni*/*np* and *i*/*np*) in Figure 7 to one-way process flexibility introduced in Figure 8. We found that process flexibility still outperforms inventory flexibility in 90.6% of the cases, and the average and maximum $P_{NI} - I_{NP}$ values are 23.9% and 84.2%, respectively.

All other things being equal, the key reason that process flexibility adds more value than inventory flexibility is as follows. To utilize inventory flexibility, the system must hold more inventory of product 1 than systems with process flexibility. Thus, the inventory levels (costs) in systems with inventory flexibility can be significantly larger than that in systems with process flexibility. Further, process flexibility (which we have characterized as controllable at the time of production as opposed to the post-production nature of inventory flexibility) offers the greater ability to align production and inventory allocation/substitution with the cost structure. Further perspective is provided in the next section.

# 7 Conclusion

We analyzed the joint control of production in a flexible process and inventory management via one-way product substitution of finished goods. We provide a detailed analysis of the properties of the joint optimal production and substitution policy for the case of homogeneous production rates. We also propose a multi-threshold policy as an effective but simpler surrogate for the optimal policy for the more complex case of heterogeneous production rates. Our extensive numerical study shows that our multi-threshold policy is nearly optimal. We provide insights into the value of inventory flexibility generated by the one-way product substitution option and process flexibility generated by a multi-functional facility. We also compare the value of inventory flexibility to both the value of two-way process flexibility generated by a multi-functional facility as well as one-way process flexibility in a system with a dedicated and a flexible facility.

We found that inventory flexibility is not effective in every situation; however, it shouldn't be overlooked as its value can be substantial. The value of inventory flexibility increases as any of the following occur: (1) substitution cost decreases, (2,3) the production capacity for the higher-quality (lower-quality) product increases (decreases), or (4,5) the difference between lost sales costs or inventory holding costs decrease.

Our numerical study showed that inventory flexibility and process flexibility typically complement each other, i.e., the benefit of inventory flexibility is higher in systems that already have production flexibility compared to systems without production flexibility and vice versa. Moreover, we found this to be true for both two-way and one-way process flexibilities.

Finally, our numerical study suggests that in an overwhelming majority of cases, the value of process flexibility exceeds the value of inventory flexibility in the form of one-way firm-driven product substitution. Both capacity and cost considerations play an important role in the value of substitution. Two-way process flexibility benefits from the value of capacity control and pooling, which in our study exceeds the value of inventory flexibility in the majority of the cases. However, it should be noted that the value of substitution could exceed the value of process flexibility in rare situations favoring substitution. Because the model of process flexibility was two-way, we also created a special model of one-way process flexibility and once again observed the dominance of even one-way process flexibility over inventory flexibility.

To summarize the intuition, we find it helpful to characterize process flexibility as occurring at the time of production and prior to demand realization. In contrast, firm-driven inventory flexibility via substitution occurs post-production and post-demand. Our numerical studies reveal that the strength of process flexibility lies in its greater ability to proactively build more accurately sized inventories in alignment with the firm's cost structure. On the other hand, it is intuitive that one-way inventory flexibility, which is post-production, is more limited in it's ability to shape inventory to reduce cost. The weakness of process flexibility is that it is anticipative of future demand, which allows inventory flexibility as a post-demand mechanism, to complement process flexibility. Thus, our models suggest that firms should consider to employ both process as well as inventory flexibility as complementary forms of flexibility.

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# APPENDIX: Design of the Numerical Experiment

To investigate the performance of the multi-threshold policy (and also numerically investigate process and inventory flexibility), we examined the problems generated from the combination of the following set of parameters: $\lambda_i \in \{6, 8, 10\}$ and $\mu_i \in \{3, 4, 6, 8, 10, 13, 16\}$, $P_2 \in \{0.10P_1, 0.25P_1, 0.50P_1, 0.75P_1, 0.95P_1\}$, $P_1 \in \{5, 10, 25, 50, 100\}$, $h_1 \in \{1, 1.5, 2.5, 5\}$, $h_2 \in \{1\}$, and $C \in \{0, 1, 2.5, 5, 10\}$. Recall from Section 2 that $P_1 > P_2$, so for any lost sales cost of the higher quality product, $P_1$, we generate different lost sales cost values for the lower quality product by taking $10\%, 25\%, 50\%, 75\%$, and $95\%$ of $P_1$. Since simultaneously increasing the holding costs and the lost sales costs of both of the products does not change the insights, we set the inventory holding cost of the lower quality product, $h_2$, to be 1 and vary $h_1, P_1$, and $P_2$. To avoid trivial cases where it is optimal not to produce item $i$, we require the lost sales cost for item $i$, $P_i$, to observe $P_i \geq h_i$. We eliminate those cases with $C > P_1 - P_2$ to avoid testing cases in which the substitution cost is relatively very large and thereby avoid biasing the test suite against the use of substitution. Overall, our test suite consists of more than 75,000 cases.

The test suite is designed to cover a wide range for the ratio of lost sales cost to holding cost, $P_i/h_i$, and the ratio of lost sales cost to substitution cost, $P_i/C$ for $i = 1, 2$. In our test suite $P_1/h_1$ ranges from 1 to 100 and $P_2/h_2$ ranges from 1 to 95. Excluding cases where $C = 0$, $P_1/C$ ranges from 0.5 to 100, whereas $P_2/C$ ranges from 0.1 to 95. Due to the asymmetry in the items and therefore the model parameters, the aforementioned ratios for the products 1 and 2 have slightly different but comparable ranges. Some of the product arrival rate and production rate combinations in the test suite lead to extreme and unrealistic cases, therefore we limit our analysis to the cases with $0.5 \leq \lambda_i/\mu_i \leq 2$ for $i = 1, 2$. Thus, we include low utilization through high utilization and even systems in which nearly half of the arrivals overflow and are lost in the absence of process flexibility or inventory flexibility.

The test suite is also used to numerically investigate process and inventory flexibility. For this purpose, each parameter combination is taken as a dedicated system and the comparable flexible system is obtained by (16). The resulting dedicated system and flexible system pair is considered as a valid case if each system is valid under the assumptions discussed above. Overall, our test suite to investigate process and inventory flexibility consists of more than 23,400 case pairs. As the performance measures for each case of the test suite are calculated by using the results of a code that converges, performance measures and the difference of performance measures that are within 0.01% of zero are taken as zero.

# ONLINE APPENDICES

## ONLINE APPENDIX A

Recall that, for any real-valued function $v(\mathbf{n})$, with $\mathbf{n} \in \mathcal{S}$, the first difference operator $\mathbf{D}_i$ and the second difference operators $\mathbf{D}_{ij}$, with $i, j \in \{1, 2\}$, were defined as:

$$\begin{aligned} \mathbf{D}_i v(\mathbf{n}) &= v(\mathbf{n} + \mathbf{e}_i) - v(\mathbf{n}) \quad i \in \{1, 2\} \\ \mathbf{D}_{ij} v(\mathbf{n}) &= \mathbf{D}_i(\mathbf{D}_j v(\mathbf{n})) \quad i, j \in \{1, 2\}. \end{aligned}$$

Furthermore, $\mathcal{U}$ was defined as the set of functions $v$ on $\mathcal{S}$ with the following properties:

| | | |
|---|---|---|
| **P1** | Supermodularity: | $\mathbf{D}_{12} v(\mathbf{n}) \geq 0$ |
| **P2** | Diagonal dominance: | $\mathbf{D}_{11} v(\mathbf{n}) \geq \mathbf{D}_{12} v(\mathbf{n}), \quad \mathbf{D}_{22} v(\mathbf{n}) \geq \mathbf{D}_{12} v(\mathbf{n})$ |
| **P3** | Convexity: | $\mathbf{D}_{11} v(\mathbf{n}) \geq 0, \quad \mathbf{D}_{22} v(\mathbf{n}) \geq 0$ |
| **P4** | Lower bound: | $\mathbf{D}_1 v(\mathbf{n}) \geq -P_1, \quad \mathbf{D}_2 v(\mathbf{n}) \geq -P_2.$ |

It should be noted that $\mathbf{D}_{12} v \geq 0$, $\mathbf{D}_{11} v \geq \mathbf{D}_{12} v$, and $\mathbf{D}_{22} v \geq \mathbf{D}_{12} v$ imply $\mathbf{D}_{11} v \geq 0$ and $\mathbf{D}_{22} v \geq 0$. Therefore, supermodularity (**P1**) and diagonal dominance (**P2**) together imply convexity in both coordinates $n_1$ and $n_2$ (**P3**).

By (8), (4), and (7) we have the following operators on any function $v \in \mathcal{U}$:

$$\begin{aligned} \mathbf{T}_p\, v(\mathbf{n}) &= \min\Big\{ v(\mathbf{n}), v(\mathbf{n} + \mathbf{e}_1), v(\mathbf{n} + \mathbf{e}_2) \Big\} \\ \mathbf{T}_s\, v(\mathbf{n}) &= \min\Big\{ v(\mathbf{n} - \mathbf{e}_2 \mathbf{I}_{(n_2 > 0)}) + P_2 \mathbf{I}_{(n_2 = 0)}, v(\mathbf{n} - \mathbf{e}_1) + C \Big\} \\ \mathbf{T}\, v(\mathbf{n}) &= h(\mathbf{n}) + \lambda_1 \Big( v(\mathbf{n} - \mathbf{e}_1 \mathbf{I}_{(n_1 > 0)}) + P_1 \mathbf{I}_{(n_1 = 0)} \Big) + \lambda_2 \mathbf{T}_s\, v(\mathbf{n}) + \mu \mathbf{T}_p\, v(\mathbf{n}). \end{aligned}$$

so that for the equal production rate case, where $\mu_1 = \mu_2 = \mu$, the optimality equation (7) becomes:

$$V_\alpha(\mathbf{n}) = \mathbf{T}\, V_\alpha(\mathbf{n}).$$

**PROOF OF PROPOSITION 1:**

To prove Proposition 1, we will show that, if $v \in \mathcal{U}$, then $\mathbf{T}_p\, v \in \mathcal{U}$, $\mathbf{T}_s\, v \in \mathcal{U}$, and $\mathbf{T}\, v \in \mathcal{U}$. This way, we show that the operators $\mathbf{T}_p$, $\mathbf{T}_s$, and $\mathbf{T}$ preserve the properties **P1**-**P4**. Then, similar to Véricourt et al. (2002) and Ha (2000), a direct application of value iteration on the optimality equation $V_\alpha(\mathbf{n}) = \mathbf{T}\, V_\alpha(\mathbf{n})$ implies that the optimal cost function $V_\alpha$ belongs to $\mathcal{U}$. To see this, note that $V_0 \in \mathcal{U}$, where $V_0$ is the zero function on the state space $\mathcal{S}$, and the optimal cost function $V_\alpha$ is given by $\lim_{n \to \infty} \mathbf{T}^n V_0$.

We will now show that, if $v \in \mathcal{U}$, then $\mathbf{T}_p\, v \in \mathcal{U}$, $\mathbf{T}_s\, v \in \mathcal{U}$, and $\mathbf{T}\, v \in \mathcal{U}$. The proof is similar to that of Lemma 2 in Ha (1997). The proof is presented in three parts:

**Proof for $\mathbf{T}_p\, v \in \mathcal{U}$:**

We first show that, if $v \in \mathcal{U}$, then $\mathbf{T}_p\, v \in \mathcal{U}$. The proof of Lemma 2 in Ha (1997) shows that when $v$ is supermodular and has diagonal dominance, the operator $\mathbf{T}_p\, v$ is supermodular and has diagonal dominance (**P1** and **P2**). **P1** and **P2** together implies **P3**, the convexity of $\mathbf{T}_p v$ in both coordinates. Therefore, we need to show that the lower bound is preserved under the minimization operation, giving us **P4**.

Take any $v \in \mathcal{U}$ and let $w$ be a function of $(u, \mathbf{n})$, where $u \in \{0, 1, 2\}$ and $\mathbf{n} = (n_1, n_2) \in \mathcal{S}$, defined as

$$\begin{aligned} w(u, \mathbf{n}) &= (1/2)(1-u)(2-u)v(\mathbf{n}) + u(2-u)v(\mathbf{n}+\mathbf{e}_1) + (1/2)u(u-1)v(\mathbf{n}+\mathbf{e}_2) \\ &= \begin{cases} v(\mathbf{n}) & : \text{ if } u = 0, \\ v(\mathbf{n}+\mathbf{e}_1) & : \text{ if } u = 1, \\ v(\mathbf{n}+\mathbf{e}_2) & : \text{ if } u = 2. \end{cases} \end{aligned}$$

Furthermore, for a fixed $u$,

$$\mathbf{D}_1 w(u, \mathbf{n}) = w(u, \mathbf{n}+\mathbf{e}_1) - w(u, \mathbf{n}) = \begin{cases} \mathbf{D}_1 v(\mathbf{n}) & : \text{ if } u = 0, \\ \mathbf{D}_1 v(\mathbf{n}+\mathbf{e}_1) & : \text{ if } u = 1, \\ \mathbf{D}_1 v(\mathbf{n}+\mathbf{e}_2) & : \text{ if } u = 2. \end{cases}$$

Therefore, for any $u$, $\mathbf{D}_1 w(u, \mathbf{n}) \geq -P_1$ as $v \in \mathcal{U}$. Similarly, $\mathbf{D}_2 w(u, \mathbf{n}) \geq -P_2$ and hence, $w$ has **P4**.

By definition, we have

$$\mathbf{T}_p v(\mathbf{n}) = \min_{u \in \{0,1,2\}} w(u, \mathbf{n}).$$

We need to show that for any $\mathbf{n}$, $\mathbf{D}_1 \mathbf{T}_p v(\mathbf{n}) \geq -P_1$ and $\mathbf{D}_2 \mathbf{T}_p v(\mathbf{n}) \geq -P_2$. Let $u_1, u_2 \in \{0, 1, 2\}$ be the optimal indices at $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_1$, respectively. Then,

$$\begin{aligned} \mathbf{T}_p v(\mathbf{n}) &= \min_{u \in \{0,1,2\}} w(u, \mathbf{n}) = w(u_1, \mathbf{n}) \\ \mathbf{T}_p v(\mathbf{n}+\mathbf{e}_1) &= w(u_2, \mathbf{n}+\mathbf{e}_1). \end{aligned}$$

We have,

$$\begin{aligned} \mathbf{D}_1 \mathbf{T}_p v(\mathbf{n}) &= w(u_2, \mathbf{n}+\mathbf{e}_1) - w(u_1, \mathbf{n}) \\ &\geq w(u_2, \mathbf{n}+\mathbf{e}_1) - w(u_2, \mathbf{n}) \qquad (\text{as } w(u_1, \mathbf{n}) \leq w(u_2, \mathbf{n})) \\ &= \mathbf{D}_1 w(u_2, \mathbf{n}) \geq -P_1. \end{aligned}$$

Similarly, let $u_1, u_2 \in \{0, 1, 2\}$ be the optimal indices at $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_2$, respectively. Then,

$$\mathbf{T}_p v(\mathbf{n}) = w(u_1, \mathbf{n}) \quad \text{and} \quad \mathbf{T}_p v(\mathbf{n}+\mathbf{e}_2) = w(u_2, \mathbf{n}+\mathbf{e}_2).$$

We have,

$$\begin{aligned} \mathbf{D}_2 \mathbf{T}_p v(\mathbf{n}) &= w(u_2, \mathbf{n}+\mathbf{e}_2) - w(u_1, \mathbf{n}) \\ &\geq w(u_2, \mathbf{n}+\mathbf{e}_2) - w(u_2, \mathbf{n}) \qquad (\text{as } w(u_1, \mathbf{n}) \leq w(u_2, \mathbf{n})) \\ &= \mathbf{D}_2 w(u_2, \mathbf{n}) \geq -P_2. \end{aligned}$$

**Proof for $\mathbf{T}_s v \in \mathcal{U}$:**

Take any function $v \in \mathcal{U}$ and let $w$ be a function of $(u, \mathbf{n})$ with $\mathbf{n} \in \mathcal{S}$ and $u \in \{0, 1\}$ if $n_1 > 0$, and $u = 0$ if $n_1 = 0$, defined as follows

$$\begin{aligned} w(u, \mathbf{n}) &= (1-u)\Big[v(\mathbf{n} - \mathbf{e}_2 \mathbf{I}_{(n_2>0)}) + P_2 \mathbf{I}_{(n_2=0)}\Big] + u\Big[v(\mathbf{n}-\mathbf{e}_1) + C\Big] \\ &= \begin{cases} v(\mathbf{n} - \mathbf{e}_2 \mathbf{I}_{(n_2>0)}) + P_2 \mathbf{I}_{(n_2=0)} & : \text{ if } u = 0, \\ v(\mathbf{n}-\mathbf{e}_1) + C & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \end{aligned} \tag{17}$$

By the definition of the domain of $u$, when $n_1 = 0$, $u = 0$ is the only possibility and the function $w$ becomes:

$$w(0, 0, n_2) = v(\mathbf{n} - \mathbf{e}_2 \mathbf{I}_{(n_2 > 0)}) + P_2 \mathbf{I}_{(n_2 = 0)}$$

For any $u$,

$$\mathbf{D}_1 w(u, \mathbf{n}) = w(u, \mathbf{n} + \mathbf{e}_1) - w(u, \mathbf{n}) = \begin{cases} \mathbf{D}_1 v(\mathbf{n}) & : \text{ if } u = 0 \text{ and } n_2 = 0, \\ \mathbf{D}_1 v(\mathbf{n} - \mathbf{e}_2) & : \text{ if } u = 0 \text{ and } n_2 > 0, \\ \mathbf{D}_1 v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \tag{18}$$

$$\mathbf{D}_2 w(u, \mathbf{n}) = w(u, \mathbf{n} + \mathbf{e}_2) - w(u, \mathbf{n}) = \begin{cases} -P_2 & : \text{ if } u = 0 \text{ and } n_2 = 0, \\ \mathbf{D}_2 v(\mathbf{n} - \mathbf{e}_2) & : \text{ if } u = 0 \text{ and } n_2 > 0, \\ \mathbf{D}_2 v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \tag{19}$$

Therefore, for any $u$, $v \in \mathcal{U}$ implies that the function $w$ satisfies **P4**. Furthermore, when $n_1 > 0$, $w$ is supermodular in $(u, n_2)$ as $\mathbf{D}_2 w(u, \mathbf{n})$ is increasing in $u$ (see Topkis (1978)). To see this, notice that when $n_1 > 0$,

$$-P_2 \leq \mathbf{D}_2 v(\mathbf{n} - \mathbf{e}_2) \leq \mathbf{D}_2 v(\mathbf{n} - \mathbf{e}_1) \quad \text{by } \mathbf{P4} \text{ and } \mathbf{P2}\text{, respectively.}$$

From (19) we have,

$$\mathbf{D}_{12} w(u, \mathbf{n}) = \mathbf{D}_{21} w(u, \mathbf{n}) = \begin{cases} 0 & : \text{ if } u = 0 \text{ and } n_2 = 0, \\ \mathbf{D}_{12} v(\mathbf{n} - \mathbf{e}_2) & : \text{ if } u = 0 \text{ and } n_2 > 0, \\ \mathbf{D}_{12} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \tag{20}$$

Since $v$ is supermodular, for any $u$, $w$ is also supermodular (**P1**) in $\mathbf{n}$. From (18) and (19), we have

$$\mathbf{D}_{11} w(u, \mathbf{n}) = \begin{cases} \mathbf{D}_{11} v(\mathbf{n}) & : \text{ if } u = 0 \text{ and } n_2 = 0, \\ \mathbf{D}_{11} v(\mathbf{n} - \mathbf{e}_2) & : \text{ if } u = 0 \text{ and } n_2 > 0, \\ \mathbf{D}_{11} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \tag{21}$$

$$\mathbf{D}_{22} w(u, \mathbf{n}) = \begin{cases} \mathbf{D}_2 v(\mathbf{n}) + P_2 & : \text{ if } u = 0 \text{ and } n_2 = 0, \\ \mathbf{D}_{22} v(\mathbf{n} - \mathbf{e}_2) & : \text{ if } u = 0 \text{ and } n_2 > 0, \\ \mathbf{D}_{22} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } u = 1 \text{ and } n_1 > 0. \end{cases} \tag{22}$$

By the convexity of $v$, we have $\mathbf{D}_{11} v(\mathbf{n}) \geq 0$ and by **P4** we have $\mathbf{D}_2 v(\mathbf{n}) \geq -P_2$. Furthermore, $v$ has diagonal dominance. Therefore, for any $u$, $w$ has diagonal dominance (**P2**) in $\mathbf{n}$. As $w$ is supermodular and has diagonal dominance, for any $u$, $w$ is convex (**P3**) in $\mathbf{n}$. For any $u$, $w$ satisfy **P1** to **P4** and therefore $w \in \mathcal{U}$.

By definition,

$$\mathbf{T}_s\, v(\mathbf{n}) = \min\nolimits_u w(u, \mathbf{n}). \tag{23}$$

We need to show that supermodularity, diagonal dominance, and lower bound are preserved under the minimization operation.

**Supermodularity of $\mathbf{T}_s\, v$**

We need to show that for any $\mathbf{n}$,

$$\begin{aligned} \mathbf{D}_{12} \mathbf{T}_s\, v(\mathbf{n}) &\geq 0 \quad \Rightarrow \mathbf{D}_1(\mathbf{D}_2 \mathbf{T}_s\, v(\mathbf{n})) \geq 0 \\ \mathbf{T}_s\, v(\mathbf{n} + \mathbf{e}_1 + \mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}) &\geq \mathbf{T}_s\, v(\mathbf{n} + \mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n} + \mathbf{e}_2) \end{aligned} \tag{24}$$

Let $u_1$ and $u_2$ be the minimizers at $(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)$ and $\mathbf{n}$, respectively. Therefore, $u_1, u_2 \in \{0,1\}$ ($u_2 = 0$ if $n_1 = 0$) and

$$\mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) = w(u_1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) \quad \text{and} \quad \mathbf{T}_s\, v(\mathbf{n}) = w(u_2, \mathbf{n}). \tag{25}$$

We consider two cases: $(i)$ $u_1 \geq u_2$ and $(ii)$ $u_1 < u_2$.

**Case 1:** $u_1 \geq u_2$**,**

$$\begin{aligned}
\mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) &\leq w(u_1, \mathbf{n}+\mathbf{e}_1) + w(u_2, \mathbf{n}+\mathbf{e}_2) \qquad \text{by definition (23)}, \\
&= w(u_1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + w(u_2, \mathbf{n}) - \mathbf{D}_2 w(u_1, \mathbf{n}+\mathbf{e}_1) + \mathbf{D}_2 w(u_2, \mathbf{n}) \\
&\leq w(u_1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + w(u_2, \mathbf{n}) - \mathbf{D}_2 w(u_2, \mathbf{n}+\mathbf{e}_1) + \mathbf{D}_2 w(u_2, \mathbf{n}) \\
&\quad \text{as } \mathbf{D}_2 w(u, \mathbf{n}+\mathbf{e}_1) \text{ is increasing in } u \ (n_1 > 0 \text{ at } \mathbf{n}+\mathbf{e}_1), \\
&= w(u_1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + w(u_2, \mathbf{n}) - \mathbf{D}_{21} w(u_2, \mathbf{n}) \\
&\leq w(u_1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + w(u_2, \mathbf{n}) \\
&\quad \text{as for any } u,\ w \text{ is supermodular in } \mathbf{n}, \\
&= \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}) \qquad \text{by (25)}.
\end{aligned}$$

The proof above is still valid when $n_1 = 0$.

**Case 2:** $u_1 < u_2$**,**

As $u_1, u_2 \in \{0,1\}$, we have $u_1 = 0$ and $u_2 = 1$. Furthermore, $n_1 > 0$ as otherwise $u_2 = 0$ and this case does not exist.

$$\begin{aligned}
\mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) &\leq w(1, \mathbf{n}+\mathbf{e}_1) + w(0, \mathbf{n}+\mathbf{e}_2) \qquad \text{by definition (23)}, \\
&= 2v(\mathbf{n}) + C \qquad \text{by (17)}, \\
&= v(\mathbf{n}+\mathbf{e}_1) - \mathbf{D}_1 v(\mathbf{n}) + v(\mathbf{n}-\mathbf{e}_1) + \mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_1) + C \\
&= v(\mathbf{n}+\mathbf{e}_1) + v(\mathbf{n}-\mathbf{e}_1) + C - \mathbf{D}_{11} v(\mathbf{n}-\mathbf{e}_1) \\
&\leq v(\mathbf{n}+\mathbf{e}_1) + v(\mathbf{n}-\mathbf{e}_1) + C \qquad \text{as } v \text{ is convex, } \mathbf{D}_{11} v \geq 0, \\
&= \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}) \qquad \text{by (17) and (25)}.
\end{aligned}$$

**Diagonal dominance of $\mathbf{T}_s\, v$**

We need to show that for any $\mathbf{n}$, $\mathbf{D}_{11}\mathbf{T}_s\, v(\mathbf{n}) \geq \mathbf{D}_{12}\mathbf{T}_s\, v(\mathbf{n})$ and $\mathbf{D}_{22}\mathbf{T}_s\, v(\mathbf{n}) \geq \mathbf{D}_{21}\mathbf{T}_s\, v(\mathbf{n})$. This is equivalent to showing:

$$\begin{aligned}
\mathbf{D}_{22}\mathbf{T}_s\, v(\mathbf{n}) &\geq \mathbf{D}_{21}\mathbf{T}_s\, v(\mathbf{n}) \\
\mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) &\geq \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2)
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\mathbf{D}_{11}\mathbf{T}_s\, v(\mathbf{n}) &\geq \mathbf{D}_{12}\mathbf{T}_s\, v(\mathbf{n}) \\
\mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) &\geq \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1)
\end{aligned} \tag{27}$$

Verification of both (26) and (27) is similar to that of (24). We start by showing (26). Let $u_1$ and $u_2$ be the minimizers at $(\mathbf{n}+2\mathbf{e}_2)$ and $(\mathbf{n}+\mathbf{e}_1)$, respectively. Therefore, $u_1, u_2 \in \{0,1\}$ ($u_1 = 0$ if $n_1 = 0$) and

$$\mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_2) = w(u_1, \mathbf{n}+2\mathbf{e}_2) \quad \text{and} \quad \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) = w(u_2, \mathbf{n}+\mathbf{e}_1). \tag{28}$$

We consider two cases: $(i)$ $u_1 \geq u_2$ and $(ii)$ $u_1 < u_2$.

**Case 1:** $u_1 \geq u_2$**,**

$$\begin{aligned}
\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)+\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_2) &\leq w(u_2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)+w(u_1,\mathbf{n}+\mathbf{e}_2) \qquad \text{by definition (23)},\\
&= \mathbf{D}_2 w(u_2,\mathbf{n}+\mathbf{e}_1)+w(u_2,\mathbf{n}+\mathbf{e}_1)+w(u_1,\mathbf{n}+\mathbf{e}_2)\\
&\leq \mathbf{D}_2 w(u_2,\mathbf{n}+\mathbf{e}_2)+w(u_2,\mathbf{n}+\mathbf{e}_1)+w(u_1,\mathbf{n}+\mathbf{e}_2)\\
&\quad \text{as for any } u,\ w \text{ is supermodular in } \mathbf{n},\\
&\leq \mathbf{D}_2 w(u_1,\mathbf{n}+\mathbf{e}_2)+w(u_2,\mathbf{n}+\mathbf{e}_1)+w(u_1,\mathbf{n}+\mathbf{e}_2)\\
&\quad \text{as } \mathbf{D}_2 w(u,\mathbf{n}+\mathbf{e}_2) \text{ is increasing in } u \text{ when } n_1>0,\\
&= w(u_1,\mathbf{n}+2\mathbf{e}_2)+w(u_2,\mathbf{n}+\mathbf{e}_1)\\
&= \mathbf{T}_s v(\mathbf{n}+2\mathbf{e}_2)+\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_1) \qquad \text{by (28)}.
\end{aligned}$$

The proof above is still valid when $n_1 = 0$. To see this, note that when $n_1 = 0$, we have $u_1 = 0$ and $u_2 = 0$ (as $u_1 \geq u_2$ in this case). Therefore, $\mathbf{D}_2 w(u_2, \mathbf{n}+\mathbf{e}_2)$ is defined and $\mathbf{D}_2 w(u_2 = 0, \mathbf{n}+\mathbf{e}_2) = \mathbf{D}_2 w(u_1 = 0, \mathbf{n}+\mathbf{e}_2)$ when $n_1 = 0$.

**Case 2:** $u_1 < u_2$**,**

In this case, $u_1 = 0$ and $u_2 = 1$.

$$\begin{aligned}
\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)+\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_2) &\leq w(1,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2)+w(0,\mathbf{n}+\mathbf{e}_2) \qquad \text{by definition (23)},\\
&= v(\mathbf{n}+\mathbf{e}_2)+C+v(\mathbf{n}) \qquad \text{by (17)},\\
&= w(0,\mathbf{n}+2\mathbf{e}_2)+w(1,\mathbf{n}+\mathbf{e}_1) \qquad \text{by (17)},\\
&= \mathbf{T}_s v(\mathbf{n}+2\mathbf{e}_2)+\mathbf{T}_s v(\mathbf{n}+\mathbf{e}_1) \qquad \text{by (28)}.
\end{aligned}$$

The proof above is still valid when $n_1 = 0$, as we are using $u = 1$ only for states with $n_1 > 0$.

To verify (27), we take the function $w$ in (17) and redefine it by interchanging the roles of the indices $u = 0$ and $u = 1$. For any function $v \in \mathcal{U}$, let $\hat{w}$ be a function of $(u, \mathbf{n})$ with $\mathbf{n} \in \mathcal{S}$ and $u \in \{0, 1\}$ if $n_1 > 0$, and $u = 1$ if $n_1 = 0$, defined as follows

$$\begin{aligned}
\hat{w}(u,\mathbf{n}) &= u\Big[v(\mathbf{n}-\mathbf{e}_2\mathbf{I}_{(n_2>0)})+P_2\mathbf{I}_{(n_2=0)}\Big]+(1-u)\Big[v(\mathbf{n}-\mathbf{e}_1)+C\Big]\\
&= \begin{cases} v(\mathbf{n}-\mathbf{e}_2\mathbf{I}_{(n_2>0)})+P_2\mathbf{I}_{(n_2=0)} & : \text{ if } u=1,\\ v(\mathbf{n}-\mathbf{e}_1)+C & : \text{ if } u=0 \text{ and } n_1>0. \end{cases}
\end{aligned} \tag{29}$$

By the definition of the domain of $u$, when $n_1 = 0$, $u = 1$ is the only possibility and the function $\hat{w}$ becomes:

$$\hat{w}(1,0,n_2) = v(\mathbf{n}-\mathbf{e}_2\mathbf{I}_{(n_2>0)})+P_2\mathbf{I}_{(n_2=0)}$$

In equations (18) to (22), if we replace $u = 0$ with $u = 1$, and vice versa, they still hold for $\hat{w}$. Therefore, for any $u$, $v \in \mathcal{U}$ implies that $\hat{w} \in \mathcal{U}$. Furthermore, when $n_1 > 0$, $\hat{w}$ is supermodular in $(u, n_1)$ as $\mathbf{D}_1\hat{w}(u, \mathbf{n})$ is increasing in $u$. To see this, notice that the equivalent of (18) for $\hat{w}$ is:

$$\mathbf{D}_1\hat{w}(u,\mathbf{n}) = \begin{cases} \mathbf{D}_1 v(\mathbf{n}) & : \text{ if } u=1 \text{ and } n_2=0,\\ \mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_2) & : \text{ if } u=1 \text{ and } n_2>0,\\ \mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_1) & : \text{ if } u=0 \text{ and } n_1>0. \end{cases}$$

To see this, notice that when $n_1 > 0$,

$$\mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_1) \leq \mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_2) \quad \text{by } \mathbf{P2} \text{ and } \quad \mathbf{D}_1 v(\mathbf{n}-\mathbf{e}_1) \leq \mathbf{D}_1 v(\mathbf{n}) \quad \text{by } \mathbf{P3}.$$

For the function $\hat{w}$, we have

$$\mathbf{T}_s\, v(\mathbf{n}) = \min_u \hat{w}(u, \mathbf{n}). \tag{30}$$

To verify (27), let $u_1$ and $u_2$ be the minimizers at $(\mathbf{n}+2\mathbf{e}_1)$ and $(\mathbf{n}+\mathbf{e}_2)$, respectively. Therefore, $u_1, u_2 \in \{0,1\}$ ($u_2 = 1$ if $n_1 = 0$) and

$$\mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_1) = \hat{w}(u_1, \mathbf{n}+2\mathbf{e}_1) \quad \text{and} \quad \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) = \hat{w}(u_2, \mathbf{n}+\mathbf{e}_2). \tag{31}$$

We consider two cases: $(i)$ $u_1 \geq u_2$ and $(ii)$ $u_1 < u_2$.

**Case 1:** $u_1 \geq u_2$**,**

$$\begin{aligned}
\mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) &\leq \hat{w}(u_2, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \hat{w}(u_1, \mathbf{n}+\mathbf{e}_1) \qquad \text{by definition (30)},\\
&= \mathbf{D}_1\hat{w}(u_2, \mathbf{n}+\mathbf{e}_2) + \hat{w}(u_2, \mathbf{n}+\mathbf{e}_2) + \hat{w}(u_1, \mathbf{n}+\mathbf{e}_1)\\
&\leq \mathbf{D}_1\hat{w}(u_2, \mathbf{n}+\mathbf{e}_1) + \hat{w}(u_2, \mathbf{n}+\mathbf{e}_2) + \hat{w}(u_1, \mathbf{n}+\mathbf{e}_1)\\
&\phantom{=}\ \text{as for any } u,\ \hat{w} \text{ is supermodular in } \mathbf{n},\\
&\leq \mathbf{D}_1\hat{w}(u_1, \mathbf{n}+\mathbf{e}_1) + \hat{w}(u_2, \mathbf{n}+\mathbf{e}_2) + \hat{w}(u_1, \mathbf{n}+\mathbf{e}_1)\\
&\phantom{=}\ \text{as } \mathbf{D}_1\hat{w}(u, \mathbf{n}+\mathbf{e}_1) \text{ is increasing in } u \text{ and } n_1 > 0 \text{ at } \mathbf{n}+\mathbf{e}_1,\\
&= \hat{w}(u_1, \mathbf{n}+2\mathbf{e}_1) + \hat{w}(u_2, \mathbf{n}+\mathbf{e}_2)\\
&= \mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) \qquad \text{by (31)}.
\end{aligned}$$

The proof above is still valid when $n_1 = 0$, as $u_1 = u_2 = 1$ in this case and we have $\mathbf{D}_1\hat{w}(u_2 = 1, \mathbf{n}+\mathbf{e}_1) = \mathbf{D}_1\hat{w}(u_1 = 1, \mathbf{n}+\mathbf{e}_1)$.

**Case 2:** $u_1 < u_2$**,**

In this case, $u_1 = 0$ and $u_2 = 1$.

$$\begin{aligned}
\mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) &\leq \hat{w}(1, \mathbf{n}+\mathbf{e}_1+\mathbf{e}_2) + \hat{w}(0, \mathbf{n}+\mathbf{e}_1) \qquad \text{by definition (30)},\\
&= v(\mathbf{n}+\mathbf{e}_1) + C + v(\mathbf{n}) \qquad \text{by (29)},\\
&= \hat{w}(0, \mathbf{n}+2\mathbf{e}_1) + \hat{w}(1, \mathbf{n}+\mathbf{e}_2) \qquad \text{by (29)},\\
&= \mathbf{T}_s\, v(\mathbf{n}+2\mathbf{e}_1) + \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) \qquad \text{by (31)}.
\end{aligned}$$

The proof above is still valid when $n_1 = 0$, as we are using $u = 0$ only for states with $n_1 > 0$. This concludes our proof for the diagonal dominance of $\mathbf{T}_s\, v$.

**Lower bound of $\mathbf{T}_s\, v$**

The proof is similar to the lower bound proof of $\mathbf{T}_p\, v$. We need to show that for any $\mathbf{n}$, $\mathbf{D}_1\mathbf{T}_s\, v(\mathbf{n}) \geq -P_1$ and $\mathbf{D}_2\mathbf{T}_s\, v(\mathbf{n}) \geq -P_2$. Let $u_1$ and $u_2$ be the minimizers at $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_1$, respectively. Therefore, $u_1, u_2 \in \{0,1\}$ ($u_1 = 0$ if $n_1 = 0$) and,

$$\mathbf{T}_s\, v(\mathbf{n}) = w(u_1, \mathbf{n}) \quad \text{and} \quad \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_1) = w(u_2, \mathbf{n}+\mathbf{e}_1).$$

We have,

$$\begin{aligned}
\mathbf{D}_1\mathbf{T}_s\, v(\mathbf{n}) &= w(u_2, \mathbf{n}+\mathbf{e}_1) - w(u_1, \mathbf{n}) \geq w(u_1, \mathbf{n}+\mathbf{e}_1) - w(u_1, \mathbf{n})\\
&\phantom{=}\ \text{as } u_2 \text{ is the minimizer at } \mathbf{n}+\mathbf{e}_1,\\
&= \mathbf{D}_1 w(u_1, \mathbf{n}) \geq -P_1.
\end{aligned}$$

Similarly, let $u_1$ and $u_2$ be the optimal indices at $\mathbf{n}$ and $\mathbf{n}+\mathbf{e}_2$, respectively. Therefore, $u_1, u_2 \in \{0,1\}$ ($u_1 = u_2 = 0$ if $n_1 = 0$) and,

$$\mathbf{T}_s\, v(\mathbf{n}) = w(u_1, \mathbf{n}) \quad \text{and} \quad \mathbf{T}_s\, v(\mathbf{n}+\mathbf{e}_2) = w(u_2, \mathbf{n}+\mathbf{e}_2).$$

We have,

$$\begin{aligned} \mathbf{D}_2\mathbf{T}_s\, v(\mathbf{n}) &= w(u_2, \mathbf{n}+\mathbf{e}_2) - w(u_1, \mathbf{n}) \geq w(u_2, \mathbf{n}+\mathbf{e}_2) - w(u_2, \mathbf{n}) \\ &\quad \text{as } u_1 \text{ is the minimizer at } \mathbf{n}, \\ &= \mathbf{D}_2 w(u_2, \mathbf{n}) \geq -P_2. \end{aligned}$$

The proof still holds when $n_1 = 0$, as we have $u_1 = u_2 = 0$ in that case.

**Proof for $\mathbf{T}\, v \in \mathcal{U}$:**

We have shown that, if $v \in \mathcal{U}$, then $\mathbf{T}_p\, v \in \mathcal{U}$ and $\mathbf{T}_s\, v \in \mathcal{U}$. We conclude the proof by showing that, if $v \in \mathcal{U}$, then we also have $\mathbf{T}\, v \in \mathcal{U}$. Now define

$$g(\mathbf{n}) = v(\mathbf{n} - \mathbf{e}_1 \mathbf{I}_{(n_1>0)}) + P_1 \mathbf{I}_{(n_1=0)}$$

then we have,

$$\mathbf{T}\, v(\mathbf{n}) = h(\mathbf{n}) + \lambda_1 g(\mathbf{n}) + \lambda_2 \mathbf{T}_s\, v(\mathbf{n}) + \mu \mathbf{T}_p\, v(\mathbf{n}).$$

We show that, if $v \in \mathcal{U}$ then $g \in \mathcal{U}$. Basically, $g(\mathbf{n})$ is $w(0, \mathbf{n})$ (as defined in (17)) with the indices 2 replaced by 1. As $v$ has the properties **P1-P4** jointly for both dimensions, the result follows. To see this, note that

$$\mathbf{D}_1 g(\mathbf{n}) = \begin{cases} -P_1 & : \text{ if } n_1 = 0, \\ \mathbf{D}_1 v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } n_1 > 0. \end{cases} \qquad \mathbf{D}_2 g(\mathbf{n}) = \begin{cases} \mathbf{D}_2 v(\mathbf{n}) & : \text{ if } n_1 = 0, \\ \mathbf{D}_2 v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } n_1 > 0. \end{cases}$$

$$\mathbf{D}_{11} g(\mathbf{n}) = \begin{cases} \mathbf{D}_1 v(\mathbf{n}) + P_1 & : \text{ if } n_1 = 0, \\ \mathbf{D}_{11} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } n_1 > 0. \end{cases} \qquad \mathbf{D}_{22} g(\mathbf{n}) = \begin{cases} \mathbf{D}_{22} v(\mathbf{n}) & : \text{ if } n_1 = 0, \\ \mathbf{D}_{22} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } n_1 > 0. \end{cases}$$

$$\mathbf{D}_{12} g(\mathbf{n}) = \begin{cases} 0 & : \text{ if } n_1 = 0, \\ \mathbf{D}_{12} v(\mathbf{n} - \mathbf{e}_1) & : \text{ if } n_1 > 0. \end{cases}$$

$\mathcal{U}$ is closed under addition and multiplication by nonnegative scalars. Therefore, for any function $h \in \mathcal{U}$, we have $\mathbf{T}\, v \in \mathcal{U}$ whenever $v \in \mathcal{U}$. A linear function $h$ with nonnegative constants is an element of $\mathcal{U}$, as $h(\mathbf{n}) = h_1 n_1 + h_2 n_2$, with $h_1, h_2 \geq 0$, is supermodular, has diagonal dominance, and has a lower bound of zero. ■

**PROOF OF THEOREM 1:**

All costs are nonnegative and the set of feasible actions is finite for every state $\mathbf{n} \in \mathcal{S}$. Therefore, by Proposition 11 in Section 5.4 of Bertsekas (1987), there exists an optimal stationary policy.

*Proof of (i):*

Recall that, by (8)

$$\begin{aligned} \mathbf{T}_p V_\alpha(\mathbf{n}) &= \min\Big\{V_\alpha(\mathbf{n}), V_\alpha(\mathbf{n}+\mathbf{e}_1), V_\alpha(\mathbf{n}+\mathbf{e}_2)\Big\} \\ &= \min\Big\{0, \mathbf{D}_1 V_\alpha(\mathbf{n}), \mathbf{D}_2 V_\alpha(\mathbf{n})\Big\} + V_\alpha(\mathbf{n}) \end{aligned} \tag{32}$$

Therefore, if idling is optimal at state $\mathbf{n}$, we have:

$$\min\Big\{0, \mathbf{D}_1 V_\alpha(\mathbf{n}), \mathbf{D}_2 V_\alpha(\mathbf{n})\Big\} = 0 \quad \Leftrightarrow \quad \mathbf{D}_1 V_\alpha(\mathbf{n}) \geq 0 \text{ and } \mathbf{D}_2 V_\alpha(\mathbf{n}) \geq 0. \tag{33}$$

As $V_\alpha \in \mathcal{U}$, $V_\alpha$ satisfies **P1** to **P4**. We have,

$$\begin{aligned} &(33) \text{ and } \mathbf{P3} \quad \Rightarrow \quad \mathbf{D}_1 V_\alpha(\mathbf{n}+\mathbf{e}_1) \geq \mathbf{D}_1 V_\alpha(\mathbf{n}) \geq 0 \\ &(33) \text{ and } \mathbf{P1} \quad \Rightarrow \quad \mathbf{D}_2 V_\alpha(\mathbf{n}+\mathbf{e}_1) \geq \mathbf{D}_2 V_\alpha(\mathbf{n}) \geq 0. \end{aligned}$$

Therefore, idling is optimal at state $\mathbf{n}+\mathbf{e}_1$. Similarly, idling is also optimal at state $\mathbf{n}+\mathbf{e}_2$ because

$$\begin{aligned} &(33) \text{ and } \mathbf{P1} \quad \Rightarrow \quad \mathbf{D}_1 V_\alpha(\mathbf{n}+\mathbf{e}_2) \geq \mathbf{D}_1 V_\alpha(\mathbf{n}) \geq 0 \\ &(33) \text{ and } \mathbf{P3} \quad \Rightarrow \quad \mathbf{D}_2 V_\alpha(\mathbf{n}+\mathbf{e}_2) \geq \mathbf{D}_2 V_\alpha(\mathbf{n}) \geq 0. \end{aligned}$$

*Proof of (ii) and (iii):*

If producing item 1 is optimal at state $\mathbf{n}$, then by (32):

$$\min\Big\{0, \mathbf{D}_1 V_\alpha(\mathbf{n}), \mathbf{D}_2 V_\alpha(\mathbf{n})\Big\} = \mathbf{D}_1 V_\alpha(\mathbf{n}) \quad \Rightarrow \quad \mathbf{D}_1 V_\alpha(\mathbf{n}) \leq \mathbf{D}_2 V_\alpha(\mathbf{n}). \tag{34}$$

Note that, idling can not be optimal at $\mathbf{n}-\mathbf{e}_1$ by Theorem 1, Part $(ii)$. Hence, producing item 1 is optimal at state $\mathbf{n}-\mathbf{e}_1$ if and only if the following holds:

$$\mathbf{D}_1 V_\alpha(\mathbf{n}-\mathbf{e}_1) \leq \mathbf{D}_2 V_\alpha(\mathbf{n}-\mathbf{e}_1) \tag{35}$$

By using $\mathbf{D}_{12} V_\alpha(\mathbf{n}) = \mathbf{D}_{21} V_\alpha(\mathbf{n})$ and diagonal dominance, **P2**, we obtain the following inequality:

$$\mathbf{D}_1 V_\alpha(\mathbf{n}) - \mathbf{D}_2 V_\alpha(\mathbf{n}) \leq \mathbf{D}_1 V_\alpha(\mathbf{n}+\mathbf{e}_1) - \mathbf{D}_2 V_\alpha(\mathbf{n}+\mathbf{e}_1) \quad \forall\, \mathbf{n} \in \mathcal{S}. \tag{36}$$

Replacing $\mathbf{n}$ by $\mathbf{n}-\mathbf{e}_1$ in (36) and using (34), we show that (35) holds

$$\mathbf{D}_1 V_\alpha(\mathbf{n}-\mathbf{e}_1) - \mathbf{D}_2 V_\alpha(\mathbf{n}-\mathbf{e}_1) \leq \mathbf{D}_1 V_\alpha(\mathbf{n}) - \mathbf{D}_2 V_\alpha(\mathbf{n}) \leq 0.$$

Therefore, if it is optimal to produce item 1 at $\mathbf{n}$ with $n_1 > 0$, then it is also optimal to produce item 1 at $\mathbf{n}-\mathbf{e}_1$.

We now show that, if producing item 2 is optimal at state $\mathbf{n}$, then it is also optimal to produce item 2 at state $\mathbf{n}-\mathbf{e}_2$. If producing item 2 is optimal at state $\mathbf{n}$, then:

$$\min\Big\{0, \mathbf{D}_1 V_\alpha(\mathbf{n}), \mathbf{D}_2 V_\alpha(\mathbf{n})\Big\} = \mathbf{D}_2 V_\alpha(\mathbf{n}) \quad \Rightarrow \quad \mathbf{D}_2 V_\alpha(\mathbf{n}) \leq \mathbf{D}_1 V_\alpha(\mathbf{n}). \tag{37}$$

Note that, idling can not be optimal at $\mathbf{n}-\mathbf{e}_2$ by Theorem 1, Part $(ii)$. Hence, producing item 2 is optimal at state $\mathbf{n}-\mathbf{e}_2$ if and only if the following holds:

$$\mathbf{D}_2 V_\alpha(\mathbf{n}-\mathbf{e}_2) \leq \mathbf{D}_1 V_\alpha(\mathbf{n}-\mathbf{e}_2) \tag{38}$$

By using $\mathbf{D}_{12}V_\alpha(\mathbf{n}) = \mathbf{D}_{21}V_\alpha(\mathbf{n})$ and diagonal dominance, **P2**, we obtain the following inequality:

$$\mathbf{D}_1V_\alpha(\mathbf{n}) - \mathbf{D}_2V_\alpha(\mathbf{n}) \geq \mathbf{D}_1V_\alpha(\mathbf{n}+\mathbf{e}_2) - \mathbf{D}_2V_\alpha(\mathbf{n}+\mathbf{e}_2) \quad \forall\ \mathbf{n} \in \mathcal{S}. \tag{39}$$

Replacing $\mathbf{n}$ by $\mathbf{n}-\mathbf{e}_2$ in (39) and using (37), we show that (38) holds:

$$\mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_2) - \mathbf{D}_2V_\alpha(\mathbf{n}-\mathbf{e}_2) \geq \mathbf{D}_1V_\alpha(\mathbf{n}) - \mathbf{D}_2V_\alpha(\mathbf{n}) \geq 0.$$

Therefore, if it is optimal to produce item 2 at $\mathbf{n}$ with $n_2 > 0$, then it is also optimal to produce item 2 at $\mathbf{n}-\mathbf{e}_2$.

We now go back to our statement about item 1. We have shown that if it is optimal to produce item 1 at $\mathbf{n}$ with $n_1 > 0$, then it is also optimal to produce item 1 at $\mathbf{n}-\mathbf{e}_1$. Furthermore, in this case, producing item 2 at $\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2$ can not be optimal. Because, we have also shown that if it is optimal to produce item 2 at $\mathbf{n}$ with $n_2 > 0$, then it is also optimal to produce item 2 at $\mathbf{n}-\mathbf{e}_2$. Hence, to prove that producing item 1 is optimal at $\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2$ whenever it is optimal at $\mathbf{n}$, it suffices to show that

$$\mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2) \leq 0 \tag{40}$$

As producing item 1 is optimal at $\mathbf{n}$, we know that $\mathbf{D}_1V_\alpha(\mathbf{n}) \leq 0$. Combining this with diagonal dominance **P2** applied to state $\mathbf{n}-\mathbf{e_1}$, we obtain (40) as follows:

$$\mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_1+\mathbf{e}_2) \leq \mathbf{D}_1V_\alpha(\mathbf{n}) \leq 0$$

We now go back to our statement about item 2. We have shown that if it is optimal to produce item 2 at $\mathbf{n}$ with $n_2 > 0$, then it is also optimal to produce item 2 at $\mathbf{n}-\mathbf{e}_2$. Furthermore, in this case, producing item 1 at $\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2$ can not be optimal. Because, we have also shown that if it is optimal to produce item 1 at $\mathbf{n}$ with $n_1 > 0$, then it is also optimal to produce item 1 at $\mathbf{n}-\mathbf{e}_1$. Hence, to prove that producing item 2 is optimal at $\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2$ whenever it is optimal at $\mathbf{n}$, it suffices to show that

$$\mathbf{D}_2V_\alpha(\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2) \leq 0 \tag{41}$$

As producing item 2 is optimal at $\mathbf{n}$, we know that $\mathbf{D}_2V_\alpha(\mathbf{n}) \leq 0$. Combining this with diagonal dominance **P2** applied to state $\mathbf{n}-\mathbf{e_2}$, we obtain (41) as follows:

$$\mathbf{D}_2V_\alpha(\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2) \leq \mathbf{D}_2V_\alpha(\mathbf{n}) \leq 0$$

*Proof of (iv):*

If upgrading is optimal at state $\mathbf{n}$ with $n_1 > 0$, then:

$$V_\alpha(\mathbf{n}) + P_2 \geq V_\alpha(\mathbf{n}-\mathbf{e_1}) + C \Rightarrow C - P_2 \leq \mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_1) \quad \text{if } n_2 = 0, \tag{42}$$

$$V_\alpha(\mathbf{n}-\mathbf{e_2}) \geq V_\alpha(\mathbf{n}-\mathbf{e_1}) + C \quad \text{if } n_2 > 0. \tag{43}$$

At state $\mathbf{n}+\mathbf{e}_1$, we need to show that:

$$C - P_2 \leq \mathbf{D}_1V_\alpha(\mathbf{n}) \quad \text{if } n_2 = 0, \tag{44}$$

$$V_\alpha(\mathbf{n}+\mathbf{e}_1-\mathbf{e_2}) \geq V_\alpha(\mathbf{n}) + C \quad \text{if } n_2 > 0. \tag{45}$$

(42) and **P3** imply (44). **P2** applied to state $\mathbf{n}-\mathbf{e_1}-\mathbf{e_2}$ and (43) give (45) as follows:

$$\begin{aligned}\mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_1) \leq \mathbf{D}_1V_\alpha(\mathbf{n}-\mathbf{e}_2) \quad &\Rightarrow \quad V_\alpha(\mathbf{n}) - V_\alpha(\mathbf{n}-\mathbf{e_1}) \leq V_\alpha(\mathbf{n}+\mathbf{e}_1-\mathbf{e_2}) - V_\alpha(\mathbf{n}-\mathbf{e_2}) \\ &\Rightarrow \quad V_\alpha(\mathbf{n}+\mathbf{e}_1-\mathbf{e_2}) - V_\alpha(\mathbf{n}) \geq V_\alpha(\mathbf{n}-\mathbf{e_2}) - V_\alpha(\mathbf{n}-\mathbf{e_1}) \geq C.\end{aligned}$$

We are left with showing that if upgrading is optimal at state $\mathbf{n}$, then it is also optimal at state $\mathbf{n} - \mathbf{e}_2$. To prove this, we will show that whenever upgrading is not optimal at a state $\mathbf{n}$, it is also not optimal at state $\mathbf{n} + \mathbf{e}_2$. If upgrading is not optimal at state $\mathbf{n}$, then:

$$V_\alpha(\mathbf{n}) + P_2 \leq V_\alpha(\mathbf{n} - \mathbf{e_1}) + C \Rightarrow C - P_2 \geq \mathbf{D}_1 V_\alpha(\mathbf{n} - \mathbf{e}_1) \quad \text{if } n_2 = 0, \tag{46}$$

$$V_\alpha(\mathbf{n} - \mathbf{e_2}) \leq V_\alpha(\mathbf{n} - \mathbf{e_1}) + C \quad \text{if } n_2 > 0. \tag{47}$$

At state $\mathbf{n} + \mathbf{e}_2$, we need to show that:

$$V_\alpha(\mathbf{n}) \leq V_\alpha(\mathbf{n} - \mathbf{e}_1 + \mathbf{e_2}) + C \tag{48}$$

If $n_2 = 0$, then (46) and **P4** applied to state $\mathbf{n} - \mathbf{e}_1$ give (48) as follows:

$$\begin{aligned} & \mathbf{D}_1 V_\alpha(\mathbf{n} - \mathbf{e}_1) \leq C - P_2 \leq C + \mathbf{D}_2 V_\alpha(\mathbf{n} - \mathbf{e}_1) \\ \Rightarrow \quad & \mathbf{D}_1 V_\alpha(\mathbf{n} - \mathbf{e}_1) - \mathbf{D}_2 V_\alpha(\mathbf{n} - \mathbf{e}_1) \leq C \Rightarrow V_\alpha(\mathbf{n}) \leq V_\alpha(\mathbf{n} - \mathbf{e}_1 + \mathbf{e_2}) + C. \end{aligned}$$

If $n_2 > 0$, then **P2** applied to state $\mathbf{n} - \mathbf{e}_1 - \mathbf{e}_2$ and (47) give (48) as follows:

$$\begin{aligned} & \mathbf{D}_2 V_\alpha(\mathbf{n} - \mathbf{e}_2) \leq \mathbf{D}_2 V_\alpha(\mathbf{n} - \mathbf{e}_1) \Rightarrow V_\alpha(\mathbf{n}) - V_\alpha(\mathbf{n} - \mathbf{e}_1 + \mathbf{e}_2) \leq V_\alpha(\mathbf{n} - \mathbf{e}_2) - V_\alpha(\mathbf{n} - \mathbf{e}_1) \\ \text{and} \quad & V_\alpha(\mathbf{n} - \mathbf{e}_2) - V_\alpha(\mathbf{n} - \mathbf{e}_1) \leq C \Rightarrow V_\alpha(\mathbf{n}) - V_\alpha(\mathbf{n} - \mathbf{e}_1 + \mathbf{e}_2) \leq C \quad \blacksquare \end{aligned}$$

# ONLINE APPENDIX B

**PROOF OF PROPOSITION 2:**

The balance equations of the proposed policy when $0 < B_1 < A_2 \leq A_1 \leq A_2 + A_3$ and $0 \leq A_3 \leq B_2$ are:

$$\mu_1 \pi_{0,0} = \lambda_1 \pi_{1,0} + \lambda_2 \pi_{0,1} \tag{49}$$

$$(\mu_1 + \lambda_1)\pi_{i,0} = \lambda_1 \pi_{i+1,0} + \lambda_2 \pi_{i,1} + \mu_1 \pi_{i-1,0} \qquad i = 1,..,B_1 - 1 \text{ (if } B_1 > 1) \tag{50}$$

$$(\mu_1 + \lambda_2)\pi_{0,j} = \lambda_1 \pi_{1,j} + \mathbf{I}_{(j \leq A_3)} \lambda_2 \pi_{0,j+1} \qquad j = 1,..,A_3 + 1 \tag{51}$$

$$(\mu_1 + \lambda_1 + \lambda_2)\pi_{i,j} = \lambda_1 \pi_{i+1,j} + \mathbf{I}_{(j \leq A_3)} \lambda_2 \pi_{i,j+1} + \mu_1 \pi_{i-1,j} \qquad i = 1,..,B_1 - 1, \; j = 1,..,A_3 + 1 \text{ (if } B_1 > 1) \tag{52}$$

$$(\mu_1 + \lambda_1)\pi_{B_1,0} = (\lambda_1 + \lambda_2)\pi_{B_1+1,0} + \lambda_2 \pi_{B_1,1} + \mu_1 \pi_{B_1-1,0} \tag{53}$$

$$(\mu_1 + \lambda_1 + \lambda_2)\pi_{B_1,j} = (\lambda_1 + \lambda_2)\pi_{B_1+1,j} + \mathbf{I}_{(j \leq A_3)} \lambda_2 \pi_{B_1,j+1} + \mu_1 \pi_{B_1-1,j} \qquad j = 1,..,A_3 + 1 \tag{54}$$

$$(\mu_1 + \lambda_1 + \lambda_2)\pi_{i,j} = (\lambda_1 + \lambda_2)\pi_{i+1,j} + \mu_1 \pi_{i-1,j} \qquad i = B_1 + 1,..,A_2 - 1, \; j = 0,..,A_3 + 1 \text{ (if } B_1 + 1 < A_2) \tag{55}$$

$$(\mu_2 + \lambda_1 + \lambda_2)\pi_{A_2,j} = \mu_1 \pi_{A_2-1,j} + \mathbf{I}_{(j \geq 1)} \mu_2 \pi_{A_2,j-1} \qquad j = 0,..,A_3 \tag{56}$$

$$(\lambda_1 + \lambda_2)\pi_{A_2,A_3+1} = \mu_1 \pi_{A_2-1,A_3+1} + \mu_2 \pi_{A_2,A_3}. \tag{57}$$

Note that (50) and (52) exist only when $B_1 > 1$. Similarly, (55) exists only when $B_1 + 1 < A_2$. If we multiply both sides of each equation related to state $(i,j)$ with $v^i$ and $w^j$, and add up the similar equations (e.g., we sum (50) over $i = 1, .., B_1 - 1$), we obtain the following:

$$
\begin{aligned}
\mu_1 \pi_{0,0} &= \lambda_1 \pi_{1,0} + \lambda_2 \pi_{0,1} \\
(\mu_1 + \lambda_1) \sum_{i=1}^{B_1-1} v^i \pi_{i,0} &= \lambda_1 \sum_{i=1}^{B_1-1} v^i \pi_{i+1,0} + \lambda_2 \sum_{i=1}^{B_1-1} v^i \pi_{i,1} + \mu_1 \sum_{i=1}^{B_1-1} v^i \pi_{i-1,0} \\
&\quad (\text{if } B_1 > 1) \\
(\mu_1 + \lambda_2) \sum_{j=1}^{A_3+1} w^j \pi_{0,j} &= \lambda_1 \sum_{j=1}^{A_3+1} w^j \pi_{1,j} + \lambda_2 \sum_{j=1}^{A_3} w^j \pi_{0,j+1} \\
(\mu_1 + \lambda_1 + \lambda_2) \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^i w^j \pi_{i,j} &= \lambda_1 \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^i w^j \pi_{i+1,j} + \lambda_2 \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3} v^i w^j \pi_{i,j+1} \\
&\quad + \mu_1 \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^i w^j \pi_{i-1,j} \quad (\text{if } B_1 > 1) \\
(\mu_1 + \lambda_1) v^{B_1} \pi_{B_1,0} &= (\lambda_1 + \lambda_2) v^{B_1} \pi_{B_1+1,0} + \lambda_2 v^{B_1} \pi_{B_1,1} + \mu_1 v^{B_1} \pi_{B_1-1,0} \\
(\mu_1 + \lambda_1 + \lambda_2) \sum_{j=1}^{A_3+1} v^{B_1} w^j \pi_{B_1,j} &= (\lambda_1 + \lambda_2) \sum_{j=1}^{A_3+1} v^{B_1} w^j \pi_{B_1+1,j} + \lambda_2 \sum_{j=1}^{A_3} v^{B_1} w^j \pi_{B_1,j+1} \\
&\quad + \mu_1 \sum_{j=1}^{A_3+1} v^{B_1} w^j \pi_{B_1-1,j} \\
(\mu_1 + \lambda_1 + \lambda_2) \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j} &= (\lambda_1 + \lambda_2) \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i+1,j} \\
&\quad + \mu_1 \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i-1,j} \quad (\text{if } B_1 + 1 < A_2) \\
(\mu_2 + \lambda_1 + \lambda_2) \sum_{j=0}^{A_3} v^{A_2} w^j \pi_{A_2,j} &= \mu_1 \sum_{j=0}^{A_3} v^{A_2} w^j \pi_{A_2-1,j} + \mu_2 \sum_{j=1}^{A_3} v^{A_2} w^j \pi_{A_2,j-1} \\
(\lambda_1 + \lambda_2) v^{A_2} w^{A_3+1} \pi_{A_2,A_3+1} &= \mu_1 v^{A_2} w^{A_3+1} \pi_{A_2-1,A_3+1} + \mu_2 v^{A_2} w^{A_3+1} \pi_{A_2,A_3}.
\end{aligned}
$$

Then, if necessary, we add terms to both sides of each equation to make the multiplier of left-hand sides of each equation equal to $(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)$. Furthermore, we arrange the terms in right-hand side so that $\pi_{i,j}$, the steady-state probability of state $(i, j)$, is multiplied with $v^i w^j$.

Let $\Lambda = \mu_1 + \mu_2 + \lambda_1 + \lambda_2$. Then, we obtain:

$$\begin{aligned}
\Lambda \pi_{0,0} &= (\lambda_1+\lambda_2)\pi_{0,0} + (\frac{\lambda_1}{v})v\pi_{1,0} + (\frac{\lambda_2}{w})w\pi_{0,1} + \mu_2\pi_{0,0} \\
\Lambda \sum_{i=1}^{B_1-1} v^i \pi_{i,0} &= \lambda_2 \sum_{i=1}^{B_1-1} v^i\pi_{i,0} + (\frac{\lambda_1}{v}) \sum_{i=1}^{B_1-1} v^{i+1}\pi_{i+1,0} + (\frac{\lambda_2}{w}) \sum_{i=1}^{B_1-1} v^i w \pi_{i,1} \\
&\quad +(\mu_1 v) \sum_{i=1}^{B_1-1} v^{i-1}\pi_{i-1,0} + \mu_2 \sum_{i=1}^{B_1-1} v^i\pi_{i,0} \quad (\text{if } B_1 > 1) \\
\Lambda \sum_{j=1}^{A_3+1} w^j \pi_{0,j} &= \lambda_1 \sum_{j=1}^{A_3+1} w^j\pi_{0,j} + (\frac{\lambda_1}{v}) \sum_{j=1}^{A_3+1} v w^j \pi_{1,j} + (\frac{\lambda_2}{w}) \sum_{j=1}^{A_3} w^{j+1}\pi_{0,j+1} \\
&\quad +\mu_2 \sum_{j=1}^{A_3+1} w^j \pi_{0,j} \\
\Lambda \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^i w^j \pi_{i,j} &= (\frac{\lambda_1}{v}) \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^{i+1} w^j \pi_{i+1,j} + (\frac{\lambda_2}{w}) \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3} v^i w^{j+1}\pi_{i,j+1} \\
&\quad +(\mu_1 v) \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^{i-1} w^j \pi_{i-1,j} + \mu_2 \sum_{i=1}^{B_1-1} \sum_{j=1}^{A_3+1} v^i w^j \pi_{i,j} \\
&\quad (\text{if } B_1 > 1) \\
\Lambda v^{B_1} \pi_{B_1,0} &= \lambda_2 v^{B_1} \pi_{B_1,0} + (\frac{\lambda_1+\lambda_2}{v}) v^{B_1+1} \pi_{B_1+1,0} + (\frac{\lambda_2}{w}) v^{B_1} w \pi_{B_1,1} \\
&\quad +(\mu_1 v) v^{B_1-1} \pi_{B_1-1,0} + \mu_2 v^{B_1}\pi_{B_1,0} \\
\Lambda \sum_{j=1}^{A_3+1} v^{B_1} w^j \pi_{B_1,j} &= (\frac{\lambda_1+\lambda_2}{v}) \sum_{j=1}^{A_3+1} v^{B_1+1} w^j \pi_{B_1+1,j} + (\frac{\lambda_2}{w}) \sum_{j=1}^{A_3} v^{B_1} w^{j+1} \pi_{B_1,j+1} \\
&\quad +(\mu_1 v) \sum_{j=1}^{A_3+1} v^{B_1-1} w^j \pi_{B_1-1,j} + \mu_2 \sum_{j=1}^{A_3+1} v^{B_1} w^j \pi_{B_1,j} \\
\Lambda \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j} &= (\frac{\lambda_1+\lambda_2}{v}) \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^{i+1} w^j \pi_{i+1,j} \\
&\quad +(\mu_1 v) \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^{i-1} w^j \pi_{i-1,j} + \mu_2 \sum_{i=B_1+1}^{A_2-1} \sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j} \\
&\quad (\text{if } B_1 + 1 < A_2) \\
\Lambda \sum_{j=0}^{A_3} v^{A_2} w^j \pi_{A_2,j} &= (\mu_1 v) \sum_{j=0}^{A_3} v^{A_2-1} w^j \pi_{A_2-1,j} + (\mu_2 w) \sum_{j=1}^{A_3} v^{A_2} w^{j-1} \pi_{A_2,j-1} \\
&\quad +\mu_1 \sum_{j=0}^{A_3} v^{A_2} w^j \pi_{A_2,j} \\
\Lambda v^{A_2} w^{A_3+1} \pi_{A_2,A_3+1} &= \mu_1 v^{A_2} w^{A_3+1} \pi_{A_2,A_3+1} + (\mu_1 v) v^{A_2-1} w^{A_3+1} \pi_{A_2-1,A_3+1} \\
&\quad +(\mu_2 w) v^{A_2} w^{A_3} \pi_{A_2,A_3} + \mu_2 v^{A_2} w^{A_3+1} \pi_{A_2,A_3+1}.
\end{aligned}$$

We add all the equations and obtain a single equation. By the definition of the generating function, left-hand side of the equations add up to $\Lambda\mathbf{\Pi}(v,w) = (\mu_1+\mu_2+\lambda_1+\lambda_2)\mathbf{\Pi}(v,w)$

$$\begin{aligned}(\mu_1+\mu_2+\lambda_1+\lambda_2)\mathbf{\Pi}(v,w) &= \lambda_1\sum_{j=0}^{A_3+1} w^j\pi_{0,j} + (\frac{\lambda_1}{v})\sum_{i=1}^{A_2}\sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j} + \lambda_2\sum_{i=0}^{B_1} v^i\pi_{i,0} \quad (58)\\ &+(\frac{\lambda_2}{w})\sum_{i=0}^{B_1}\sum_{j=1}^{A_3+1} v^i w^j\pi_{i,j} + (\frac{\lambda_2}{v})\sum_{i=B_1+1}^{A_2}\sum_{j=0}^{A_3+1} v^i w^j \pi_{i,j}\\ &+\mu_1\sum_{j=0}^{A_3+1} v^{A_2}w^j\pi_{A_2,j} + (\mu_1 v)\sum_{i=0}^{A_2-1}\sum_{j=0}^{A_3+1} v^i w^j\pi_{i,j}\\ &+\mu_2\big(\Pi(v,w) - \sum_{j=0}^{A_3} v^{A_2}w^j\pi_{A_2,j}\big) + (\mu_2 w)\sum_{j=0}^{A_3} v^{A_2}w^j\pi_{A_2,j}.\end{aligned}$$

We now use $\mathcal{N}_k(v,w)$, defined by (11), in (58) and obtain

$$\begin{aligned}(\mu_1(1-v)+\lambda_1(1-\frac{1}{v})+\lambda_2(1-\frac{1}{w}))\mathbf{\Pi}(v,w) &= \lambda_1(1-\frac{1}{v})\mathcal{N}_1(v,w)+\lambda_2(1-\frac{1}{w})\mathcal{N}_2(v,w)\\ &\lambda_2(\frac{1}{v}-\frac{1}{w})\mathcal{N}_3(v,w)+\mu_2(w-1)\mathcal{N}_4(v,w)\\ &+\mu_1(1-v)\big(\mathcal{N}_4(v,w)+v^{A_2}w^{A_3+1}\pi_{A_2,A_3+1}\big).\end{aligned}$$

After multiplying both sides with $vw$, we obtain the generating function presented in Proposition 2:

$$\begin{aligned}\mathbf{\Pi}(v,w) &= \frac{1}{(\mu_1 vw(1-v)+\lambda_1 w(v-1)+\lambda_2 v(w-1))}\Big\{\lambda_1 w(v-1)\mathcal{N}_1(v,w) \quad (59)\\ &+\lambda_2\big(v(w-1)\mathcal{N}_2(v,w)+(w-v)\mathcal{N}_3(v,w)\big)\\ &+\mu_1 vw(1-v)\big(\mathcal{N}_4(v,w)+v^{A_2}w^{A_3+1}\pi_{A_2,A_3+1}\big)+\mu_2 vw(w-1)\mathcal{N}_4(v,w)\Big\}\\ \mathbf{\Pi}(v,w) &= \frac{G(v,w)}{H(v,w)} \quad (60)\\ \text{where } G(v,w) &= \lambda_1 w(v-1)\mathcal{N}_1(v,w)+\lambda_2 v(w-1)\mathcal{N}_2(v,w)+\lambda_2(w-v)\mathcal{N}_3(v,w)\\ &+\mu_1 vw(1-v)\big(\mathcal{N}_4(v,w)+v^{A_2}w^{A_3+1}\pi_{A_2,A_3+1}\big)+\mu_2 vw(w-1)\mathcal{N}_4(v,w)\\ \text{and } H(v,w) &= \mu_1 vw(1-v)+\lambda_1 w(v-1)+\lambda_2 v(w-1)) \quad \text{as given in (10).}\end{aligned}$$

The generating function $\mathbf{\Pi}(v,w)$ given in (59) includes $(A_2-B_1+1)(A_3+2)+B_1$ unknown steady state probabilities. Below, we present a 3-step algorithm that shows how to obtain the steady-state probabilities in $\mathbf{\Pi}(v,w)$ by solving $(A_2-B_1+1)(A_3+2)+B_1$ equations instead of solving all of the $(A_2+1)(A_3+2)$ balance equations.

**Step 1:** In this step, we obtain two equations by using the L'Hospital's rule and the properties of the generating functions. The unknowns in these two equations are the unknowns in the generating function $\mathbf{\Pi}(v,w)$ in (59), therefore no new unknowns are introduced.

Let $G(v,w)$ be the numerator of the generating function $\Pi(v,w)$ in (60). According to the law of total probability, $\Pi(1,1)=1$. However, both the numerator and denominator of $\Pi(v,w)$ go to zero when $(v,w)$ approaches to $(1,1)$. We apply L'Hospital's rule on $\Pi(v,w)$ with respect to $v$

and $w$ separately and obtain 2 equations as follows:

$$\begin{aligned}
\Pi(1,1) &= lim_{v\to 1}\Pi(v,1) = lim_{v\to 1}\frac{G(v,1)}{H(v,1)} = \frac{lim_{v\to 1}\frac{\partial G(v,1)}{\partial v}}{lim_{v\to 1}\frac{\partial H(v,1)}{\partial v}} = 1 \\
\frac{\partial G(v,1)}{\partial v} &= \lambda_1\big(\mathcal{N}_1(v,1) + (v-1)\frac{\partial \mathcal{N}_1(v,1)}{\partial v}\big) + \lambda_2\big(-\mathcal{N}_3(v,1) + (1-v)\frac{\partial \mathcal{N}_3(v,1)}{\partial v}\big) \\
&\quad + \mu_1\Big((1-2v)(v^{A_2}\pi_{A_2,A_3+1} + \mathcal{N}_4(v,1)) \\
&\quad + (v-v^2)(A_2 v^{A_2-1}\pi_{A_2,A_3+1} + \frac{\partial \mathcal{N}_4(v,1)}{\partial v})\Big)
\end{aligned} \tag{61}$$

$$\begin{aligned}
\frac{\partial G(v,1)}{\partial v}\Big|_{v=1} &= \lambda_1\mathcal{N}_1(1,1) - \lambda_2\mathcal{N}_3(1,1) - \mu_1\big(\pi_{A_2,A_3+1} + \mathcal{N}_4(1,1)\big) \\
H(v,1) &= \mu_1 v(1-v) + \lambda_1(v-1) \quad \Rightarrow \quad \frac{\partial H(v,1)}{\partial v}\Big|_{v=1} = \lambda_1 - \mu_1.
\end{aligned}$$

Thus,

$$\lambda_1\mathcal{N}_1(1,1) - \lambda_2\mathcal{N}_3(1,1) - \mu_1\big[\pi_{A_2,A_3+1} + \mathcal{N}_4(1,1)\big] = \lambda_1 - \mu_1. \tag{62}$$

On the other hand,

$$\begin{aligned}
\Pi(1,1) &= lim_{w\to 1}\Pi(1,w) = lim_{w\to 1}\frac{G(1,w)}{H(1,w)} = \frac{lim_{w\to 1}\frac{\partial G(1,w)}{\partial w}}{lim_{w\to 1}\frac{\partial H(1,w)}{\partial w}} = 1 \\
\frac{\partial G(1,w)}{\partial w} &= \lambda_2\Big(\mathcal{N}_2(1,w) + \mathcal{N}_3(1,w) + (w-1)(\frac{\partial \mathcal{N}_2(1,w)}{\partial w} + \frac{\partial \mathcal{N}_3(1,w)}{\partial w})\Big) \\
&\quad + \mu_2\Big((2w-1)\mathcal{N}_4(1,w)) + (w^2-w)\frac{\partial \mathcal{N}_4(1,w)}{\partial w})\Big) \\
\frac{\partial G(1,w)}{\partial w}\Big|_{w=1} &= \lambda_2\big(\mathcal{N}_2(1,1) + \mathcal{N}_3(1,1)\big) + \mu_2\mathcal{N}_4(1,1)) \\
H(1,w) &= \lambda_2(w-1) \quad \Rightarrow \quad \frac{\partial H(1,w)}{\partial w}\Big|_{w=1} = \lambda_2.
\end{aligned}$$

Therefore,

$$\lambda_2\big[\mathcal{N}_2(1,1) + \mathcal{N}_3(1,1)\big] + \mu_2\mathcal{N}_4(1,1) = \lambda_2. \tag{63}$$

**Step 2:** In this step, we obtain $(A_2 - B_1 - 1)(A_3 + 2)$ equations from the balance equations of the continuous time Markov chain representation of the multi-threshold policy. The balance equations of the multi-threshold policy for the case we are analyzing are provided in (49) to (57). The unknowns in these $(A_2 - B_1 - 1)(A_3 + 2)$ linearly independent equations are the unknowns in the generating function $\mathbf{\Pi}(v, w)$ in (59), therefore no new unknowns are introduced.

If $A_2 = B_1 + 1$, we skip Step 2 and execute Step 3. If $A_2 \geq B_1 + 2$, then equations (56) when $j = 0, \ldots, A_3$ and equation (57) provide $A_3 + 2$ equations in total. If $A_2 > B_1 + 2$ also holds, then equations (55) when $i = B_1 + 2, \ldots, A_2 - 1,\ j = 0, \ldots, A_3 + 1$ provide additional $(A_2 - B_1 - 2)(A_3 + 2)$ equations. Notice that we choose $(A_2 - B_1 - 1)(A_3 + 2)$ linearly independent equations and they do not introduce new unknowns as the steady state probabilities they contain are already in $\Pi(v, w)$.

**Step 3:** In this step we obtain $2A_3 + B_1 + 2$ additional equations by using the properties of the generating functions. The total number of equations obtained in Steps 1 to 3 is equal to the number of unknowns, i.e., $(A_2 - B_1 + 1)(A_3 + 2) + B_1$.

Since $\Pi(v, w)$ is bounded when $v$ and $w$ are bounded, any finite $v$ and $w$ pair which is the root of the denominator $H(v, w)$, should also be the root of the numerator, i.e., should solve $G(v, w) = 0$. If we choose $2A_3 + B_1 + 2$ linearly independent root pairs $(v, w)$ of $H(v, w)$ and substitute them into $G(v, w)$, we obtain $2A_3 + B_1 + 2$ equations in the form $G(v, w) = 0$. For a chosen $v_k$, the root of $H(v, w)$ can be found as follows:

$$\begin{aligned} H(v_k, w_k) &= \lambda_1 w_k(v_k - 1) + \lambda_2 v_k(w_k - 1) + \mu_1 v_k w_k(1 - v_k) = 0 \quad \Rightarrow \\ w_k &= \frac{\lambda_2 v_k}{-\mu_1 v_k^2 + (\mu + \lambda_1 + \lambda_2)v_k - \lambda_1} \\ \text{and } G(v_k, w_k) &= 0. \end{aligned}$$

Through Steps 1,2 and 3, we have constructed a linear system with $(A_2 - B_1 + 1)(A_3 + 2) + B_1$ equations and same number of unknowns. This system of equations can be solved to obtain the steady state probabilities involved. Note that obtaining the probabilities by solving the balance equations requires solving $(A_2+1)(A_3+2)$ equations, therefore the algorithm given above requires $(A_3 + 1)B_1$ fewer equations.

# ONLINE APPENDIX C

Recall that the balance equations of the proposed policy when $0 < B_1 < A_2 \leq A_1 \leq A_2 + A_3$ and $0 \leq A_3 \leq B_2$ are provided in online Appendix B with (49) to (57). Note that (50) and (52) exist only when $B_1 > 1$. Similarly, (55) exists only when $B_1 + 1 < A_2$. The steady state probabilities $\pi$ can be obtained by the algorithm presented below. Let $\hat{\pi}_{i,j}$ be the temporary, un-normalized value of the probability $\pi_{i,j}$ obtained via the algorithm. We start by setting $\hat{\pi}_{0,A_3+1} = 1$ and obtain $\hat{\pi}_{i,j}$ for the rest of the states iteratively. Afterwards, the actual values of the state probabilities are obtained by normalizing the $\hat{\pi}$ values as follows:

$$\pi_{i,j} = \frac{\hat{\pi}_{i,j}}{\sum_{i=0}^{A_2} \sum_{j=0}^{A_3+1} \hat{\pi}_{i,j}}, \qquad i = 0, .., A_2, j = 0, .., A_3 + 1. \tag{64}$$

**Step 1:**

In this step, starting from $\hat{\pi}_{0,A_3+1}$, we find the values of $\hat{\pi}_{i,A_3+1}$, for $i = 0, .., A_2$, and lastly find $\hat{\pi}_{A_2,A_3}$. Let $\hat{\pi}_{0,A_3+1} = 1$. (51) implies

$$\lambda_1 \pi_{1,A_3+1} = (\mu_1 + \lambda_2)\hat{\pi}_{0,A_3+1} \Rightarrow \hat{\pi}_{1,A_3+1} = \frac{\mu_1 + \lambda_2}{\lambda_1}.$$

If $B_1 = 1$, then by (54):

$$\begin{aligned}
(\lambda_1 + \lambda_2)\pi_{B_1+1,A_3+1} &= (\mu_1 + \lambda_1 + \lambda_2)\hat{\pi}_{B_1,A_3+1} - \mu_1 \hat{\pi}_{B_1-1,A_3+1} \Rightarrow \\
\pi_{2,A_3+1} &= \hat{\pi}_{1,A_3+1} + \frac{\mu_1}{\lambda_1 + \lambda_2}(\hat{\pi}_{1,A_3+1} - \hat{\pi}_{0,A_3+1}) \\
\hat{\pi}_{2,A_3+1} &= \frac{\mu_1 + \lambda_2}{\lambda_1} + \frac{\mu_1}{\lambda_1 + \lambda_2}\Big(\frac{\mu_1 + \lambda_2}{\lambda_1} - 1\Big).
\end{aligned}$$

If $B_1 > 1$, then we find $\hat{\pi}_{i,A_3+1}$, $i = 2, .., B_1$ by using (52) successively as follows:

$$\begin{aligned}
\lambda_1 \pi_{i,A_3+1} &= (\mu_1 + \lambda_1 + \lambda_2)\hat{\pi}_{i-1,A_3+1} - \mu_1 \hat{\pi}_{i-2,A_3+1} \Rightarrow \\
\pi_{i,A_3+1} &= \Big(1 + \frac{\mu_1 + \lambda_2}{\lambda_1}\Big)\hat{\pi}_{i-1,A_3+1} - \frac{\mu_1}{\lambda_1}\hat{\pi}_{i-2,A_3+1}.
\end{aligned}$$

Balance equations are for the multi-threshold policy when $A_2 > B_1$. By using (54) and (55), we find $\hat{\pi}_{i,A_3+1}$, for $i = B_1 + 1, .., A_2$, as follows:

$$\begin{aligned}
(\lambda_1 + \lambda_2)\pi_{i,A_3+1} &= (\mu_1 + \lambda_1 + \lambda_2)\pi_{i-1,A_3+1} - \mu_1 \pi_{i-2,A_3+1} \Rightarrow \\
\pi_{i,A_3+1} - \pi_{i-1,A_3+1} &= \frac{\mu_1}{\lambda_1 + \lambda_2}(\pi_{i-1,A_3+1} - \pi_{i-2,A_3+1}) \quad \text{for } i = B_1 + 1, .., A_2 \\
\pi_{B_1+1,A_3+1} - \hat{\pi}_{B_1,A_3+1} &= \frac{\mu_1}{\lambda_1 + \lambda_2}(\hat{\pi}_{B_1,A_3+1} - \hat{\pi}_{B_1-1,A_3+1}) \\
\pi_{i,A_3+1} - \pi_{i-1,A_3+1} &= \Big(\frac{\mu_1}{\lambda_1 + \lambda_2}\Big)^{i-B_1}(\hat{\pi}_{B_1,A_3+1} - \hat{\pi}_{B_1-1,A_3+1}) \quad \text{for } i = B_1 + 1, .., A_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi_{i,A_3+1} &= \hat{\pi}_{B_1,A_3+1} + \sum_{k=1}^{i-B_1}\Big(\frac{\mu_1}{\lambda_1 + \lambda_2}\Big)^k(\hat{\pi}_{B_1,A_3+1} - \hat{\pi}_{B_1-1,A_3+1}) \quad \text{for } i = B_1 + 1, .., A_2 \\
\pi_{i,A_3+1} &= \sum_{k=0}^{i-B_1}\Big(\frac{\mu_1}{\lambda_1 + \lambda_2}\Big)^k \hat{\pi}_{B_1,A_3+1} - \sum_{k=1}^{i-B_1}\Big(\frac{\mu_1}{\lambda_1 + \lambda_2}\Big)^k \hat{\pi}_{B_1-1,A_3+1}.
\end{aligned} \tag{65}$$

Starting by setting $\hat{\pi}_{0,A_3+1}$ to 1, we showed how to obtain the values of $\hat{\pi}_{i,A_3+1}$, $i = 1,..,A_2$, sequentially by only using the probabilities calculated earlier in the algorithm. Lastly, $\hat{\pi}_{A_2,A_3}$ can be obtained from (57) as follows:

$$\pi_{A_2,A_3} = \frac{\lambda_1+\lambda_2}{\mu_2}\hat{\pi}_{A_2,A_3+1} - \frac{\mu_1}{\mu_2}\hat{\pi}_{A_2-1,A_3+1}. \tag{66}$$

**Step 2:**

Let $j = A_3$ be the initial value of the index $j$. If $j$ is zero, then we skip Step 2 and go to Step 3, otherwise we execute Step 2. After each execution of this step, we reduce the value of the index $j$ by one and repeat the step until $j$ becomes zero. At the beginning of each Step 2 execution, the values of $\hat{\pi}_{i,k}$ (for $i = 0,..,A_2$ and $k \geq j+1$) and $\hat{\pi}_{A_2,j}$ are known. With each execution of Step 2, we calculate the values of $\hat{\pi}_{i,j}$, for $i = 0,..,A_2-1$ (balance equations are for the multi-threshold policy when $A_2 > 0$), and also obtain $\hat{\pi}_{A_2,j-1}$. In this step, for a given $j$, we represent the unknown probabilities $\pi_{i,j}$, for $i = 0,..,A_2-2$, in terms of the unknown $\pi_{A_2-1,j}$. Then, we solve for $\pi_{A_2-1,j}$ which gives us $\hat{\pi}_{i,j}$, for $i = 0,..,A_2-1$. Lastly, we calculate $\hat{\pi}_{A_2,j-1}$.

Balance equations are for the multi-threshold policy when $A_2 > B_1$ Therefore, either $A_2 > B_1+1$ (balance equations (55) exist) or $A_2 = B_1+1$ holds. If $A_2 > B_1+1$, then by (55) we have:

$$\begin{aligned}
\mu_1\pi_{i-1,j} &= (\mu_1+\lambda_1+\lambda_2)\pi_{i,j} - (\lambda_1+\lambda_2)\pi_{i+1,j} \quad \text{for } i = B_1+1,..,A_2-1\\
\Rightarrow \pi_{A_2-2,j} - \pi_{A_2-1,j} &= \frac{\lambda_1+\lambda_2}{\mu_1}(\pi_{A_2-1,j} - \hat{\pi}_{A_2,j})\\
\pi_{i-1,j} - \pi_{i,j} &= \frac{\lambda_1+\lambda_2}{\mu_1}(\pi_{i,j} - \pi_{i+1,j}) = \left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^{A_2-i}(\pi_{A_2-1,j} - \hat{\pi}_{A_2,j})\\
&\quad \text{for } i = B_1+1,..,A_2-1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi_{i,j} - \pi_{A_2-1,j} &= \sum_{k=1}^{A_2-(i+1)}\left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^k(\pi_{A_2-1,j} - \hat{\pi}_{A_2,j}) \quad \text{for } i = B_1,..,A_2-2\\
\pi_{i,j} &= \sum_{k=0}^{A_2-(i+1)}\left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^k\pi_{A_2-1,j} - \sum_{k=1}^{A_2-(i+1)}\left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^k\hat{\pi}_{A_2,j} \qquad (67)\\
&\quad \text{for } i = B_1,..,A_2-2.
\end{aligned}$$

If $A_2 = B_1+1$, instead of (67), we simply have the following:

$$\pi_{B_1,j} = \pi_{A_2-1,j} \quad \text{and} \quad \pi_{B_1+1,j} = \hat{\pi}_{A_2,j}. \tag{68}$$

Balance equation (54) gives

$$\pi_{B_1-1,j} = \pi_{B_1,j} + \frac{\lambda_1+\lambda_2}{\mu_1}(\pi_{B_1,j} - \pi_{B_1+1,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1,j+1}. \tag{69}$$

Equation (67) (or (68) if $A_2 = B_1+1$), combined with (69) gives us $\pi_{B_1-1,j}$ in terms of $\pi_{A_2-1,j}$, i.e., we have the following:

$$\begin{aligned}
\pi_{B_1-1,j} - \pi_{B_1,j} &= \left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^{A_2-B_1}(\pi_{A_2-1,j} - \hat{\pi}_{A_2,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1,j+1} \qquad (70)\\
\pi_{B_1-1,j} &= \sum_{k=0}^{A_2-B_1}\left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^k\pi_{A_2-1,j} - \sum_{k=1}^{A_2-B_1}\left(\frac{\lambda_1+\lambda_2}{\mu_1}\right)^k\hat{\pi}_{A_2,j} - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1,j+1}. \qquad (71)
\end{aligned}$$

We have obtained $\pi_{i,j}$, for $i = B_1 - 1, .., A_2 - 2$ in terms of $\pi_{A_2-1,j}$. We now proceed to find $\pi_{i,j}$, for $i = 0, .., B_1 - 2$, in terms of $\pi_{B_1-1,j}$ and $\pi_{B_1,j}$. As the balance equations are for the multi-threshold policy when $B_1 > 0$, either $B_1 = 1$, or $B_1 > 1$ holds.

<u>**Case 1:**</u>

If $B_1 = 1$, (51) implies

$$\pi_{0,j} = \pi_{B_1-1,j} = \frac{\lambda_1}{\mu_1 + \lambda_2}\pi_{B_1,j} + \frac{\lambda_2}{\mu_1 + \lambda_2}\hat{\pi}_{0,j+1}. \tag{72}$$

In this case, by simply setting (71) = (72) and using (67) (or (68) if $A_2 = B_1 + 1$) for $\pi_{B_1,j}$, we obtain $\hat{\pi}_{A_2-1,j}$, as $\pi_{A_2-1,j}$ is the only unknown. Rest of the values $\hat{\pi}_{i,j}$, for $i = 1, .., A_2 - 2$ (if $A_2 - 1 > B_1$) can be obtained by using (67) and $\hat{\pi}_{0,j}$ can be obtained from (72).
We finish Step 2 by obtaining $\hat{\pi}_{A_2,j-1}$ from the balance equation (56):

$$\pi_{A_2,j-1} = \Big(1 + \frac{\lambda_1 + \lambda_2}{\mu_2}\Big)\hat{\pi}_{A_2,j} - \frac{\mu_1}{\mu_2}\hat{\pi}_{A_2-1,j}.$$

<u>**Case 2:**</u>

If $B_1 > 1$, then from (52) we have:

$$\pi_{i-1,j} = \frac{\mu_1 + \lambda_2}{\mu_1}\pi_{i,j} + \frac{\lambda_1}{\mu_1}(\pi_{i,j} - \pi_{i+1,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{i,j+1} \quad \text{for } i = 1, .., B_1 - 1 \tag{73}$$

$$\pi_{i-1,j} - \pi_{i,j} = \frac{\lambda_2}{\mu_1}\pi_{i,j} + \frac{\lambda_1}{\mu_1}(\pi_{i,j} - \pi_{i+1,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{i,j+1} \quad \text{for } i = 1, .., B_1 - 1. \tag{74}$$

By using (73) and (74), we present an iterative method that gives $\pi_{i,j}$, for $i = 0, .., B_1 - 2$, in terms of $\pi_{B_1-1,j}$ and $\pi_{B_1,j}$. By (73) and (74), we have:

$$\pi_{B_1-2,j} = \frac{\mu_1 + \lambda_2}{\mu_1}\pi_{B_1-1,j} + \frac{\lambda_1}{\mu_1}(\pi_{B_1-1,j} - \pi_{B_1,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1-1,j+1}$$

$$\pi_{B_1-2,j} - \pi_{B_1-1,j} = \frac{\lambda_2}{\mu_1}\pi_{B_1-1,j} + \frac{\lambda_1}{\mu_1}(\pi_{B_1-1,j} - \pi_{B_1,j}) - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1-1,j+1}.$$

Let $\mathbf{K}^{(n)}$, for $n \in \mathbb{N}_0$, be $2 \times 3$ matrices with $\mathbf{K}^{(0)}$ a matrix of zeros and $\mathbf{K}^{(n)}$, with $n \geq 1$ defined as follows:

$$\mathbf{K}^{(1)} = \begin{bmatrix} \frac{\mu_1+\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} & \frac{\lambda_2}{\mu_1} \\ \frac{\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} & \frac{\lambda_2}{\mu_1} \end{bmatrix} \quad \text{and} \quad \mathbf{K}^{(n)} = \begin{bmatrix} \frac{\mu_1+\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} \\ \frac{\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} \end{bmatrix} \mathbf{K}^{(n-1)} \quad \text{for } n \geq 2.$$

Then,

$$\begin{bmatrix} \pi_{B_1-2,j} \\ \pi_{B_1-2,j} - \pi_{B_1-1,j} \end{bmatrix} = \begin{bmatrix} \frac{\mu_1+\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} & \frac{\lambda_2}{\mu_1} \\ \frac{\lambda_2}{\mu_1} & \frac{\lambda_1}{\mu_1} & \frac{\lambda_2}{\mu_1} \end{bmatrix} \begin{bmatrix} \pi_{B_1-1,j} \\ \pi_{B_1-1,j} - \pi_{B_1,j} \\ -\hat{\pi}_{B_1-1,j+1} \end{bmatrix} = \mathbf{K}^{(1)} \begin{bmatrix} \pi_{B_1-1,j} \\ \pi_{B_1-1,j} - \pi_{B_1,j} \\ -\hat{\pi}_{B_1-1,j+1} \end{bmatrix}.$$

After some algebra, it can be shown that the iterative solution to the difference equation is:

$$\begin{bmatrix} \pi_{i,j} \\ \pi_{i,j} - \pi_{i+1,j} \end{bmatrix} = \mathbf{K}^{((B_1-1)-i)} \begin{bmatrix} \pi_{B_1-1,j} \\ \pi_{B_1-1,j} - \pi_{B_1,j} \\ -\hat{\pi}_{B_1-1,j+1} \end{bmatrix} - \sum_{k=2}^{B_1-i} \mathbf{K}_3^{(B_1-k-i)}\hat{\pi}_{B_1-k,j+1} \tag{75}$$
$$\text{for } i = 0, .., B_1 - 2.$$

where $\mathbf{K}_3^n$ denote the $3^{rd}$ column of the matrix $\mathbf{K}^n$. $\mathbf{K}^n$, for $n = 1, .., B_1 - 1$, can easily be obtained by programs like Excel. Furthermore, we need to calculate the $\mathbf{K}$ matrices once for the whole algorithm, as they are independent of the row index $j$.

It should be noted that with equations (70) - (71) we have obtained $\pi_{B_1-1,j}$ and $\pi_{B_1-1,j} - \pi_{B_1,j}$ in terms of the unknown $\pi_{A_2-1,j}$ and the known $\hat{\pi}$ values. Therefore, (75) gives $\pi_{i,j}$, for $i = 0, .., B_1 - 2$, in terms of the unknown $\pi_{A_2-1,j}$ and the known $\hat{\pi}$ values. By (75), we have:

$$\pi_{0,j} = \mathbf{K}_{1,1}^{(B_1-1)} \pi_{B_1-1,j} + \mathbf{K}_{1,2}^{(B_1-1)} (\pi_{B_1-1,j} - \pi_{B_1,j}) - \sum_{k=1}^{B_1} \mathbf{K}_3^{(B_1-k)} \hat{\pi}_{B_1-k,j+1} \quad (76)$$

$$\pi_{1,j} = \mathbf{K}_{1,1}^{(B_1-2)} \pi_{B_1-1,j} + \mathbf{K}_{1,2}^{(B_1-2)} (\pi_{B_1-1,j} - \pi_{B_1,j}) - \sum_{k=1}^{B_1-1} \mathbf{K}_3^{(B_1-1-k)} \hat{\pi}_{B_1-k,j+1}, \quad (77)$$

where $\mathbf{K}_{r,c}^{(n)}$ is the $r^{th}$ row, $c^{th}$ column element of $\mathbf{K}^{(n)}$. Balance equation (51) implies:

$$\pi_{0,j} = \frac{\lambda_1}{\mu_1 + \lambda_2} \pi_{1,j} + \frac{\lambda_2}{\mu_1 + \lambda_2} \hat{\pi}_{0,j+1}. \quad (78)$$

By using (76), (77) and (78), we obtain:

$$C_1 \pi_{B_1-1,j} + C_2 (\pi_{B_1-1,j} - \pi_{B_1,j}) = C_3, \quad (79)$$

where the constants $C_1$ - $C_3$ are as follows:

$$C_1 = \left( \mathbf{K}_{1,1}^{(B_1-1)} - \frac{\lambda_1}{\mu_1 + \lambda_2} \mathbf{K}_{1,1}^{(B_1-2)} \right)$$

$$C_2 = \left( \mathbf{K}_{1,2}^{(B_1-1)} - \frac{\lambda_1}{\mu_1 + \lambda_2} \mathbf{K}_{1,2}^{(B_1-2)} \right)$$

$$C_3 = \sum_{k=1}^{B_1} \mathbf{K}_3^{(B_1-k)} \hat{\pi}_{B_1-k,j+1} - \frac{\lambda_1}{\mu_1 + \lambda_2} \sum_{k=1}^{B_1-1} \mathbf{K}_3^{(B_1-1-k)} \hat{\pi}_{B_1-k,j+1} + \frac{\lambda_2}{\mu_1 + \lambda_2} \hat{\pi}_{0,j+1}.$$

We replace $\pi_{B_1-1,j}$ and $(\pi_{B_1-1,j} - \pi_{B_1,j})$ in (79) with (71) and (70), respectively, and obtain $\pi_{A_2-1,j}$, as it is the only unknown in (79).

Now that we know $\hat{\pi}_{A_2-1,j}$, we obtain $\hat{\pi}_{i,j}$, for $i = B_1, .., A_2 - 2$, by using (67) (or by (68) if $A_2 = B_1 + 1$). By using $\hat{\pi}_{A_2-1,j}$ in (71), we obtain $\hat{\pi}_{B_1-1,j}$. And by using $\hat{\pi}_{B_1-1,j}$ and $\hat{\pi}_{B_1,j}$ in (75), we obtain $\hat{\pi}_{i,j}$, for $i = 0, .., B_1 - 2$.

Lastly, by the balance equation (56), we have:

$$\pi_{A_2,j-1} = \left(1 + \frac{\lambda_1 + \lambda_2}{\mu_2}\right) \hat{\pi}_{A_2,j} - \frac{\mu_1}{\mu_2} \hat{\pi}_{A_2-1,j}.$$

**Step 3:**

In this step, we will finish the algorithm by finding the $\hat{\pi}_{i,0}$ values for $i = 0, .., A_2 - 1$. At this point, we already have all of the $\hat{\pi}_{i,j}$ values for $j \geq 1$ and we also have the $\hat{\pi}_{A_2,0}$ value.

We obtain $\hat{\pi}_{A_2-1,0}$ by using the balance equation (56):

$$\pi_{A_2-1,0} = \frac{\mu_2 + \lambda_1 + \lambda_2}{\mu_1} \hat{\pi}_{A_2,0}.$$

As we already have $\hat{\pi}_{i,1}$, $\hat{\pi}_{A_2,0}$, and $\hat{\pi}_{A_2-1,0}$ values, unlike Step 2, we can obtain the remaining $\hat{\pi}_{i,0}$ values directly by using the related balance equations successively in the descending order of $i$.

If $A_2 > B_1 + 1$, then we can obtain $\hat{\pi}_{i,0}$, for $i = A_2 - 2, .., B_1$ by using (55) successively as follows:

$$\pi_{i-1,0} = \Big(1 + \frac{\lambda_1 + \lambda_2}{\mu_1}\Big)\hat{\pi}_{i,0} - \Big(\frac{\lambda_1 + \lambda_2}{\mu_1}\Big)\hat{\pi}_{i+1,0} \quad \text{for } i = B_1 + 1, .., A_2 - 1,$$

or we can use (67), as it still holds when $j = 0$,

$$\pi_{i,0} = \sum_{k=0}^{A_2-(i+1)} \Big(\frac{\lambda_1 + \lambda_2}{\mu_1}\Big)^k \hat{\pi}_{A_2-1,0} - \sum_{k=1}^{A_2-(i+1)} \Big(\frac{\lambda_1 + \lambda_2}{\mu_1}\Big)^k \hat{\pi}_{A_2,0} \qquad \text{for } i = B_1, .., A_2 - 2.$$

If $B_1 = 1$, then we complete the algorithm by getting $\pi_{0,0}$ from (49),

$$\pi_{0,0} = \frac{\lambda_1}{\mu_1}\hat{\pi}_{1,0} + \frac{\lambda_2}{\mu_1}\hat{\pi}_{0,1}.$$

If $B_1 > 1$, then we get $\pi_{B_1-1,0}$ from (53) and get $\pi_{i,0}$ starting from $i = B_1 - 2$ to $i = 0$ from (50),

$$\begin{aligned} \pi_{B_1-1,0} &= \frac{\mu_1 + \lambda_1}{\mu_1}\hat{\pi}_{B_1,0} - \frac{\lambda_1 + \lambda_2}{\mu_1}\hat{\pi}_{B_1+1,0} - \frac{\lambda_2}{\mu_1}\hat{\pi}_{B_1,1} \\ \pi_{i,0} &= \frac{\mu_1 + \lambda_1}{\mu_1}\hat{\pi}_{i+1,0} - \frac{\lambda_1}{\mu_1}\hat{\pi}_{i+2,0} - \frac{\lambda_2}{\mu_1}\hat{\pi}_{i+1,1} \qquad i = 0, .., B_1 - 2. \end{aligned}$$