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Optimal server scheduling in nonpreemptive finite-population queueing systems

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Abstract We consider a finite-population queueing system with heterogeneous classes of customers and a single server. For the case of nonpreemptive service, we fully characterize the structure of the server's optimal service policy that minimizes the total average customer waiting costs. We show that the optimal service policy may never serve some classes of customers. For those classes that are served, we show that the optimal service policy is a simple static priority policy. We also derive sufficient conditions that determine the optimal priority sequence.

Keywords Machine-interference problems · Finite population queueing systems · Markov decision process

1 Introduction

Finite-population queueing systems are systems in which the number of customers who use the system is limited; the customer arrival rates therefore depend on the number of customers in the system. The machine-repairman problem is one of the most well-known finite-population queueing systems (see Stecke and Aronson (1985) and Stecke (1992) for the reviews of literature on the machine-repairman problem). In a *general machine-repairman problem*,

- there are $N (1 \leq N < \infty)$ different groups of machines, where group i includes N_i identical machines of type i $(i = 1, 2, \ldots, N)$;
- all machines of type i have failure and repair rates λ_i and μ_i, respectively;
- the downtime of a type-i machine incurs the cost of c_i per unit time;
- there is/are one or more repairmen who can repair all machines, or a subset of the machines.

When the problem involves only one type of machine (i.e., identical machines, $N = 1$), system performance measures such as the average number of broken machines, or the average machine downtime or cost, are not affected by the way repairmen choose which machine to repair next. However, when there is more than one machine type ($N > 1$), these performance measures are indeed affected by the service discipline (i.e., repair policy).

In this paper we study a machine repairman problem with N different machine types and one repairman where repair operations are nonpreemptive. Under a nonpreemptive repair, when repair operations starts, it cannot be interrupted until the repair is completed. We show that the optimal repair policy that minimizes the total expected machine downtime cost may never repair some machine types. For those machines types that are repaired, we show that the optimal repair policy is a simple static priority policy, and we introduce conditions that determine the optimal priority sequence among those machines. Furthermore, we derive sufficient conditions that determine a subset of machines that are never repaired under the optimal policy.

2 Literature review

The machine-repairman problem is one of the earliest applications of operations research and queueing theory. Even though a large body of literature is available on machine-repairman problems, only a few study these problems with heterogeneous machines. One of the earliest works

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on *performance analysis* of systems with heterogeneous machines is Hodgson and Hebble (1967). For a static machine-priority policy, they find the steady state probabilities and the performance measures for a nonpreemptive repair system with heterogeneous machines and a single repairman. A machine-repairman problem with a single repairman, general repair time distributions, and a constant failure rate was studied by Chandra and Sargent (1983). They develop a numerical procedure to obtain the performance measures of the system with heterogeneous machines and a fixed nonpreemptive priority discipline. In another paper, Chandra (1986) considers the same problem with Erlang, exponential, and hyperexponential repair time distributions in order to study the effect of repair time variability, as well as the first-come-first-served repair discipline on the performance measures of the system.

The *optimization analysis* of machine-repairman problems with identical machines and a single repairman is studied in Hsieh (1996) and Elsayed (1981). Hsieh studies the optimal preemptive repair policy of such a machine-repairman problem with two modes of failures. Based on a numerical study, Hsieh conjectures that the optimal policy is always a static policy. Considering a similar setting but with nonpreemptive repair, Elsayed (1981) compares the system under priority and no-priority repair policies. Under the priority policy, two modes of failures are prioritized, while under the no-priority policy, both modes of failures are equally likely to be chosen for the next repair. Using a numerical study, he compares these policies with respect to their costs, repairman efficiency, and machine availability.

In a preemptive repair setting, Lehtonen (1984) studies a single repairman problem in which $c_i = 1$, $N_i = 1$ for all i, and machines have the same failure rate but different repair rates. Lehtonen shows that choosing the machine with the largest repair rate maximizes the average number of working machines. Derman et al. (1980) study a similar problem with an identical service rate but different failure rates (i.e., $c_i = 1$, $N_i = 1$, $\mu_i = \mu\ \forall i$). They show that the least-λ policy, which assigns the server to the machine with the lowest failure rate, minimizes the average number of broken machines. Courcoubetis and Varaiya (1984), on the other hand, show that the largest-λ policy, that assigns the server to the machine with largest failure rate, maximizes the repairman's utilization.

Frostig (1993) considers n machines (i.e., $N_i = 1 \forall i$) with more than one repairman and shows that if Smallest Failure Rate (SFR) is agreeable with shortest-expected-processing-time, i.e., if $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$, then SFR policy maximizes the expected discounted machine up-time for both cases where machines have constant or increasing failure rates. Righter (1996) considers a similar problem, but with identical repair times (i.e., $\mu_i = \mu\ \forall i$), and shows that the SFR policy stochastically maximizes the up-time probability of the first k machines in the SFR order.

Koole and Vrijenhoek (1996) extend the above studies to cases with different failure and repair rates. They show that if $\lambda_1 \le \lambda_2 \cdots \le \lambda_N$ and $c_1\mu_1 \ge c_2\mu_2 \ge \cdots \ge c_N\mu_N$, then the policy that assigns the repairman to the machine with the smallest index minimizes the average cost of the system. Iravani and Kolfal (2005) consider the same problem in a more general setting, where the machines have different population sizes ($N_i \ge 1 \forall i$). For this general system, they prove that repairman idling is not optimal and they derive conditions under which the optimal policy is a static machine-priority rule. They also show how their conditions translate into the optimality of the $c\mu$ rule.

Iravani and Krishnamurthy (2004) consider a machine-repairman problem with M heterogeneous machines, *preemptive repairs,* and K partially cross-trained repairmen (i.e., each repairman can repair only a subset of M machines, and that subset may or may not overlap with other repairmen's subsets of machines). Using numerical examples, they illustrate that the optimal repairman assignment policy has a complex structure. However, prioritizing machines based on a static priority rule results in a cost close to the optimal cost.

To the best of our knowledge, there is no literature beyond these studies that discusses the optimal repairman assignment policies in systems with heterogeneous machines and *nonpreemptive* repair. This is, therefore, the focus of this paper. In Section 3 we introduce our assumptions and formulate our problem as a Markov Decision process (MDP). Section 3 also discusses the properties of our MDP model. In Section 4 we fully characterize the structure of the optimal repair policy and show that it follows a simple static machine-priority policy. In Section 5 we derive conditions that determine the optimal priority sequence among two machine types as well as the optimality of idling policy. Section 6 focuses on systems in which idling is not allowed, and Section 7 concludes our paper and suggests directions for future research.

3 Markov decision process formulation

Consider a general machine-repairman problem with a single repairman who is trained to repair all N different machine types. Also, assume that the machines have constant failure rates and their repair operations follow an exponential distribution. Note that assuming a constant failure rate is consistent with the behavior of a machine during its useful life in the Bathtub curve. Furthermore, exponentially distributed repair times represent situations where shorter repair times are more probable than longer repair times. This often happens when most of the machine failures are minor failures that can be fixed in a short time, while major failures requiring an overhaul repair (with longer repair times) are less frequent.

In this section, we formulate our machine-repairman problem as a Markov Decision Process (MDP) model and study the properties of the MDP model. The MDP model is defined as follows:

- *State Space:* State space $\mathcal{S}$ consists of $N+1$ dimensional row vectors $(\mathbf{n}, r)$, in which $\mathbf{n} = (n_1, n_2, \ldots, n_N)$, where $n_i \in \{0, 1, 2, \ldots, N_i\}$ represents the number of broken machines of type $i \in \{1, 2, \ldots, N\}$, and r represents whether the repairman is idle ($r = 0$) or is repairing a machine of type $r (r \in \{1, 2, \ldots, N\})$.
- *Decision Epochs:* Decision epochs are machine failure and repair completion epochs.
- *Action Set:* Since preemption is not allowed, the allowable actions in state $(\mathbf{n}, r)$ are: (1) when $r = 0$, the repairman can either idle or repair a broken machine of type i, and (2) when $r \neq 0$, the only allowable action is to continue the current repair operation until the repair operation is completed.

Let $J_{(\mathbf{n})}$ be the set that includes the indices of machine types that have at least one broken machine in state $\mathbf{n}$, and let $J^0_{(\mathbf{n})} = \{0\} \cup J_{(\mathbf{n})}$. Thus, $J^0_{(\mathbf{n})}$ is the set of potential actions in state $\mathbf{n}$, and for any given state $(\mathbf{n}, r)$, we have $r \in J^0_{(\mathbf{n})}$. For example, when $N = 4$, if $\mathbf{n} = (0, 1, 2, 0)$, then $J_{(\mathbf{n})} = \{2, 3\}$ and $J^0_{(\mathbf{n})} = \{0, 2, 3\}$.

The optimality equation of the MDP model with the objective of minimizing the total average down-time cost per unit time, can be expressed as follows. When $r \neq 0$, we have

$$\frac{g}{\Upsilon} + V(\mathbf{n}, r) = \frac{1}{\Upsilon}\left[\sum_{j=1}^{N} c_j n_j + \sum_{j=1}^{N}(N_j - n_j)\lambda_j V(\mathbf{n} + \mathbf{e}^j, r) + \sum_{j=1}^{N} n_j \lambda_j V(\mathbf{n}, r) + f(\mathbf{n}, r)\right], \quad (1)$$

where

$$f(\mathbf{n}, r) = \mu_r \min_{s \in J^0_{(\mathbf{n}-\mathbf{e}^r)}} \{V(\mathbf{n} - \mathbf{e}^r, s)\} + \sum_{j=1, j\neq r}^{N} \mu_j V(\mathbf{n}, r),$$

and $V(\mathbf{n}, r)$ is the value function that corresponds to the minimum total expected cost if system starts at state $(\mathbf{n}, r)$. On the other hand, when $r = 0$

$$\frac{g}{\Upsilon} + V(\mathbf{n}, 0) = \frac{1}{\Upsilon}\left[\sum_{j=1}^{N} c_j n_j + \sum_{j=1}^{N}(N_j - n_j)\lambda_j \min_{s \in J^0_{(\mathbf{n}+\mathbf{e}^j)}} \{V(\mathbf{n} + \mathbf{e}^j, s)\} + \sum_{j=1}^{N}(n_j\lambda_j + \mu_j)V(\mathbf{n}, 0)\right], \quad (2)$$

where $\Upsilon = \sum_{j=1}^{N}(N_j\lambda_j + \mu_j)$ is the uniformization rate, and $\mathbf{e}^j$ is an N-dimensional vector with zero elements except for its j^{th} element, which is one.

Since the state and action spaces in our MDP model are finite, it is easy to show that there exists a stationary average-cost optimal policy for our general machine-repairman problem with a constant gain and the value iteration algorithm converges. Before we present our main result regarding the optimality of static priority rules, we need to present some properties of the optimality equations. For $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ in which $n_q \geq 1$, define operator D_q on function $V(\mathbf{n}, r)$ as follows:

$$D_q V(\mathbf{n}, r) = V(\mathbf{n}, r) - V(\mathbf{n} - \mathbf{e}^q, r).$$

Note that $D_q V(\mathbf{n}, q)$ only holds for $n_q \geq 2$ because $V(\mathbf{n} - \mathbf{e}^q, q)$ does not exist when $n_q = 1$.

Proposition 1. *For all $p, q, r, u \in J^0_{(\mathbf{n})}$, $i, j \in \{1, 2, \ldots, N\}$, the value functions presented in (1) and (2) have the following properties:*

P1 : *$D_j V(\mathbf{n} + \mathbf{e}^j, p)$ is non-decreasing in n_i;*
P2 : *$V(\mathbf{n}, q) - V(\mathbf{n}, p)$ is non-decreasing in n_i when $V(\mathbf{n}, q) \geq V(\mathbf{n}, p)$;*
P3 : *$\frac{c_q}{\Upsilon} \leq D_q V(\mathbf{n}, r) \leq \frac{c_q}{\lambda_q}$ when $n_q \geq 1$ and $\forall r \in J_{(\mathbf{n})}$;*
P4 : *$\frac{c_q}{\Upsilon} \leq D_q V(\mathbf{n}, 0) \leq \frac{c_q}{\lambda_q}$ when $n_q \geq 1$ and idling is optimal in state* $\mathbf{n}$;
P5 : *$V(\mathbf{n} + \mathbf{e}^r, r) - V(\mathbf{n}, u) \geq \frac{c_r}{\Upsilon}$, where $r \in J_{(\mathbf{n}+\mathbf{e}^r)}$ and $u \in J^0_{(\mathbf{n})}$ are optimal in states $\mathbf{n} + \mathbf{e}^r$ and $\mathbf{n}$, respectively.*

Proof: The proof of Proposition 1 is long and is therefore omitted. The complete proof can be found in Iravani et al. (2006), the full version of this paper. □

4 Structural results

In this section, we characterize the structure of the optimal policy in Theorems 1 and 2. Theorem 1 describes the behavior of the optimal policy with respect to idling decisions, while Theorem 2 explains how the optimal policy behaves with respect to repair decisions. The proofs of both theorems can be found in the Appendix.

Theorem 1. *(Idling Decisions)*

(i) *When there exists at least one broken machine of each type (*i.e., $n_i \geq 1, \forall i \in \{1, 2, \ldots, N\}$*),* idling is never optimal.

(ii) *Consider state* $\mathbf{n}_0$ *in which there exists at least one machine type with no broken machine. If idling is optimal in state* $\mathbf{n}_0$*, then idling is also optimal in any state* $\mathbf{n}'_0$*, where* $J_{(\mathbf{n}'_0)} \subseteq J_{(\mathbf{n}_0)}$.

Theorem 1 part (ii) implies that, if the optimal action in state $\mathbf{n}_0$ is to idle, then the repairman will never repair any machine type in $J_{(\mathbf{n}_0)}$, regardless of the number of broken machines of each type. This in turn implies that the optimal policy divides all machine types into two disjoint groups: (1) the idle group $\mathcal{I}$ ($\mathcal{I} \subset \mathcal{N}$), and (2) the repair group $\mathcal{R}$ ($\mathcal{R} \subseteq \mathcal{N}$), where $\mathcal{I} \cap \mathcal{R} = \emptyset$, $\mathcal{I} \cup \mathcal{R} = \mathcal{N}$, and $\mathcal{N}$ is the set of all machine types, i.e., $\mathcal{N} = \{1, 2, 3, \ldots, N\}$. The idle group $\mathcal{I}$ consists of machine types that are never repaired under the optimal policy (regardless of the number of broken machines in the system). On the other hand, the repair group $\mathcal{R}$ consists of machine types that are repaired under the optimal policy. When all machines in the repair group are repaired, the optimal policy is idling until a machine in the repair group breaks.

Theorem 2. *(Repair Decisions)*
If it is optimal to repair machines of type p *in state* $\mathbf{n}$*, it is also optimal to repair machines of type* p *in any state* $\mathbf{n}'$*, where* $p \in J_{(\mathbf{n}')}$*, and* $J_{(\mathbf{n}')} \subseteq J_{(\mathbf{n})}$.

Theorem 2 implies that the optimal dynamic policy is in fact a simple static machine-priority policy. To illustrate this, suppose repairing a machine of type p is optimal in state $\mathbf{n}$ in which there exists at least one broken machine of each type. Then according to Theorem 2, it is always optimal to repair a machine of type p, regardless of the number of broken machines of other types. This implies that machine type p has the highest priority. Now consider state $\mathbf{n}_{\bar{p}}$ where there exists at least one broken machine of each type, except for type p which has no broken machine (i.e., $n_p = 0$, and $n_j \geq 1$ for all $j \neq p$). Without losing generality, assume that repairing machine type q is optimal in state $\mathbf{n}_{\bar{p}}$. According to Theorem 2, then repairing machine type q is always optimal as long as there is no broken machine of type p. This implies that machine type q has the second highest priority. Following the same line of argumentation, it becomes clear that the optimal repair policy is indeed a simple static priority policy.

5 Parametric results

In the previous section we showed that the structure of the optimal policy follows a simple static priority policy when idling is not optimal. However, finding the optimal static priority rule, in general, is difficult and depends on the parameters of the system. In this section, we present sufficient conditions by which one can find the optimal priority sequence among different machine types. Specifically, in Theorem 3 we introduce two conditions that determine whether repairing a machine of type p has a higher priority than repairing a machine of type q.

Theorem 3. *Consider machine types* p *and* q ($p \neq q$) *where* $\mu_p \geq \mu_q$*. If either* **A1** *or* **A2** *holds between machine type* p *and machine type* q*, then machine type* p *has higher priority than machine type* q.

$$\mathbf{A1}: \quad c_p\mu_p \geq \frac{\lambda_p}{\lambda_q} c_q\mu_q \quad \textit{and} \quad \lambda_p \geq \lambda_q$$

$$\mathbf{A2}: \quad c_p\mu_p \geq \left\{1 - \frac{\lambda_q - \lambda_p}{\Upsilon}\right\} c_q\mu_q \quad \textit{and} \quad \lambda_p < \lambda_q$$

Proof: The proof is presented in the Appendix. □

Note that conditions **A1** and **A2** do not depend on the number of machines in the system. Furthermore, it is not necessary that the same condition, **A1** or **A2**, should hold between type p and all the other machine types. For example, if there are three machine types ($N = 3$) in which **A1** holds between type 1 and 2, and **A2** holds between machine type 1 and 3, then machine type 1 has higher priority over the other two machines.

The intuition behind **A1** and **A2** can be described by comparing the dynamics of the arrival rate in finite and infinite-population queueing systems. It is known that in multi-class, infinite-population queueing systems, the static priority discipline which gives a higher priority to customers with a larger $c_i\mu_i$ minimizes the total average waiting cost in the system. Specifically, type-p customers have higher priority than type-q customers, if $c_p\mu_p \geq c_q\mu_q$. The intuition behind the $c\mu$ rule is easy to grasp. By giving higher priority to customers with larger service rates (i.e., shorter average service times) and larger waiting costs, the system's cost can be reduced faster, which results in a lower total average cost. However, this may not be true in finite-population queueing systems.

As opposed to an infinite-population queueing system in which the customer arrival rates are independent of the number of customers in the system, the arrival rates in a finite-population queueing systems increase as the number of customers in the system decreases. Therefore, to decrease the number of arrivals (machine failures) per unit time, the optimal policy tends to give higher priority to customers with smaller arrival rates. This has been shown by the optimality of the least-λ rule in systems with identical $c_i\mu_i = c\mu$ for all machine types (see Derman et al. (1980) and Iravani and Kolfal (2005)). Thus, if $c_p\mu_p \geq c_q\mu_q$ (which favors giving higher priority to machine type p), but $\lambda_p \geq \lambda_q$ (which favors giving higher priority to machine of type q), the value of $c_p\mu_p$ should be much larger than $c_q\mu_q$ for machine type p

to have a higher priority than machine type q. This is exactly what Condition **A1** implies. As **A1** shows, $c_p\mu_p$ should be at least $\frac{\lambda_p}{\lambda_q}$ times larger than $c_q\mu_q$ to guarantee that machine type p has a higher priority than machine type q in the optimal policy. (Note that $\frac{\lambda_p}{\lambda_q}$ is a number greater than or equal to one.)

On the other hand, when $\lambda_p < \lambda_q$ (which favors giving higher priority to machine of type p), machine type p can have a higher priority than machine type q, even when $c_p\mu_p$ is smaller than $c_q\mu_q$. However, $c_p\mu_p$ cannot be $[1 - \frac{\lambda_q-\lambda_p}{\Upsilon}]$ times smaller than $c_q\mu_q$, where $[1 - \frac{\lambda_q-\lambda_p}{\Upsilon}]$ is a number smaller than one.

In proposition 2 we describe that conditions **A1** and **A2** are also sufficient conditions that can be used to determine the priority between two machine types in systems where *preemption* is allowed.

Proposition 2. **A1** *and* **A2** *can be also used to prioritize machine types in systems with preemptive repair.*

Proof: The proof is presented in the Appendix. □

Note that conditions **A1** and **A2** identify whether repairing a machine of type p has a higher priority than repairing a machine of type q. However, it does not guarantee that repairing a machine of type p is better than idling. In other words, it does not guarantee that machine type q does not belong to idle group $\mathcal{I}$.

In general, one must use the MDP model to determine the idle and repair groups $\mathcal{I}$ and $\mathcal{R}$. In Theorem 4, however, we present sufficient conditions that guarantee a machine type to be in idle group $\mathcal{I}$. Before we present theorem 4, we need to introduce the following notation. We define Ψ_q as the set of machine types with the higher priority than machine type q, and $\bar{\Psi}_q$ as the set of machine types with lower than or same priority as machine type q (including type q), where $\Psi_q \cup \bar{\Psi}_q = \mathcal{N}$.

Theorem 4. *If all machine types can be prioritized by* **A1** *or* **A2**, *then if* **A3** *holds, where*

$$\textbf{A3}: \frac{c_q\mu_q}{\lambda_q} \le \frac{\sum_{j\in\Psi_q} N_j\lambda_j c_j\mu_j}{\sum_{j\in\Psi_q} N_j\lambda_j^2 + \Upsilon^2}$$

then machine type q and all machines of lower priority than q belong to the idle group (i.e., $\bar{\Psi}_q \subseteq \mathcal{I}$).

Proof: The proof is presented in the Appendix. □

Condition **A3** has a complex expression which includes the uniformization rate $\Upsilon = \sum_j (N_j\lambda_j + \mu_j)$ and Υ^2. Thus, it does not seem to have a simple intuitive interpretation. It introduces a threshold for $c_q\mu_q/\lambda_q$ that guarantees if $c_q\mu_q/\lambda_q$ is small enough to fall below that threshold, then machine type q will never be repaired under the optimal policy.

As an example, consider a simple case with two machine types, where $N_1 = N_2 = 2$, $\lambda_1 = 10$, $\mu_1 = 15$, $c_1 = 1$, and $\lambda_2 = 0.1$, $\mu_2 = 0.15$, $c_2 = 0.1$. Since $\frac{c_1\mu_1}{\lambda_1} = 1.5 \ge 0.15 = \frac{c_2\mu_2}{\lambda_2}$, Condition **A1** holds, and machine type 1 has higher priority than machine type 2. On the other hand, since $\frac{c_2\mu_2}{\lambda_2} = 0.15 \le 0.21 = \frac{N_1\lambda_1 c_1\mu_1}{N_1\lambda_1^2 + \Upsilon^2}$, then **A3** holds, and thus machine type 2 is in idle group $\mathcal{I}$. In this system, machines of type 2 will never be repaired under the optimal policy.

Note that Condition **A3** is a *sufficient condition* that, if it holds, guarantees that machines of type q are in the idle group. Therefore, if **A3** does not hold, it does not necessarily mean that machines of type q are not in the idle group.

6 Systems with non-idling policies

In this section we focus on systems in which idling is not permitted, i.e., systems in which the repairman never idles as long as there is at least one broken machine in the system. This is often the case in practice, since repairmen do not intentionally idle when there is a broken machine in the system. In our MDP formulation, this is equivalent to excluding action "idling" from the set of possible actions, which results in the omission of the optimality equation (2). However, it is easy to show that Theorem 2 and Theorem 3 still hold. Thus, we have the following corollary:

Corollary 1. *If idling is not allowed, then all machines are in the repair group (i.e., $\mathcal{R} = \mathcal{N}$), and a static priority rule is optimal.*

In systems in which idling is not allowed, the following corollaries show how our conditions **A1** and **A2** result in simple machine-priority rules in some special cases.

Corollary 2. *If machines can be renumbered such that $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N$, and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$, and we have $\frac{c_1\mu_1}{\lambda_1} \ge \frac{c_2\mu_2}{\lambda_2} \ge \cdots \ge \frac{c_N\mu_N}{\lambda_N}$, then machines of type j have higher priority than machines of type $j+1$, for $j = 1, 2, \ldots, N-1$.*

Corollary 3. *If machines can be renumbered such that $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N$, and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$, and we have $c_1\mu_1 \ge c_2\mu_2 \ge \cdots \ge c_N\mu_N$, then machines of type j have higher priority than machines of type $j+1$, for $j = 1, 2, \ldots, N-1$.*

Corollary 4. *If all machine types have the same failure rate (i.e., $\lambda_j = \lambda$ for $j = 1, 2, \ldots, N$), and machines can be renumbered such that $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_N$ and $c_1\mu_1 \ge c_2\mu_2 \ge \cdots \ge c_N\mu_N$, then machines of type j have* *higher priority than machines of type $j+1$, for $j = 1, 2, \dots, N-1$.*

Corollary 5. *If all machine types have the same down time cost and repair rates (i.e., $c_j = c$ and $\mu_j = \mu$ for $j = 1, 2, \dots, N$), then the least-λ rule is optimal. That is, machines must be prioritized such that the machine type with the lowest failure rate has the highest priority, and the machine type with the highest failure rate has the lowest priority.*

7 Conclusion

In this paper, we have studied the optimal service policy in finite population queueing systems in the context of the optimal repair policy in a machine-repairman problem with one repairman and heterogeneous machines. For the case of nonpreemptive repair, and with the objective of minimizing the total average downtime cost, we formulated the problem as an MDP model and fully characterized the structure of the optimal repair and idling policies. We showed that the optimal repair policy is a simple static priority policy, and we derived conditions under which one can determine the optimal priority sequence among machines of different types. We also showed that our conditions are valid in systems in which idling is not allowed as well as in systems in which preemption is allowed.

Future research should focus on systems with general repair times and time to failure. Other research directions include the study of the optimality of static-priority policies in machine-repairman problems with multiple repairmen, or with repairmen who are trained to repair only a subset of machines.

Appendix

Proof of Theorem 1:

PART (i): We use the sample path method and contradiction to prove that idling is not optimal when there is at least one broken machine in each machine type. Suppose policy Φ is the optimal policy that idles at time t_1, when there are at least one broken machine of each type. Specifically, suppose that policy Φ idles at time t_1 until time t_2 and repairs a machine of type r which takes τ_r units of time to finish the repair. Therefore, the repair of machine type r is finished at time t_3 ($t_3 = t_2 + \tau_r$).

Now consider policy Γ that behaves exactly the same as policy Φ from time 0 to time t_1. However, policy Γ repairs the same machine of type r first at time t_1 and finishes its repair at time $t_1 + \tau_r$. Then policy Γ idles until time t_3, and behaves exactly the same as policy Φ after time t_3. Note that the condition of having at least one broken machine of each type at time t_1 makes it possible for policy Γ to repair a machine of type r.

Define $E[TC(\Gamma)]$ and $E[TC(\Phi)]$ as the expected total down-time cost under Policies Γ and Φ, respectively. Then, we get

$$E[TC(\Gamma)] = E[TC(\Gamma)]_0^{t_1} + E[TC(\Gamma)]_{t_1}^{t_3} + E[TC(\Gamma)]_{t_3}^{\infty}$$

$$E[TC(\Phi)] = E[TC(\Phi)]_0^{t_1} + E[TC(\Phi)]_{t_1}^{t_3} + E[TC(\Phi)]_{t_3}^{\infty}$$

Since these two policies are the same before time t_1 and after time t_3, then $E[TC(\Gamma)]_0^{t_1} = E[TC(\Phi)]_0^{t_1}$, $E[TC(\Gamma)]_{t_3}^{\infty} = E[TC(\Phi)]_{t_3}^{\infty}$. Therefore, we have

$$\begin{aligned} E[TC(\Gamma)] - E[TC(\Phi)] &= E[TC(\Gamma)]_{t_1}^{t_3} - E[TC(\Phi)]_{t_1}^{t_3} \\ &= \left[\sum_{j=1}^{N} c_j n_j (t_3 - t_1) - c_r (t_2 - t_1)\right] \\ &\quad - \left[\sum_{j=1}^{N} c_j n_j (t_3 - t_1)\right] = -c_r (t_2 - t_1) \le 0 \end{aligned}$$

which implies that policy Φ cannot be optimal, a contradiction! Thus, it is not optimal to idle at time t_1 where there is at least one broken machine of each machine type.

PART (ii): We use contradiction to prove part (*ii*). Let us assume that it is optimal to be idle in state $\mathbf{n}_0$. However, the optimal action is to repair machine type q, $q \in J_{(\mathbf{n}'_0)}$, in state $\mathbf{n}'_0$. Consider another state $\mathbf{n}''_0$, where $n''_{0i} = min\{n_{0i}, n'_{0i}\}$ for $i = 1, 2, \dots, N$.

There are three possible cases for the optimal action in state $\mathbf{n}''_0$.

Case T1-1 :	Idling is optimal:
Case T1-2 :	Repairing a machine of type q is optimal in state $\mathbf{n}''_0$;
Case T1-3 :	Repairing a machine of type r ($r \ne q$, $r \in J_{(\mathbf{n}''_0)}$) is optimal in state $\mathbf{n}''_0$.

- **Case T1-1**: If idling is optimal in state $\mathbf{n}''_0$, we get

  $$V(\mathbf{n}''_0, q) - V(\mathbf{n}''_0, 0) \ge 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}''_0, q) - V(\mathbf{n}''_0, 0)$ is non-decreasing in n''_{0i}. Thus, since $n'_{0i} \ge n''_{0i}$ we get $V(\mathbf{n}'_0, q) - V(\mathbf{n}'_0, 0) \ge 0$, which implies repairing a machine of type q cannot be optimal in state $\mathbf{n}'_0$. This is a contradiction to our assumption!

- **Case T1-2**: If repairing a machine of type q is optimal in state $\mathbf{n}_0''$, we get

  $$V(\mathbf{n}_0'', 0) - V(\mathbf{n}_0'', q) \geq 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}_0'', 0) - V(\mathbf{n}_0'', q)$ is non-decreasing in n_{0i}. Thus, we get $V(\mathbf{n}_0, 0) - V(\mathbf{n}_0, q) \geq 0$, which implies that idling cannot be not optimal in state $\mathbf{n}_0$. This is a contradiction to our assumption!
- **Case T1-3**: If repairing a machine type r is optimal in state $\mathbf{n}_0''$, we get

  $$V(\mathbf{n}_0'', 0) - V(\mathbf{n}_0'', r) \geq 0 \text{ and } V(\mathbf{n}_0'', q) - V(\mathbf{n}_0'', r) \geq 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}_0'', 0) - V(\mathbf{n}_0'', r)$ and $V(\mathbf{n}_0'', q) - V(\mathbf{n}_0'', r)$ are non-decreasing in n_{0i}. Thus, we get $V(\mathbf{n}_0, 0) - V(\mathbf{n}_0, r) \geq 0$ and $V(\mathbf{n}_0', q) - V(\mathbf{n}_0', r) \geq 0$. These imply that idling is not optimal in $\mathbf{n}_0$ and repairing a machine of type q is not optimal in $\mathbf{n}_0'$. This is a contradiction to our assumption!

Therefore, the optimal actions in states $\mathbf{n}_0$ and $\mathbf{n}_0'$ should be the same. This concludes the proof of Theorem 1.

Proof of Theorem 2:

We use contradiction to prove Theorem 2. Let us assume that it is optimal to repair a machine of type p in state $\mathbf{n}$. However, the optimal action is to repair machine of type q ($q \neq p$) and $q \in J_{(\mathbf{n}')}$, in state $\mathbf{n}'$. Consider another state $\mathbf{n}''$, where $n_i'' = min\{n_i, n_i'\}$ for $i = 1, 2, \ldots, N$.
There are three possible cases for the optimal action in state $\mathbf{n}''$.

Case T2-1 : Repairing a machine of type p is optimal in state $\mathbf{n}''$;
Case T2-2 : Repairing a machine of type q ($q \neq p$) is optimal in state $\mathbf{n}''$;
Case T2-3 : Repairing a machine of type r ($r \neq p, q$) or idling is optimal in state $\mathbf{n}''$.

- **Case T2-1**: If repairing a machine of type p is optimal in state $\mathbf{n}''$, we get

  $$V(\mathbf{n}'', q) - V(\mathbf{n}'', p) \geq 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}'', q) - V(\mathbf{n}'', p)$ is non-decreasing in n_i''. Thus, since $n_i' \geq n_i''$ we get $V(\mathbf{n}', q) - V(\mathbf{n}', p) \geq 0$, which implies that repairing a machine of type q cannot be optimal in state $\mathbf{n}'$. This is a contradiction to our assumption!
- **Case T2-2**: If repairing a machine of type q is optimal in state $\mathbf{n}''$, we get

  $$V(\mathbf{n}'', p) - V(\mathbf{n}'', q) \geq 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}'', p) - V(\mathbf{n}'', q)$ is non-decreasing in n_i''. Thus, we get $V(\mathbf{n}, p) - V(\mathbf{n}, q) \geq 0$, which implies that repairing a machine of type p cannot be not optimal in state $\mathbf{n}$. This is a contradiction to our assumption!
- **Case T2-3**: If repairing a machine of type r ($r \neq p, q$) or idling (i.e., $r = 0$) is optimal in state $\mathbf{n}''$, we get

  $$V(\mathbf{n}'', p) - V(\mathbf{n}'', r) \geq 0 \text{ and } V(\mathbf{n}'', q) - V(\mathbf{n}'', r) \geq 0.$$

  On the other hand, according to **P2**, $V(\mathbf{n}'', p) - V(\mathbf{n}'', r)$ and $V(\mathbf{n}'', q) - V(\mathbf{n}_0'', r)$ are non-decreasing in n_i''. Thus, we get $V(\mathbf{n}, p) - V(\mathbf{n}, r) \geq 0$ and $V(\mathbf{n}', q) - V(\mathbf{n}', r) \geq 0$. These imply that repairing a machine of type p and repairing a machine of type q are not optimal in states $\mathbf{n}_0$ and $\mathbf{n}_0'$, respectively. A contradiction to our assumption!

This concludes the proof of Theorem 2. ■

Before we present the proof of Theorem 3, we need to introduce the following Lemma. The proof of this lemma can be found in Iravani et al. (2006), the full version of this paper.

Lemma 1. *If* **A1** *holds between machine type p and q, then*

$$\Upsilon[\mu_p D_p V(\boldsymbol{n}, r) - \mu_q D_q V(\boldsymbol{n}, r)] \geq c_p\mu_p - \frac{\lambda_p}{\lambda_q} c_q\mu_q \geq 0.$$

If **A2** *holds between machine type p and q, then*

$$\Upsilon[\mu_p D_p V(\boldsymbol{n}, r) - \mu_q D_q V(\boldsymbol{n}, r)] \geq c_p\mu_p - \left(1 - \frac{\lambda_q - \lambda_p}{\Upsilon}\right) c_q\mu_q \geq 0.$$

Thus, if either **A1** *or* **A2** *holds,* $\mu_p D_p V(\boldsymbol{n}, r) - \mu_q D_q V(\boldsymbol{n}, r) \geq 0$.

Proof of Theorem 3:

To prove that machine type p has a higher priority than machine type q ($q \neq p$), we need to show $V(\mathbf{n}, q) \geq V(\mathbf{n}, p)$ for $p, q \in J_{(\mathbf{n})}$. We will use induction and the value iteration algorithm to prove this. It is clear that $V_0(\mathbf{n}, q) \geq V_0(\mathbf{n}, p)$ at $k = 0$. We assume $V_{k-1}(\mathbf{n}, q) \geq V_{k-1}(\mathbf{n}, p)$ holds at iteration $k - 1$ and we will prove that it also holds at iteration k.

To show $V_k(\mathbf{n}, q) \geq V_k(\mathbf{n}, p)$, we need to discuss the optimal actions in states $\mathbf{n} - \mathbf{e}^p$ and $\mathbf{n} - \mathbf{e}^q$. The list of all possible combinations of the optimal actions can be divided into two cases, where $r \neq p$, and $r \neq 0$.

Case	n_p	Optimal action in $\mathbf{n}-\mathbf{e}^p$	Optimal action in $\mathbf{n}-\mathbf{e}^q$
Case T3-1	$n_p \geq 2$	p	p
		r	r
		0	0
	$n_p = 1$	r	r
Case T3-2	$n_p = 1$	r	p
		0	p

Case T3-1: In this case, without loss of generality, we assume that action r, $r \in J^0_{(\mathbf{n}-\mathbf{e}^q)} \cap J^0_{(\mathbf{n}-\mathbf{e}^p)}$, is optimal in both states $\mathbf{n}-\mathbf{e}^q$ and $\mathbf{n}-\mathbf{e}^p$, i.e.,

$$\min_{s \in J^0_{(\mathbf{n}-\mathbf{e}^q)}} \{V_{k-1}(\mathbf{n}-\mathbf{e}^q, s)\} = V_{k-1}(\mathbf{n}-\mathbf{e}^q, r) \quad \text{and}$$

$$\min_{s \in J^0_{(\mathbf{n}-\mathbf{e}^p)}} \{V_{k-1}(\mathbf{n}-\mathbf{e}^p, s)\} = V_{k-1}(\mathbf{n}-\mathbf{e}^p, r).$$

Therefore, after some algebra we get:

$$\begin{aligned} &V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \\ &= \frac{1}{\Upsilon}\sum_{j=1}^{N}(N_j - n_j)\lambda_j\big[V_{k-1}(\mathbf{n}+\mathbf{e}^j, q) - V_{k-1}(\mathbf{n}+\mathbf{e}^j, p)\big] \\ &\quad + \frac{1}{\Upsilon}\sum_{j=1}^{N} n_j\lambda_j[V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}, p)] \\ &\quad + \frac{1}{\Upsilon}\sum_{j=1, j\neq p, j\neq q}^{N} \mu_j[V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}, p)] \\ &\quad + \frac{\mu_q}{\Upsilon}\big[V_{k-1}(\mathbf{n}-\mathbf{e}^q, r) - V_{k-1}(\mathbf{n}, p)\big] \\ &\quad + \frac{\mu_p}{\Upsilon}\big[V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)\big]. \end{aligned} \tag{3}$$

From iteration $k-1$, we have $V_{k-1}(\mathbf{n}+\mathbf{e}^i, q) - V_{k-1}(\mathbf{n}+\mathbf{e}^i, p) \geq 0$, and $V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}, p) \geq 0$. Then we can rewrite (3) as:

$$\begin{aligned} V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq &\frac{1}{\Upsilon}\big(\mu_p\big[V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)\big] \\ &-\mu_q\big[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}-\mathbf{e}^q, r)\big]\big) \end{aligned} \tag{4}$$

Since we have $V_{k-1}(\mathbf{n}, q) \geq V_{k-1}(\mathbf{n}, p)$, we replace $V_{k-1}(\mathbf{n}, q) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)$ with $V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)$ in inequality (4), and we get

$$\begin{aligned} V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq &\frac{1}{\Upsilon}\big(\mu_p\big[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)\big] \\ &- \mu_q\big[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}-\mathbf{e}^q, r)\big]\big) \\ = &\frac{1}{\Upsilon}\big(\mu_p[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}, r)] \\ &- \mu_q[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}, r)]\big) \\ &+ \frac{1}{\Upsilon}\big(\mu_p\big[V_{k-1}(\mathbf{n}, r) - V_{k-1}(\mathbf{n}-\mathbf{e}^p, r)\big] \\ &- \mu_q\big[V_{k-1}(\mathbf{n}, r) - V_{k-1}(\mathbf{n}-\mathbf{e}^q, r)\big]\big) \\ = &\frac{1}{\Upsilon}(\mu_p - \mu_q)[V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}, r)] \\ &+ \frac{1}{\Upsilon}[\mu_p D_p V_{k-1}(\mathbf{n}, r) - \mu_q D_q V_{k-1}(\mathbf{n}, r)] \end{aligned} \tag{5}$$

Because action r is optimal in both states $\mathbf{n}-\mathbf{e}^q$ and $\mathbf{n}-\mathbf{e}^p$, then action r is also optimal in state $\mathbf{n}$, which implies $V_{k-1}(\mathbf{n}, p) - V_{k-1}(\mathbf{n}, r) \geq 0$. Then we can rewrite (5) as:

$$\begin{aligned} V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq &\frac{1}{\Upsilon}[\mu_p D_p V_{k-1}(\mathbf{n}, r) \\ &- \mu_q D_q V_{k-1}(\mathbf{n}, r)]. \end{aligned} \tag{6}$$

Since Lemma 1 holds at iteration $k-1$, the right hand side of inequality (6) is non-negative since **A1** or **A2** holds at iteration $k-1$. Therefore, $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq 0$, which means that machine type p has a higher priority than q at iteration k. This concludes the proof for Case T3-1.

Case T3-2: First we use contradiction to show that if $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$, then $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq 0$. Suppose $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$, but $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \leq 0$. According to **P2**, if $V_k(\mathbf{n}, p) - V_k(\mathbf{n}, q) \geq 0$, then

$$V_k(\mathbf{n}+\mathbf{e}^p, p) - V_k(\mathbf{n}+\mathbf{e}^p, q) \geq V_k(\mathbf{n}, p) - V_k(\mathbf{n}, q) \geq 0,$$

which contradicts $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$. Thus, when $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$, we will get $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq 0$.

We can prove that $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq 0$ by showing that $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$. The proof of $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$ is similar to that for Case T3-1 with states $\mathbf{n}-\mathbf{e}^p$ and $\mathbf{n}-\mathbf{e}^q$ replaced with $\mathbf{n}$ and

$\mathbf{n} + \mathbf{e}^p - \mathbf{e}^q$. The major difference is that

$$\min_{s \in J^0_{(\mathbf{n})}} \{V_{k-1}(\mathbf{n}, s)\} = V_{k-1}(\mathbf{n}, p), \quad \text{and}$$
$$\min_{s \in J^0_{(\mathbf{n}+\mathbf{e}^p-\mathbf{e}^q)}} \left\{V_{k-1}(\mathbf{n} + \mathbf{e}^p - \mathbf{e}^q, s)\right\} = V_{k-1}(\mathbf{n} + \mathbf{e}^p - \mathbf{e}^q, p)$$

This is because, according to Theorem 2, the optimal action in states $\mathbf{n}$ and $\mathbf{n} + \mathbf{e}^p - \mathbf{e}^q$ are to repair a machine of type p since in Case T3-2, the optimal action in state $\mathbf{n} - \mathbf{e}^q$ is to repair a machine of type p.

For Case T3-2 we will get the following inequality similar to (6):

$$V_k(\mathbf{n} + \mathbf{e}^p, q) - V_k(\mathbf{n} + \mathbf{e}^p, p) \geq \frac{1}{\Upsilon}\left[\mu_p D_p V_{k-1}(\mathbf{n} + \mathbf{e}^p, p) - \mu_q D_q V_{k-1}(\mathbf{n} + \mathbf{e}^p, p)\right]. \tag{7}$$

Since Lemma 1 holds at iteration $k-1$, the right-hand-side of the above inequality is non-negative because **A1** or **A2** holds at iteration $k-1$. Therefore, $V_k(\mathbf{n}+\mathbf{e}^p, q) - V_k(\mathbf{n}+\mathbf{e}^p, p) \geq 0$, that is, $V_k(\mathbf{n}, q) - V_k(\mathbf{n}, p) \geq 0$ which means that machine type p has higher priority than machine type q at iteration k. This concludes the proof for Case T3-2 and Theorem 3. ■

Proof of Proposition 2:

When preemption is allowed, it is shown in Iravani and Kolfal (2004), that if $\frac{c_p\mu_p}{\lambda_p} \geq \frac{c_q\mu_q}{\lambda_q}$ and either Conditions **C1** or **C2** holds, then machines of type p have higher priority than machines of type q, where

$$\mathbf{C1}: c_p\mu_p \geq c_q\mu_q$$
$$\mathbf{C2}: c_p\mu_p < c_q\mu_q, \quad \lambda_p < \lambda_q, \quad \text{and}$$
$$c_p\mu_p \geq \left\{1 - \frac{\lambda_q - \lambda_p}{\Upsilon}\right\} c_q\mu_q$$

We will show that if **A1** holds, then **C1** holds, and if **A2** holds, then one of Conditions **C1** or **C2** holds.

If **A1** holds, we have $\frac{c_p\mu_p}{\lambda_p} \geq \frac{c_q\mu_q}{\lambda_q}$. On the other hand, since $\lambda_p \geq \lambda_q$, then $\frac{\lambda_p}{\lambda_q} \geq 1$. Hence, Condition **A1** also implies that $c_p\mu_p \geq c_q\mu_q$. Consequently, if **A1** holds, then **C1** holds.

If **A2** holds, then we have two cases, Case I: **A2** holds and $c_p\mu_p \geq c_q\mu_q$, and Case II: **A2** holds and $c_p\mu_p < c_q\mu_q$.

Case I: If **A2** holds, then we have $\lambda_p < \lambda_q$, which implies that

$$\left\{1 - \frac{\lambda_q - \lambda_p}{\Upsilon}\right\} \geq \frac{\lambda_p}{\lambda_q}$$

Therefore, from **A2** we can conclude

$$c_p\mu_p \geq \left\{1 - \frac{\lambda_q - \lambda_p}{\Upsilon}\right\} c_q\mu_q \geq \frac{\lambda_p}{\lambda_q} c_q\mu_q \implies \frac{c_p\mu_p}{\lambda_p} \geq \frac{c_q\mu_q}{\lambda_q}$$

Hence, if **A2** holds and $c_p\mu_p \geq c_q\mu_q$, then we will also have $\frac{c_p\mu_p}{\lambda_p} \geq \frac{c_q\mu_q}{\lambda_q}$, which implies that Condition **C1** holds.

Case II: If **A2** holds and $c_p\mu_p < c_q\mu_q$, then **C2** holds, since as was shown in Case I, if **A2** holds, we have $\frac{c_p\mu_p}{\lambda_p} \geq \frac{c_q\mu_q}{\lambda_q}$.

Therefore, Conditions **A1** and **A2** can also be used in systems with preemption to prioritize machines of different types. ■

Proof of Theorem 4:

It is easy to show that, if q belongs to idle group (i.e., $q \in \mathcal{I}$), then all machine types with lower priority than q will also be in the idle group (i.e., $\bar{\Psi}_q \subseteq \mathcal{I}$). On the other hand, to show that $q \in \mathcal{I}$, we only need to show that in state $\mathbf{n}^*_\mathbf{q}$, where $n^*_j = 0$ for all $j \in \Psi_q$ and $n_q \geq 1$, idling is optimal, i.e.,

$$V(\mathbf{n}^*_\mathbf{q}, q) - V(\mathbf{n}^*_\mathbf{q}, 0) \geq 0. \tag{8}$$

Define state $\mathbf{n_q}$ where $n_j = n^*_j = 0, \forall j \in \Psi_q$, and $n_j = N_j, \forall j \in \bar{\Psi}_q$. To prove (8), we only need to prove that

$$V(\mathbf{n_q}, q) - V(\mathbf{n_q}, 0) \geq 0. \tag{9}$$

The reason is that according to Theorem 1, if (9) is true which implies that idling is optimal in state $\mathbf{n_q}$, then idling will also be optimal in state $\mathbf{n}^*_\mathbf{q}$.

We use induction and the value iteration algorithm to prove $V(\mathbf{n_q}, q) - V(\mathbf{n_q}, 0) \geq 0$ if **A3** holds. It is clear that $V_0(\mathbf{n_q}, q) - V_0(\mathbf{n_q}, 0) \geq 0$ at iteration 0. We assume $V_{k-1}(\mathbf{n_q}, q) - V_{k-1}(\mathbf{n_q}, 0) \geq 0$ holds at iteration $k-1$, and we will prove that it also holds at iteration k when **A3** holds.

From equation (2), we get

$$V_k(\mathbf{n_q},q)-V_k(\mathbf{n_q},0)=\frac{1}{\Upsilon}\sum_{j\in\Psi_q}N_j\lambda_j\Big[V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-\min_{s\in J^0_{(\mathbf{n}+\mathbf{e}^j)}}\big\{V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,s\big)\big\}\Big]$$
$$+\frac{1}{\Upsilon}\Bigg[\sum_{j\in\bar{\Psi}_q}^{N}N_j\lambda_j+\sum_{j=1,j\neq q}^{N}\mu_j\Bigg]\times[V_{k-1}(\mathbf{n_q},q)-V_{k-1}(\mathbf{n_q},0)]$$
$$+\frac{\mu_q}{\Upsilon}\Big[\min_{s\in J^0_{(\mathbf{n}-\mathbf{e}^q)}}\big\{V_{k-1}\big(\mathbf{n_q}-\mathbf{e}^q,s\big)\big\}-V_{k-1}(\mathbf{n_q},0)\Big]$$

In state $\mathbf{n_q}+\mathbf{e}^j$, we have $V_{k-1}(\mathbf{n_q}+\mathbf{e}^j,j)\geq\min_{s\in J^0_{(\mathbf{n}+\mathbf{e}^j)}}\{V_{k-1}(\mathbf{n_q}+\mathbf{e}^j,s)\}$.

Since at iteration $k-1$ we have $V_{k-1}(\mathbf{n_q},q)-V_{k-1}(\mathbf{n_q},0)\geq 0$, then idling is optimal in state $\mathbf{n_q}$. Thus, according to Theorem 1 part (ii), idling is also optimal in state $\mathbf{n_q}-\mathbf{e}^q$; that is, $\min_{s\in J^0_{(\mathbf{n}-\mathbf{e}^q)}}\{V_{k-1}(\mathbf{n_q}-\mathbf{e}^q,s)\}=V_{k-1}(\mathbf{n_q}-\mathbf{e}^q,0)$. Therefore,

$$V_k(\mathbf{n_q},q)-V_k(\mathbf{n_q},0)\geq\frac{1}{\Upsilon}\sum_{j\in\Psi_q}N_j\lambda_j\big[V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,j\big)\big]$$
$$+\frac{1}{\Upsilon}\Bigg[\sum_{j\in\bar{\Psi}_q}^{N}N_j\lambda_j+\sum_{j=1,j\neq q}^{N}\mu_j\Bigg]\times[V_{k-1}(\mathbf{n_q},q)-V_{k-1}(\mathbf{n_q},0)]$$
$$+\frac{\mu_q}{\Upsilon}\big[V_{k-1}\big(\mathbf{n_q}-\mathbf{e}^q,0\big)-V_{k-1}(\mathbf{n_q},0)\big].\tag{10}$$

From iteration $k-1$, we have $V_{k-1}(\mathbf{n_q},q)-V_{k-1}(\mathbf{n_q},0)\geq 0$. Hence, the second term on the right-hand-side of (10) is non-negative, and (10) can be reduced to:

$$V_k(\mathbf{n_q},q)-V_k(\mathbf{n_q},0)\geq\sum_{j\in\Psi_q}\frac{N_j\lambda_j}{\Upsilon}\big[V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,j\big)\big]$$
$$+\frac{\mu_q}{\Upsilon}\big[V_{k-1}\big(\mathbf{n_q}-\mathbf{e}^q,0\big)-V_{k-1}(\mathbf{n_q},0)\big].\tag{11}$$

Now we find a lower bound for $V_{k-1}(\mathbf{n_q}+\mathbf{e}^j,q)-V_{k-1}(\mathbf{n_q}+\mathbf{e}^j,j)$. Since machine type j ($j\in\Psi_q$) has higher priority than machine type q, according to inequality (6) which was obtained in the proof of Theorem 3, we have

$$V_k(\mathbf{n}+\mathbf{e}^j,q)-V_k(\mathbf{n}+\mathbf{e}^j,j)\geq\frac{1}{\Upsilon}\big[\mu_jD_jV_{k-1}(\mathbf{n}+\mathbf{e}^j,r)-\mu_qD_qV_{k-1}(\mathbf{n}+\mathbf{e}^j,r)\big].\tag{12}$$

If $\lambda_j\geq\lambda_q$, then **A1** holds between machine j and q. According to Lemma 1, we can rewrite (12) as:

$$V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,j\big)\geq\frac{1}{\Upsilon^2}\Big[c_j\mu_j-\frac{\lambda_j}{\lambda_q}c_q\mu_q\Big].\tag{13}$$

If $\lambda_j<\lambda_q$, then **A2** holds between machine j and q. According to Lemma 1 we can rewrite (12) as:

$$V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,j\big)\geq\frac{1}{\Upsilon^2}\Big[c_j\mu_j-\Big(1-\frac{\lambda_q-\lambda_j}{\Upsilon}\Big)c_q\mu_q\Big]\geq\frac{1}{\Upsilon^2}[c_j\mu_j-c_q\mu_q].\tag{14}$$

From **P4**, we get

$$V_{k-1}(\mathbf{n_q},0)-V_{k-1}\big(\mathbf{n_q}-\mathbf{e}^q,0\big)\leq\frac{c_q}{\lambda_q}.\tag{15}$$

We divide the set of machine types in Ψ_q into two subsets Ψ_{qH} and Ψ_{qL} ($\Psi_q=\Psi_{Hq}\cup\Psi_{Lq}$), where Ψ_{qH} is the set of machine types with higher (or equal) failure rate than machine type q, and Ψ_{qL} is the set of machine types with lower failure rate than machine type q. By plugging inequality (13), (14) and (15) into (11), we get

$$[V_k(\mathbf{n_q},q)-V_k(\mathbf{n_q},0)]\geq\sum_{j\in\Psi_q}\frac{N_j\lambda_j}{\Upsilon}\big[V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,q\big)-V_{k-1}\big(\mathbf{n_q}+\mathbf{e}^j,j\big)\big]+\frac{\mu_q}{\Upsilon}\big[V_{k-1}\big(\mathbf{n_q}-\mathbf{e}^q,0\big)-V_{k-1}(\mathbf{n_q},0)\big]$$
$$\geq\sum_{j\in\Psi_{qH}}\frac{N_j\lambda_j}{\Upsilon^3}\Big[c_j\mu_j-\frac{\lambda_j}{\lambda_q}c_q\mu_q\Big]+\sum_{j\in\Psi_{qL}}\frac{N_j\lambda_j}{\Upsilon^3}[c_j\mu_j-c_q\mu_q]-\frac{c_q\mu_q}{\Upsilon\lambda_q}.$$

That is, we have $V_k(\mathbf{n_q},q)-V_k(\mathbf{n_q},0)\geq 0$ if $\sum_{j\in\Psi_{qH}}\frac{N_j\lambda_j}{\Upsilon^3}[c_j\mu_j-\frac{\lambda_j}{\lambda_q}c_q\mu_q]+\sum_{j\in\Psi_{qL}}\frac{N_j\lambda_j}{\Upsilon^3}[c_j\mu_j-c_q\mu_q]-\frac{c_q\mu_q}{\Upsilon\lambda_q}\geq 0$, or if

$$\frac{c_q\mu_q}{\lambda_q}\leq\frac{\sum_{j\in\Psi_q}N_j\lambda_jc_j\mu_j}{\sum_{j\in\Psi_{qH}}N_j\lambda_j^2+\sum_{j\in\Psi_{qL}}N_j\lambda_j\lambda_q+\Upsilon^2}.\tag{16}$$

Since for all machine types in Ψ_{qL}, we have $\lambda_j < \lambda_q$, then, $\sum_{j\in\Psi_{qL}} N_j\lambda_j\lambda_q \geq \sum_{j\in\Psi_{qL}} N_j\lambda_j^2$, and (16) can be rewritten as

$$\frac{c_q\mu_q}{\lambda_q} \leq \frac{\sum_{j\in\Psi_q} N_j\lambda_j c_j\mu_j}{\sum_{j\in\Psi_{qH}} N_j\lambda_j^2 + \sum_{j\in\Psi_{qL}} N_j\lambda_j^2 + \Upsilon^2} \leq \frac{\sum_{j\in\Psi_q} N_j\lambda_j c_j\mu_j}{\sum_{j\in\Psi_q} N_j\lambda_j^2 + \Upsilon^2}. \quad (17)$$

Thus, when **A3** holds, machine type q will be in $\mathcal{I}$. This concludes the proof of Theorem 4. ■

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